Institute for Applied Mathematics
WS 2018/19
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Sheet 9, "Markov Processes"

Due on December 18, 2018

Exercise 1 *[7 Pt]*

Consider the setting and the notation of Section 8.3 from the lecture notes.

(i) Show that the functions $x_n = \mathbb{1}_{[1-1/n,\infty)}$ converge to the function $x_\infty = 1_{[1,\infty)}$ with respect to the metric d (cf. equation (8.3.3) from the lecture notes).

(ii) For all $n \in \mathbb{N}$, let $x_n = \mathbb{1}_{[0,1/2^n)}$.

  (a) Show that $(x_n)_n$ is a Cauchy sequence with respect to the metric d_{J_1} (cf. equation (8.3.17) from the lecture notes).

  (b) Show that $(x_n)_n$ does not converge with respect to the metric d_{J_1}.

  (c) Show that $(x_n)_n$ is not Cauchy with respect to the metric d.

Exercise 2 *[8 Pt]*

Let S be a locally compact Polish space, and let $(P_t)_{t\geq 0}$ be a Feller-Dynkin semigroup acting on the space $C_0(S)$. You may suppose that $P_t 1 = 1$ for all $t \geq 0$. We say that a probability measure μ on S is *stationary* for $(P_t)_{t\geq 0}$ if $\mu P_t = \mu$ for all $t > 0$, i.e., if

$$\int f d\mu = \int P_t f d\mu \qquad \text{for all } f \in C_0(S) \text{ and } t \geq 0.$$

Let $\mathcal{M}_1(S)$ denote the set of all probability measures on S, and let $\mathcal{J} \subset \mathcal{M}_1(S)$ denote the set of all stationary probability measures.

(i) Show that if $\mu = \lim_{t\to\infty} \nu P_t$ exists (in the weak sense) for some $\nu \in \mathcal{M}_1(S)$, then $\mu \in \mathcal{J}$.

(ii) Show that if $\mu = \lim_{n\to\infty} t_n^{-1} \int_0^{t_n} \nu P_t dt$ exists for some $\nu \in \mathcal{M}_1(S)$ and some $t_n \uparrow \infty$, then $\mu \in \mathcal{J}$.

(iii) Suppose that S is compact. Show that $\mathcal{J}$ is a compact subset (with respect to weak convergence) of $\mathcal{M}_1(S)$.

Exercise 3 [*7 Pt*]

Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1(S)$ and $\mu \in \mathcal{M}_1(S)$, where S is a Polish space. Show that the following conditions are equivalent.

(i) $\mu_n \to \mu$ weakly;

(ii) for every closed $F \subset S$, $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$;

(iii) for every open $G \subset S$, $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$.