Institute for Applied Mathematics
WS 2018/19
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Sheet 6, "Markov Processes"

Due on November 27, 2018

Exercise 1 *[4 Pt]*

A linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset B \to B$ on a Banach space B is called closable if the closure of its graph $\{(f, \mathcal{L}f) : f \in \mathcal{D}(\mathcal{L})\}$ is again the graph of a linear operator.

(i) Show that the operator $\mathcal{L}$ is closable if and only if

$$\forall (f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{L}) : \; f_n \to 0, \; \mathcal{L}f_n \to g \implies g = 0$$

(ii) Give an example of a linear operator that is not closable.

Exercise 2 *[6 Pt]*

Let the family $\{\mathbb{P}_x\}_{x\in\mathbb{R}}$ of probability measures be such that

- for $x \neq 0$, $\mathbb{P}_x$ denotes the law of the one-dimensional Brownian motion starting from x, and
- $\mathbb{P}_0$ is the law of the process, which is constant 0, i.e. $\mathbb{P}_0$ is a Dirac measure.

Let X be a process with law $\{\mathbb{P}_x\}_{x\in\mathbb{R}}$.

a) Show that X satisfies the Markov property, i.e. for all bounded random variables η,

$$\mathbb{E}_x[\theta_t \eta | \mathfrak{F}_t] = \mathbb{E}_{X_t}[\eta] \qquad \mathbb{P}_x - \text{a.s.}.$$

b) Show that X does not satisfy the strong Markov property.

Exercise 3 *[10 Pt]*

Let $\{\mathbb{P}_x\}_{x\in\mathbb{R}}$ be the law of the one-dimensional Brownian motion B. Let $0 < b < a$ and let $t \geq 0$.

a) Use the strong Markov property to show that

$$\mathbb{P}_0 \left[\sup_{0\leq s\leq t} B_s > a, B_t < b \right] = \mathbb{P}_0 \left[B_t > 2a - b \right].$$

b) Show that $\sup_{0\leq s\leq t} B_s$ and $|B_t|$ have the same distribution.

c) Show that $\sup_{0\leq s\leq t} B_s - B_t$ and $|B_t|$ have the same distribution.

d) Let $\tau = \inf\{t \geq 0 \,|\, B_t = 0\}$ and let $x \neq 0$. Show that

$$\mathbb{P}_x\left[\tau < t\right] = \int_0^t \frac{|x|}{\sqrt{2\pi z^3}}\, e^{-\frac{x^2}{2z}}\, dz.$$