Institute for Applied Mathematics
WS 2018/19
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Sheet 12, "Markov Processes"

Due on January 22, 2019

Exercise 1 *[10 Pt]*

Let B be a one dimensional Brownian motion, $x > 0$ and let $\mu, \sigma \in \mathbb{R}$ with $\sigma \neq 0$. Define the process $(X_t)_{t\in\mathbb{R}_+}$ by

$$X_t = x\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}.$$

(i) Show that X is the unique strong solution of a stochastic differential equation.

(ii) Give the precise form of a generator G such that X is a solution of the martingale problem for G.

Exercise 2 *[10 Pt]*

Consider a linear operator G that is given by

$$Gf(x) = \frac{1}{2}x(1-x)f''(x) \qquad \text{for all polynomials } f : [0,1] \to \mathbb{R}. \tag{1}$$

(i) Show that the closure $\bar{G}$ of G on $C([0,1])$ exists, and that $\bar{G}$ is the generator of a Feller-Dynkin semi-group $(P_t)_{t\in[0,\infty)}$.

(ii) Let $(X_t)_{t\in[0,\infty)}$ be the corresponding Markov process for $(P_t)_{t\in[0,\infty)}$ (cf. Theorem 4.33 of the lecture notes). Suppose that $(X_t)_{t\in[0,\infty)}$ has continuous paths, and let $\tau = \inf\{t \geq 0 \,|\, X_t = 0 \text{ or } X_t = 1\}$. Show that for all $x \in [0,1]$,

(a) $\mathbb{E}^x\left[\int_0^\infty X_t(1-X_t)\,dt\right] = x(1-x)$,

(b) $\mathbb{P}^x[X_\tau = 1] = x$,

(c) $\mathbb{E}^x[\tau] = -2x\log x - 2(1-x)\log(1-x)$.