Institute for Applied Mathematics
WS 2018/19
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Sheet 2, “Markov Processes”

Due on October 30, 2018

Exercise 1 *[6 Pt]*

Let X_t be a one-dimensional Brownian motion started at 0 and let $T = \min\{t : |X_t| = 1\}$ and $T^* = \min\{t : X_t = 1 \text{ or } X_t = -3\}$.

a) Show that X_T and T are independent random variables.

b) Show that T^* and X_{T^*} are not independent.

Exercise 2 *[8 Pt]*

Let $(X_t)_t$ be a Lévy process with respect to the filtration $(\mathcal{F}_t)_t$, and let T be a finite stopping time. Show that the process $t \mapsto Y_t := X_{T+t} - X_T$ is independent of $\mathcal{F}_T$ and that $(X_t)_t$ and $(Y_t)_t$ have the same law.

Hint: Consider the stopping times $(T_n)_n$ defined by

$$T_n(\omega) = \frac{k+1}{2^n} \qquad \text{if} \qquad \frac{k}{2^n} \leq T(\omega) < \frac{k+1}{2^n}.$$

Notice that $T_n \downarrow T$ a.s. In a first step show that for any $t_1 < t_2 < \cdots < t_m$, $m \geq 1$, any bounded and continuous function f on $\mathbb{R}^m$ and any $A \in \mathcal{F}_T$, we have

$$E\left[f(X_{T_n+t_1} - X_{T_n}, \ldots, X_{T_n+t_m} - X_{T_n}) 1_A\right] = E\left[f(X_{t_1}, \ldots, X_{t_m})\right] P[A]$$

Exercise 3 *[6 Pt]*

Let $a > 0$. Let N be a Poisson counting process with intensity $\lambda > 0$ and $N_0 = 0$. Let $T_a = \min\{t \geq 0 \,|\, N_t = a\}$. Suppose that $E[\exp(\varepsilon T_a)] < \infty$ for sufficiently small $\varepsilon > 0$.

1. Let $\theta \geq 0$. Let $W_t = \exp(-\theta N_t + \lambda t(1 - e^{-\theta}))$ for all $t \geq 0$. Show that $(W_t)_t$ is a martingale.
2. Use the optional stopping theorem to show that $\mathrm{Var}(T_a) = a\lambda^{-2}$.

   Hint: Consider the <u>*moment*</u> generating function of T_a. Use the integrability assumption on $\exp(\varepsilon T_a)$ to ensure that you are allowed to exchange limits and expectations in certain cases.