Institute for Applied Mathematics
WS 2018/19
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Sheet 11, "Markov Processes"

Due on January 15, 2019

You shall not forget to register for the exam!

Exercise 1 *[7 Pt]*

Suppose that $X_1(t), \dots, X_n(t)$ are independent Brownian motions, and let

$$Y(t) = \sqrt{\sum_{j=1}^{n} X_j^2(t)}.$$

Show that if $f \in C^2([0,\infty))$ has compact support and satisfies $f'(0) = 0$, then it belongs to the domain of the generator G of Y, and that

$$Gf(y) = \frac{1}{2} f''(y) + \beta \frac{f'(y)}{y}$$

for some choice of β, and determine β.
Remark: Processes with generators of this type are called Bessel processes - so named because one form of Bessel's differential equation can be written as $\mathcal{L}f + f = 0$.

Exercise 2 *[8 Pt]*

The semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(P_t f)(x) = (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right) e^{-y^2/2}\, dy \qquad \text{for } f \in C_0(\mathbb{R}).$$

(i) Let G denote the generator corresponding to $(P_t)_{t\in[0,\infty)}$. Show that

$$(Gf)(x) = f''(x) - xf'(x) \qquad \text{for any } f \in C^2(\mathbb{R}) \text{ with compact support.}$$

(ii) Show that the standard normal distribution γ is the unique stationary distribution for $(P_t)_{t\in[0,\infty)}$

Hint. You may use Sheet 9, Exercise 2, and you may verify pointwise convergence of $P_t f(x)$ as $t \to \infty$ for fixed $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$. The latter is also useful to show the uniqueness claim!

Exercise 3 [*5 Pt*]

Let S be a locally compact Polish space, and let G be the generator on $C_0(S)$ of a Feller-Dynkin semi-group $(P_t)_{t\in[0,\infty)}$.

(i) Let $f \in D(G)$. Show that $P_t f \in D(G)$ and

$$\frac{d}{dt} P_t f = P_t G f = G P_t f.$$

(ii) Let $f \in C_0(S)$. Show that

$$\lim_{n\to\infty} \left(1 - \frac{t}{n} G\right)^{-n} f = P_t f.$$

Hint. You may show that for $f \in D(G)$,

$$\left(1 - \frac{t}{n} G\right)^{-n} f = \mathbb{E}\left[P_{\frac{t}{n}\sum_{i=1}^{n}\xi_i} f\right],$$

where $(\xi_i)_i$ are i.i.d. exponentially distributed random variables with mean 1.