Exercises – Set 11

1. We define the (discrete) Green’s function for a finite, connected graph \( G = (V, E) \) \( \subset \mathbb{Z}^d \) as

\[
G_G(x, y) := \mathbb{E}_x\left[ \sum_{k=1}^{\tau_G} 1\{X_k = y\} \right], \quad x, y \in V,
\]

where \((X_n)_{n \in \mathbb{N}}\) is a simple random walk on \( V \) started from \( x \) and \( \tau_G := \inf\{n \in \mathbb{N} \mid X_n \notin V\} \).

(a) Show that \( G_G(x, y) = G_G(y, x) \), for all \( x, y \in V \).

(b) Show that, for fixed \( y \in V \), we have \( \Delta_y G_G(x, y) = -1\{x = y\} \), where \( \Delta_y \) denotes the (discrete) Laplacian acting on the variable \( x \), defined as

\[
\Delta f(x) := \frac{1}{2d} \sum_{(z,x) \in E} (f(z) - f(x)) \quad f : V \rightarrow \mathbb{R}.
\]

2. Let \( U \subset \mathbb{C} \) be a bounded domain whose boundary is regular for Brownian motion. Prove that \( H^1_0(U) \subset L^2(U) \), where \( H^1_0(U) \) is the Hilbert space completion of the space

\[
C_0^\infty(U) := \{f : \mathbb{C} \rightarrow \mathbb{R} \mid f \text{ is smooth and } \text{supp}(f) \subset U \text{ is compact}\}
\]

with respect to the Dirichlet inner product

\[
(f,g)_\nabla := \frac{1}{2d} \int_U \nabla f(z) \nabla g(z) \, dz.
\]

3. Let \( G = (V, E) \subset \mathbb{Z}^d \) be a finite, connected graph. Let \( h \) be the (discrete) GFF on \( G \) (with zero b.c.) and let \( A \subset V \) be a random subset of vertices, so that we have a coupling \((A, h)\) in the same probability space. We say that the coupling is local if the following holds:

For any fixed \( B \subset V \), write \( h = h_B + h^B \) using the Markov property (for \( V \setminus B \)), where

- \( h_B \) is (discrete) harmonic on \( V \setminus B \) and \( h_B \) equals \( h \) on \( B \),
- \( h^B \) is the GFF on \( V \setminus B \) (with zero b.c.), and \( h^B \) equals zero on \( B \), and
- \( h_B \) and \( h^B \) are independent.

Then the process \( h^B \) is independent of the sigma-algebra \( \sigma(h_B, \{A = B\}) \).

Check that in the following situations, the couplings are local:

(a) For all fixed \( B \subset V \), the event \( \{A = B\} \) is \( \sigma(h_B) \)-measurable.

(b) Choose \( x \in V \) uniformly at random and independently of \( h \), and let \( A_0 \) be the largest connected component of \( V \) such that \( x \in A_0 \) and \( h(x)h(y) > 0 \) for all \( y \in A_0 \) (cluster of the same sign).

Take \( A := (A_0 \cup \partial A_0) \cap V \) to be \( A_0 \) with one layer added.

4. Let \((A_1, h)\) and \((A_2, h)\) be two local couplings. Assume that conditionally on \( h \), the sets \( A_1 \) and \( A_2 \) are independent. Show that \((A_1 \cup A_2, h)\) is a local coupling.

5. Consider the (discrete) GFF \( h \) on \( G = (V, E) \subset \mathbb{Z} \) with \( V = \{-1, 0, 1\} \) and \( E = \{(-1, 0), (0, 1)\} \). Let \( \xi \) be a random variable independent of \( h \) such that \( \mathbb{P}[\xi = 1] = \frac{1}{2} = \mathbb{P}[\xi = -1] \). Let \( A_1 := \{\xi\} \) and \( A_2 := \{\xi \times \text{sign}(h(0))\} \). Show that \((A_1, h)\) and \((A_2, h)\) are local couplings, but \((A_1 \cup A_2, h)\) is not.

Advanced Topics in Stochastic Analysis
- Introduction to Schramm-Loewner evolution

Mondays 12–14 and Thursdays 8–10 in Endenicher Allee 60 - SemR 1.008