Exercises – Set 10

1. Let $\gamma$ be the SLE($\kappa$) curve for $\kappa \in (0, 4]$, and let $(g_t)_{t \geq 0}$ be the corresponding conformal maps normalized at $\infty$ and $(W_t)_{t \geq 0}$ the driving process. Let $A \subset \mathbb{H}$ be a hull such that $0 \notin A$ and $\partial A$ is a Jordan curve. Assume that $T := \inf\{t \geq 0 \mid \text{dist}(\gamma(t), A) = 0\} < \infty$, denote $z = \gamma(T) \in \partial A$, and choose $\delta > 0$ small enough such that $\ell := [z, \delta \hat{n}] \subset A$, where $\hat{n}$ is the inward unit normal vector of $\partial A$.

(a) By considering a complex Brownian motion started from $\ell$, show that there exists a constant $r > 0$ such that $g_T(\ell) - W_T \subset \{w \in \mathbb{H} \mid \text{Im}(w) \geq r|\text{Re}(w)|\}$.

(b) Show that for every $r > 0$, there exist $C, \alpha \in (0, \infty)$ such that if $\gamma: [0, 1] \to \mathbb{H}$ is a curve with $0 < |\gamma(0)| = \varepsilon < 1 = |\gamma(1)|$ and $\gamma[0, 1] \subset \{w \in \mathbb{H} \mid \text{Im}(w) \geq r|\text{Re}(w)|\}$, then for the Brownian excursion $E$ in $\mathbb{H}$, we have $\mathbb{P}_0[E[0, \infty) \cap \gamma[0, 1] = \emptyset] \leq C\varepsilon^\alpha$.

Why is the assumption that $\gamma[0, 1]$ lies inside a cone needed?

(c) For $m \in \mathbb{N}$, define the stopping times $\sigma_m := \inf\{t \geq 0 \mid |\gamma(t) - z| = \frac{1}{m}\}$, so $\sigma_m \nearrow T$ as $m \to \infty$. Deduce that for the Brownian excursion $E$ in $\mathbb{H}$, we have $\lim_{m \to \infty} \mathbb{P}_0[\gamma(\sigma_m)[E[0, \infty) \cap A = \emptyset] = 0$.

2. Let $\kappa > 0$ and $\rho > -2$. Consider the SLE($\kappa, \rho$) process, i.e., the Loewner chain with driving process $(W_t)_{t \geq 0}$ and force point $(X_t)_{t \geq 0}$ satisfying the SDE system

$$dW_t = \frac{\rho}{W_t - X_t} \ dt + \sqrt{\kappa} \ dB_t, \quad W_0 = 0,$$

$$dX_t = \frac{2}{X_t - W_t} \ dt, \quad X_0 = x \in \mathbb{R},$$

(parameterized by half-plane capacity), up to a blow-up time. Let $(Z_t)_{t \geq 0}$ be the solution to

$$dZ_t = \left(\frac{\rho + 2}{\kappa}\right) \frac{1}{Z_t} \ dt + dB_t, \quad Z_0 = 0,$$

again up to a blow-up time. Check that

$$\int_0^t \frac{ds}{Z_s} < \infty$$

and using this, deduce that the SLE($\kappa, \rho$) process is well-defined also when $x \nearrow 0$ or $x \searrow 0$. 