Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 1

1. For $z = x + \mathrm{i}y \in \mathbb{C}$, with $x, y \in \mathbb{R}$, define $e^z = e^{x+\mathrm{i}y} := e^x(\cos y + \mathrm{i} \sin y)$.

   (a) Find the modulus, argument, real part, and imaginary part of e^z.

   (b) Prove that if $z, w \in \mathbb{C}$, then $e^{z+w} = e^z e^w$.

   (c) Prove that $z \mapsto e^z$ is holomorphic on the whole $\mathbb{C}$ and its complex derivative equals e^z.

   (d) Prove that $z \mapsto e^z$ is a bijection from $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) < 2\pi\}$ to $\mathbb{C} \setminus \{0\}$.

2. Define $\log z := \log |z| + \mathrm{i} \arg(z)$, for all $z \in \mathbb{C} \setminus (-\infty, 0] = \{z \in \mathbb{C} \mid -\pi < \arg(z) < \pi\}$ (so we have chosen a branch of the logarithm function).

   (a) Prove that $z \mapsto \log z$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and its complex derivative equals $1/z$.

   (b) What is the image of $\mathbb{C} \setminus (-\infty, 0]$ under this map?

3. Let $f \colon U \to \mathbb{C}$ be holomorphic. Show that the function $g(z) := \overline{f(\overline{z})}$ is also holomorphic, for all $\{z \in \mathbb{C} \mid \overline{z} \in U\}$. Find the derivative of g.

4. Calculate the integral

$$\int_{\partial B(0,1)} \frac{e^z}{z(z-2)} \mathrm{d}z,$$

   (where the contour $\partial B(0,1)$ is oriented counterclockwise).

5. Let $f \colon U \to \mathbb{C}$ be holomorphic in a neighborhood of the closed disc $\overline{B(z_0, r)}$. Prove that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\mathrm{i}t}) \mathrm{d}t.$$

6. Show that the stereographic projection of $(x, y, z) \in S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

$$p \colon (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right),$$

   is a smooth and bijective map $S^2 \setminus \{e_n\} \to \mathbb{R}^2$, where $e_n = (0, 0, 1) \in S^2$ is the North Pole, and $\mathbb{R}^2 = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^3$ is the Equatorial Plane.

   Topologically, the image of p can be seen as $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where we identify $p(e_n) = \infty$ and $\mathbb{R}^2 \approx \mathbb{C}$. This is called the *Riemann sphere*. It is a complex manifold. Can you find coordinate charts for it?