Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and Thursdays 8–10 in Endenicher Allee 60 - SemR 1.008

Exercises – Set 8

In this exercise sheet, we will prove the following result in several steps. This result describes the probability of the SLE curve to come close to a given point $z$, in terms of the conformal radius $\text{crad}_H(z)$.

**Theorem.** Let $\kappa \in (0, 8)$. Consider the SLE($\kappa$) curve $\gamma$ in $(\mathbb{H}; 0, \infty)$. Fix $z \in \mathbb{H}$ and $\epsilon \in (0, 1/2)$. Then there exists a constant $\alpha = \alpha(\kappa) > 0$ independent of $z$ and $\epsilon$ such that

$$
P[\text{crad}_H(z) \leq \epsilon \text{ crad}_\mathbb{H}(z)] = c_* \epsilon^{2-d} (\sin(\arg z))^{4a-1} (1 + O(\epsilon^\alpha)),
$$

where

$$
a = \frac{2}{\kappa}, \quad d = 1 + \frac{\kappa}{8}, \quad c_* = 2 \left( \int_0^\pi (\sin u)^{4a} \, du \right)^{-1}
$$

are $\kappa$-dependent constants and $H(z)$ denotes the connected component of the complement $\mathbb{H} \setminus \gamma[0, \infty)$ of the whole curve that contains the point $z$.

**Notation:**

- $(g_t)_{t \geq 0}$ is the Loewner chain associated to the SLE with the following parameterization:

$$
\partial_t g_t(z) = \frac{a}{g_t(z) - W_t}, \quad g_0(z) = z, \quad z \in H_t,
$$

where $K_t$ are the hulls and $H_t := \mathbb{H} \setminus K_t$ (so $H_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$), and the driving function is $W_t = -B_t$ (here, $B$ is a standard 1D BM and the minus sign is for convenience). The swallowing time of the point $z$ is $\tau_z := \inf \{ s > 0 \mid |g_s(z) - W_s| = 0 \}$.

- The time evolution of the point $z$ is governed by the (complex-valued Bessel process) solving the SDE

$$
Z_0 = z, \quad dZ_t = \frac{a}{Z_t} \, dt + dB_t, \quad t < \tau_z.
$$

- The argument $\Theta_t := \arg(Z_t)$ of this time evolution is a useful quantity. It satisfies the SDE

$$
\Theta_0 = \arg(z), \quad d\Theta_t = (1 - 2a) \frac{\Re Z_t (\Im Z_t)}{|Z_t|^4} \, dt - \frac{\Im Z_t}{|Z_t|^2} \, dB_t, \quad t < \tau_z.
$$
Exercises:

1. Recall that the time evolution of the conformal radius is encoded in the process satisfying
\[
Υ_t = \frac{1}{2} \text{crad}_H(z), \quad \frac{dΥ_t}{Υ_t} = -\frac{2a (\text{Im} Z_t)^2}{|Z_t|^4} \, dt, \quad t < τ_z,
\]
and if τ_z < ∞, we define
\[
Υ_t := Υ_{τ_z} = \frac{1}{2} \text{crad}_H(z) \quad \text{for all } t ≥ τ_z,
\]
so that
\[
Υ_∞ := \lim_{t→∞} Υ_t = \frac{1}{2} \text{crad}_H(z).
\]
Prove that in the 
\[
σ(t) := \inf\{s > 0 \mid \log \left( \frac{Υ_s}{Υ_t} \right) = 2a t \}
\]
(radial time-paramaterization
(with B standard 1D BM) the processes ̂Υ_t := Υ_σ(t) and ̂Θ_t := Θ_σ(t) satisfy
\[
\begin{align*}
̂Υ_t &= e^{-2at}Υ_0, \\
(1 - 2a) \cot ̂Θ_t &= d̂Θ_t + d̂B_t, \quad t < T := \inf \{ s > 0 \mid ̂Θ_t \in \{0, π\} \}.
\end{align*}
\]

2. Check that when writing
\[
ε = e^{-2as} \quad \text{for certain } s = s_ε > 0,
\]
the left-hand side of Equation (1) reads
\[
P[Υ_∞ ≤ εΥ_0] = P[T ≥ s_ε].
\]

3. By finding a suitable function \( f \in C^2(0, π) \), find a local martingale of the form
\[
M_t = e^{-λt}f( ̂Θ_t),
\]
which we expect to describe the conditional probability \( P[T > s \mid ̂Θ_s = θ] \).

[Hint: use Itô to find an ODE for \( f \). It will look like something on the right-hand side of Equation (1).]

4. Given this local martingale, check that we can define a new probability measure \( P^* \) by tilting \( P \) by the martingale \( M_{t∧T} \). You will find that under the new measure,
\[
d ̂B_t = (4a - 1) \cot ̂Θ_t \, dt + dB_t^*,
\]
where \( B^* \) is a standard BM under \( P^* \). What is the worry with \( M \) not being a true martingale?

5. Prove that
\[
P[T > s] = e^{s(\frac{1}{2} - 2a)}(\sin ̂Θ_0)^{4a - 1}E^*[\sin ̂Θ_s]^{1-4a}].
\]

6. Finally, estimate the quantity \( E^*[\sin ̂Θ_s]^{1-4a} \) using knowledge about the Bessel process ̂Θ under the measure \( P^* \). Combining this with Exercise 5 you will find the right-hand side of Equation (1).