Advanced Topics in Stochastic Analysis
- Introduction to Schramm-Loewner evolution

Mondays 12–14 and Thursdays 8–10 in Endenicher Allee 60 - SemR 1.008

Exercises – Set 9

In this exercise sheet, we will discuss the ingredients to prove that SLE(κ) is almost surely generated by a (continuous transient) curve, for any \( \kappa \in (0, \infty) \setminus \{8\} \). Unfortunately, the proof fails for \( \kappa = 8 \), as we’ll see.

**Theorem.** Let \( \kappa \in (0, \infty) \setminus \{8\} \). The SLE(\( \kappa \)) is almost surely generated by a curve \( \gamma \).

**Notation:**
- \( (g_t)_{t \geq 0} \) is the Loewner chain associated to the SLE with the following parameterization:
  \[
  \partial_t g_t(z) = \frac{a}{g_t(z) - W_t}, \quad g_0(z) = z, \quad z \in H_t,
  \]
  where \( a = 2/\kappa \), the driving function is \( W_t = -B_t \), and \( K_t \) are the hulls and \( H_t := \mathbb{H} \setminus K_t \).
- \( (h_s)_{s \geq 0} \) is the solution to the reverse LE (this is almost the same as “backward LE”)
  \[
  \partial_t h_t(z) = -\frac{a}{h_t(z) - W_t}, \quad h_0(z) = z, \quad z \in \mathbb{H}.
  \]
- We denote \( f_t(z) := g_t^{-1}(z) \) and \( \hat{f}_t(z) := g_t^{-1}(z + W_t) \). Note that LE for \( g_t \) gives an ODE for \( (f_t)_{t \geq 0} \):
  \[
  \partial_t f_t(w) = -\frac{a f_t(w)}{w - W_t}, \quad f_0(w) = w, \quad w \in \mathbb{H}.
  \]
- For all \( (y,t) \in [0, \infty) \times [0, 1] \), we denote by \( V(y,t) := \hat{f}_t(iy) \).
- We make a dyadic partitioning of \( t \in [0, 1] \):
  \[
  \mathcal{D}_{2^n} := \{k2^{-2n} \mid k = 0, 1, \ldots, 2^{2n}\}, \quad n \in \mathbb{N}.
  \]
  We are going to control the values of \( V \) when \( y = 2^{-n} > 0 \) is small and the time scale is as in \( \mathcal{D}_{2^n} \).

**Our goal:** By [2] Proposition 4.28, the Theorem follows if we show that \( V \) is well-defined and continuous as \( y \nearrow 0 \), so that the curve
\[
\gamma(t) := \lim_{y \nearrow 0} V(y,t) = \lim_{y \nearrow 0} g_t^{-1}(iy + W_t)
\]

generating the hulls \( (K_t)_{t \in [0,1]} \) is well-defined and \( f_t \) extends continuously to \( \mathbb{H} \).

To establish the goal, it suffices to find a bound function \( \delta: [0, \infty) \to [0, \infty) \) such that \( \lim \delta(\epsilon) = 0 \) and \( \epsilon > 0 \)
\[
|V(y,t) - V(x,s)| \leq \delta(x + y + |t - s|), \quad t, s, x, y \in [0, 1],
\]

By [2] Lemma 4.32, it turns out that to get this estimate, the following ingredients are sufficient:

(a): There exists a sequence \( (r_n)_{n \in \mathbb{N}} \) such that \( r_n > 0 \), and \( \lim_{n \to \infty} r_n = 0 \), and \( \lim_{n \to \infty} \frac{\sqrt{n}}{r_n} = 0 \), and

(b): \( |\hat{f}_t(i2^{-n})| \leq 2^nr_n \), for all \( t \in \mathcal{D}_{2^n} \), and

(c): there exists \( c \in (0, \infty) \) such that \( |W_{t+s} - W_t| \leq c\sqrt{n}2^{-n} \), for all \( t \in [0, 1] \) and \( s \in [0, 2^{-2n}] \).

We’ll see why in Exercises 6–9 below.
Exercises, Part 1: We establish properties (a), (b), (c) for the SLE.

0. Check that for fixed time \( t \geq 0 \), the function \( z \mapsto \hat{f}_t^i(z) \) and the function \( z \mapsto h_t^i(z) \) have the same law (but it is not true that the joint law of \((\hat{f}_t^i(z))_{t \geq 0}\) and the joint law of \((h_t^i(z))_{t \geq 0}\) would be the same!). Therefore, instead of estimating \( |\hat{f}_t^i(z)| \), it suffices to estimate \( |h_t^i(z)| \).

