Institute for Applied Mathematics
WS 2018/19
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Sheet 10, "Markov Processes"

Due on January 08, 2019

We wish you a merry Christmas and a good start into the new year!

1. (Construction of the Itô-integral) *[20 Pt]*

Let B be the one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P}, (\mathfrak{F}_t)_t)$ that satisfies the conditions of the "usual setting".

(i) A stochastic process X is called *simple process* if it is of the form

$$X_t(\omega) = \sum_{i=1}^{\infty} x_i(\omega) \mathbb{1}_{t_{i-1} < t \leq t_i},$$

where $0 = t_0 < t_1 < \cdots < t_n < \ldots$, and for all $i \in \mathbb{N}$, x_i is a real-valued and $\mathfrak{F}_{t_{i-1}}$-measurable random variable. Define

$$\int_0^t X_s \, dB_s := \sum_{i : t_i \leq t} x_i (B(t_i) - B(t_{i-1})) + x_{m(t)+1}(B(t) - B(t_{m(t)})),$$

where $m(t) = \max(m : t_m \leq t)$. Suppose that X is bounded. Show that $\left(\int_0^t X_s \, dB_s\right)_t$ is a continuous square integrable martingale and that

$$\left[\int_0^\cdot X_s \, dB_s\right]_t = \int_0^t X_s^2 \, ds.$$

(ii) Show that for each stochastic process $X \in \mathcal{L}^2(ds \otimes \mathbb{P})$ there exists a sequence $(X^n)_n$ of bounded simple processes such that

$$\lim_{n \uparrow \infty} \mathbb{E}\left[\int_0^t (X_s - X_s^n)^2 \, ds\right] = 0.$$

(iii) Let $T \in \mathbb{R}_+$. Suppose that $(X^n)_{n \in \mathbb{N}}$ is a sequence of simple processes such that

$$\sum_n \left(\mathbb{E}\left[\int_0^T (X_s^n - X_s^{n+1})^2 \, ds\right]\right)^{1/2} < \infty.$$

Show that, almost surely, the sequence $\left(\int_0^\cdot X_s^n \, dB_s\right)_n$ is a Cauchy sequence with respect to uniform convergence on $[0, T]$.

(iv) Let $X \in \mathcal{L}^2(ds \otimes \mathbb{P})$. Show that there exists a unique continuous square integrable local martingale, $\left(\int_0^{\cdot} X_s \, dB_s\right)_{t\in[0,\infty)}$, such that, whenever a sequence of simple processes $(X^n)_{n\in\mathbb{N}}$ satisfies, for all $t \in \mathbb{R}_+$,

$$\sum_{n\in\mathbb{N}} \left\{ \mathbb{E}\left[\int_0^t (X_s^n - X_s)^2 ds \right] \right\}^{1/2} < \infty,$$

then, for all $T \in \mathbb{R}_+$,

$$\lim_{n\uparrow\infty} \sup_{0\le t\le T} \left| \int_0^t X_s^n \, dB_s - \int_0^t X_s \, dB_s \right| = 0 \ \text{ almost surely,}$$

and

$$\lim_{n\uparrow\infty} \mathbb{E}\left[\sup_{0\le t\le T} \left| \int_0^t X_s^n \, dB_s - \int_0^t X_s \, dB_s \right|^2 \right] = 0.$$

Moreover, show that

$$\left[\int_0^{\cdot} X_s \, dB_s \right]_t = \int_0^t X_s^2 \, ds.$$

Exercise 2 *[Without points but still very important!]*

The examination for this course will be oral. The first examination period is scheduled between February 04, 2019 - February 08, 2019, and the second examination period is scheduled between March 18, 2019 - March 22, 2019.

(i) Note that you will have to register for a time slot for the first examination period. In order to do so, you will have to come to the office of Mrs. Wang (3.036 in the MZ) at one of the following times:

- January, 7; 9:30-11:30,
- January, 8; 9:30-11:30,
- January, 10; 9:30-11:30,
- January, 14; 14:00-16:00,
- January, 15; 14:00-16:00,
- January, 17; 14:00-16:00.

Please note that you have to register for a time slot, even if you only plan to do the second exam! In that case, we give you an imaginary exam date, but we need your sign. If you have a full agenda, than please come to the registration as soon as possible in order to make sure that you get a suitable time slot for your exam. The earlier you come, the more options you have!

(ii) In the case that you want to (or have to) take the exam in the second examination period, then you have to register for a time slot again. Hence, you will have to come to the office of Mrs. Wang (3.036 in the MZ) a second time. The possible dates are the following.

- February, 11; 9:30-11:30,
- February, 12; 9:30-11:30.