

## Sheet 8, “Markov Processes”

Due on December 11, 2018

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### Exercise 1

[8 Pt]

Suppose  $c : \mathbb{R} \rightarrow (0, \infty)$  is strictly positive and continuous.

- (i) Consider a Feller-Dynkin Markov process whose generator, when restricted to  $C^2$ -functions with compact support, is given by

$$Gf(x) = \frac{1}{2}c(x)f''(x).$$

By applying the martingale problem to an appropriate function, show that if  $a < x < b$  and  $\tau$  is the hitting time of  $\{a, b\}$ , then

$$\mathbb{E}_x \tau = \int_a^b \frac{2}{c(z)} \frac{(x \wedge z - a)(b - x \vee z)}{b - a} dz.$$

*Remark.* You may suppose that  $\tau < \infty$  a.s.

- (ii) Use part (i) to show that if  $\tau$  is the hitting time of 0, then for  $x > 0$ ,  $\mathbb{E}_x \tau < \infty$  if and only if  $\int_0^\infty \frac{1}{c(z)} dz < \infty$ .

### Exercise 2

[12 Pt]

Let  $B$  be the one-dimensional Brownian motion, and let  $\tau$  be the hitting time of 0.

- (i) Let the process  $X_a$  be defined by

$$X_a(t) = \begin{cases} B(t) & : t < \tau, \\ 0 & : t \geq \tau. \end{cases}$$

Show that the generator  $(G_a, D(G_a))$  of  $X_a$  is given by

$$G_a f = \frac{1}{2} f'' \quad \text{and} \quad \mathcal{D}(G_a) = \{f \in C_0(\mathbb{R}) : f', f'' \in C_0(\mathbb{R}), f''(0) = 0\}.$$

(ii) Let the process  $X_r$  be defined by

$$X_r(t) = |B(t)|.$$

Show that the generator  $(G_r, D(G_r))$  of  $X_r$  is given by

$$G_r f = \frac{1}{2} f'' \quad \text{and} \quad \mathcal{D}(G_r) = \{f \in C_0(\mathbb{R}) : f', f'' \in C_0(\mathbb{R}), f'(0) = 0\}.$$

(iii) Define the operator  $(G, D(G))$  by

$$Gf = \frac{1}{2} f'' \quad \text{and} \quad \mathcal{D}(G) = \{f \in C_0(\mathbb{R}) : f', f'' \in C_0(\mathbb{R}), f'(0) = f''(0) = 0\}.$$

Show that  $(G, D(G))$  is not the generator of a Feller-Dynkin semi-group.