1. **Set-up:** For fixed \( z \in \mathbb{H} \), we consider the process \( Z_t = h_t(z) - W_t \) solving the SDE

\[
Z_0 = z, \quad dZ_t = -\frac{a}{Z_t} dt + dB_t, \quad t \geq 0.
\]

(Because \( t \mapsto \text{Im } Z_t \) is increasing, this is OK for all times.) This is more useful after the time-change \( \tau(t) := \inf \{ s \geq 0 \mid \text{Im } Z_s / \text{Im } (z) = e^{at} \} \). Then the imaginary part of \( \tilde{Z}_t := Z_{\tau(t)} \) is exponentially increasing:

\[
\text{Im } \tilde{Z}_t = \text{Im } (z)e^{at}, \quad d(\text{Re } \tilde{Z}_t) = -a(\text{Re } \tilde{Z}_t) dt + |\tilde{Z}_t| dB_t,
\]

where \( B \) is standard 1D BM. It is useful to consider

\[
\tilde{K}_t := \frac{\text{Re } \tilde{Z}_t}{\text{Im } \tilde{Z}_t} = e^{-at} \text{Re } \tilde{Z}_t / \text{Im}(z), \quad \tilde{L}_t := \sqrt{\tilde{K}_t^2 + 1},
\]

which satisfy the SDEs

\[
d\tilde{K}_t = -2a \tilde{K}_t dt + \tilde{L}_t d\tilde{B}_t, \quad d\tilde{L}_t = \left( \frac{1}{2} \tilde{L}_t - \left( \frac{1}{2} + 2a \right) \frac{\tilde{K}_t^2}{\tilde{L}_t} \right) dt + \tilde{K}_t d\tilde{B}_t.
\]

To simplify this, we can write

\[
\tilde{J}_t := \sinh^{-1} \tilde{K}_t \quad \Rightarrow \quad \begin{cases}
\tilde{K}_t = \sinh \tilde{J}_t \\
\tilde{L}_t = \cosh \tilde{J}_t,
\end{cases} \quad \text{and} \quad d\tilde{J}_t = - \left( \frac{1}{2} + 2a \right) \tanh \tilde{J}_t dt + d\tilde{B}_t.
\]

Finally, the process \( \tilde{h}_t := h_{\tau(t)} \) satisfies

\[
\partial_t \log |\tilde{h}_t^i(z)| = a \left( \frac{(\text{Re } \tilde{Z}_t)^2 - (\text{Im } \tilde{Z}_t)^2}{|Z_t|^2} \right) = a \left( 1 - \frac{2}{\tilde{L}_t^2} \right) = a \left( 1 - \frac{2}{(\cosh \tilde{J}_t)^2} \right) = a \left( 2 (\tanh \tilde{J}_t)^2 - 1 \right)
\]

**Task:** Prove that the following process is a martingale:

\[
\tilde{M}_t = |\tilde{h}_t^i(z)|^p (\text{Re } \tilde{Z}_t)^p (\sin \tilde{\Theta}_t)^{-2r}, \quad \text{where} \quad \tilde{\Theta}_t := \arg(\tilde{Z}_t)
\]

and \((p, r) \in \mathbb{R}^2 \) satisfy \( r^2 - (1 + 2a) r + ap = 0 \). [Hint: Identify \( \sin \tilde{\Theta}_t \) with an expression involving \( \tilde{J}_t \).]

2. **Task:** Prove that

\[
\mathbb{E} \left[ |\tilde{h}_t^i(z)|^p (\sin \tilde{\Theta}_t)^{-2r} \right] = \left( \frac{\text{Im}(z)}{|z|} \right)^{-2r} \exp \left( -at \left( p - \frac{r}{a} \right) \right)
\]

and if \( p, r \geq 0 \), then we have

\[
\mathbb{P} \left[ |\tilde{h}_t^i(z)| \geq \lambda \right] \leq \lambda^{-p} \left( \frac{\text{Im}(z)}{|z|} \right)^{-2r} \exp \left( -at \left( p - \frac{r}{a} \right) \right), \quad \lambda > 0.
\]

3. Using the estimate (3), one can obtain the following estimate for the derivative \( h_t^i(z) \) in the original time parameterization (see [2 Corollary 7.3] and [1 Corollary 5.1]): For every \( r \in [0, 1 + 2a] \), there exists a constant \( c(k, r) \in (0, \infty) \) such that for all \( t \in [0, 1] \), \( x \in \mathbb{R} \), and \( y \in (0, 1] \) and \( \lambda \in \left[ \frac{c(k, 1)}{y} \right] \), we have

\[
\mathbb{P} \left[ |h_t^i(x + iy)| \geq \lambda \right] \leq c \lambda^{-p} \left( \frac{y}{|x + iy|} \right)^{-2r} \delta(y, \lambda),
\]
where \( p = p(r) = \frac{1}{a} ((1 + 2a)r - r^2) \geq 0 \) and
\[
\delta(y, \lambda) = \begin{cases} 
\lambda^{p+\frac{p}{2}}, & p - \frac{\lambda}{a} > 0, \\
1 + \log \frac{1}{\lambda p}, & p - \frac{\lambda}{a} = 0, \\
y^{p+\frac{p}{2}}, & p - \frac{\lambda}{a} < 0.
\end{cases}
\]

