Exercises – Set 3

1. Let $U, V$ be simply connected domains whose boundaries are Jordan curves (i.e., homeomorphic images of the circle $\partial \mathbb{D}$). Let $z_1, z_2, z_3 \in \partial U$ and $w_1, w_2, w_3 \in \partial V$ appear in counterclockwise order along the boundary. Show that there exists a unique conformal bijection $f: U \rightarrow V$ such that $f(z_i) = w_i$ for $i = 1, 2, 3$.

2. Define the function classes

$S := \{ f \text{ conformal on } \mathbb{D} \mid f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \text{ in } \mathbb{D} \}$

$\Sigma := \{ g \text{ conformal on } \mathbb{D}^* \mid g(z) = z + b_1 + b_2 z^2 + \cdots \text{ in } \mathbb{D}^* \}$, \quad \mathbb{D}^* := \mathbb{C} \setminus \mathbb{D}.

(a) Let $f \in S$. Check that $g(z) := \frac{1}{f(1/z)} \in \Sigma$ and $0 \notin g(\mathbb{D}^*)$, and

$g(z) = z - a_2 + \frac{a_2^2 - a_3}{z} + \cdots .

(b) Let g \in \Sigma and 0 \notin g(\mathbb{D}^*). Check that $f(z) := \frac{1}{g(1/z)} \in S$ and

$f(z) = z - b_0 z^2 + (b_0^2 - b_1) z^3 + \cdots .

(c) Let f \in S. Show that there exists $h \in S$ such that

$h(-z) = -h(z),

(h(z))^2 = f(z^2),

h(z) = z + \frac{a_2}{2} z^3 + \cdots \text{ for all } z \in \mathbb{D}.

[Hint: Consider $g: z \mapsto \sqrt{\frac{f(z)}{z}}$, and then $z \mapsto zg(z^2).$]

3. (Schwarz reflection principle) Let $f \in Hol(B_+)$, where $B_+ = B(0, r) \cap \mathbb{H}$. Write $f(z) = u(z) + iv(z)$ for $u, v: B_+ \rightarrow \mathbb{R}$ harmonic. Suppose that

$\lim_{z \rightarrow x} v(z) = 0 \quad \text{for all} \quad x \in (-r, r).

(a) Show that $v$ has a unique harmonic extension to $B(0, r)$, and $v(z) = -v(z)$ for all $z \in B(0, r)$.

(b) Show that $f$ has a unique holomorphic extension to $B(0, r)$, and $f(z) = f(\bar{z})$ for all $z \in B(0, r)$.

4. Let $K, K_1, K_2 \subset \mathbb{H}$ be hulls and $r > 0$ and $x \in \mathbb{R}$.

(a) Show that $\text{hcap}(rK) = r^2 \text{hcap}(K)$.

(b) Show that $\text{hcap}(K + x) = \text{hcap}(K)$.

(c) Show that $\text{hcap}(K_1 \cup K_2) = \text{hcap}(K_1) + \text{hcap}(g_{\mathbb{H} \setminus K_1}^{-1}(K_2))$, where $g_{\mathbb{H} \setminus K_1}: \mathbb{H} \setminus K_1 \rightarrow \mathbb{H}$ is the conformal bijection normalized at $\infty$. 