Institute for Applied Mathematics
WS 2018/19
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Sheet 7, "Markov Processes"

Due on December 4, 2018

Exercise 1 *[6 Pt]*

Let (S, d) be a Polish space. Let X and Y be S-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is assumed to be a Polish space. Furthermore, let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose that $M \subset C_b(S)$ is separating and countable and that

$$\mathbb{E}[f(X)|\mathcal{G}] = f(Y) \qquad \text{a.s. for all } f \in M. \tag{1}$$

Show that $X = Y$ a.s.

Definition. A set $M \subset C_b(S)$ is called *separating* if whenever μ and ν are probability measures on S and

$$\int_S f\, d\mu = \int_S f\, d\nu \qquad \text{for all } f \in M, \tag{2}$$

then $\mu = \nu$.

Exercise 2 *[6 Pt]*

Let B be the one-dimensional Brownian motion, and let

$$Z = \{t \geq 0 \,|\, B_t = 0\}.$$

Use the strong Markov property to show that, almost surely, Z is a perfect subset of $\mathbb{R}$.

Definition. $A \subset \mathbb{R}$ is called *perfect* if A is equal to the set of all limit points of A.

Exercise 3 *[8 Pt]*

Consider a Feller process X on $\mathbb{R}$, whose generator is given by

$$Gf = \frac{1}{2} f'' - f'$$

for C^2-functions f with compact support. For all $b \in \mathbb{R}$ let $\tau_b = \inf\{t \geq 0 \,|\, X_t = b\}$.

(i) Use the martingale problem to show that $\lim_{b\to\infty} b\,\mathbb{P}^x(\tau_b < \tau_0) = 0$ for $x > 0$.

(iii) Use the martingale problem to compute $\mathbb{E}_x \tau_0$ for $x > 0$.