Recall that \( a = 2/\kappa \). We still have freedom to choose the parameter \( r \geq 0 \). Note that by choosing \( r = r_0 = \frac{1 + 4a}{4} = \frac{1}{4} + \frac{2}{\kappa} \), which maximises the quantity \( 2p - \frac{\kappa}{2} \), we have
\[
2p(r_0) - \frac{r_0}{a} = \kappa r_0 \left( \frac{\frac{1}{2} + \frac{4}{\kappa}}{2} \right) = \kappa r_0^2 \geq 2
\]
and \( \kappa r_0^2 = 2 \) if and only if \( \kappa = 8 \).

**Task:** Verify that if \( \kappa \in (0, \infty) \setminus \{8\} \), then choosing these \((r_0, p(r_0))\), the estimate \((\ref{eq:ineq})\) gives for \( x = 0, y = 2^{-n}, \) and \( \lambda = 2^n(1-\alpha) \), with \( n \) large enough and \( \alpha \in (0, \frac{2}{2p(r_0) - r_0/a}) \) small enough,
\[
\mathbb{P} \left[ |h'_f(i2^{-n})| \geq 2^n(1-\alpha) \right] \leq c 2^{-n(2+\varepsilon)},
\]
for some \( \varepsilon > 0 \). [NB: There are two different cases: \( \kappa < 8 \) and \( \kappa > 8 \).]

4. **Task:** Using the dyadic partitioning \( D_{2n} \) for \( t \in [0,1] \), show that \((\ref{eq:ineq})\) implies that for any \( \alpha \) small enough, there exists a random variable \( C \) such that almost surely, \( C < \infty \) and
\[
|h'_f(i2^{-n})| \leq C 2^{n(1-\alpha)}, \quad t \in D_{2n}, \quad n \in \mathbb{N}.
\]

5. **Task:** Conclude that all properties (a), (b), (c) indeed hold.

**Exercises, Part 2:** Why do properties (a), (b), (c) imply our goal?

Let’s begin by arguing backwards: Let \( t \in [0,1] \), \( s \in [0,2^{-2n}] \) and \( 0 < x, y \leq 2^{-n} \) and write
\[
|f_t(iy) - f_t+i(s)(ix)| \leq |f_t(iy) - f_t(i2^{-n})| + |f_t(i2^{-n}) - f_{t+s}(i2^{-n})| + |f_{t+s}(i2^{-n}) - f_{t+s}(i)(ix)|.
\]

6. **Task:** Estimate the middle term in \((\ref{eq:ineq})\) in terms of \( \sup_{v \in [t,t+s]} |\hat{f}_t(v)| \), by using the ODE \((\ref{eq:ode})\) for \( f_t \).

7. **Task:** Estimate the first term in \((\ref{eq:ineq})\) in terms of \( \sup_{v \in [2^{-j},2^{-j+1}]} |\hat{f}_t(v)| \), with a sum over \( j = n, n+1, \ldots \).

(The third term can be estimated similarly.)

8. **Tools:** Using property (b), the ODE \((\ref{eq:ode})\) for \( f_t \), and Gronwall’s Area theorem, one can show that
\[
|f_t(i2^{-n} + W_k 2^{-2n})| \leq e^{\beta 2^n} r_n, \quad t \in [k 2^{-2n}, (k+1) 2^{-2n}], \quad k = 0, 1, \ldots, 2^{-2n} - 1, \quad n \in \mathbb{N}.
\]

Using Koebe distortion theorem, one can show that for any conformal map \( \varphi \) on \( \mathbb{H} \), we have
\[
|\varphi'(w)| \leq 144 \frac{1}{|w|+1} |\varphi'(z)|, \quad \text{Im}(z), \text{Im}(w) \geq y > 0.
\]

**Task:** Using these facts and property (c), prove that there exists \( \beta > 0 \) such that
\[
|\hat{f}_t(i2^{-n})| \leq ce^{\beta \sqrt{n} 2^n} r_n, \quad t \in [0,1], \quad n \in \mathbb{N},
\]
and furthermore,
\[
|\hat{f}_t(iy)| \leq ce^{\beta \sqrt{n} 2^n} r_n, \quad t \in [0,1], \quad y \in [2^{-n}, 2^{-n+1}], \quad n \in \mathbb{N}.
\]

9. **Task:** Conclude using \((\ref{eq:ineq})\) that all terms in the expression \((\ref{eq:ineq})\) have the desired bound, so \((\ref{eq:ineq})\) holds.

**References**
