Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 8

Hand in until Friday, June 19, 2020

Exercise 1 [*5 Pkt*]

Let $(Y_n)_{n\in\mathbb{N}}$ be a Markov chain on the finite state space $S = \{1, \dots, m\}$ with transition matrix $P = \{p_{ij}\}$, i.e. $\mathbb{P}(Y_{n+1} = i|Y_n = j) = p_{ij}$. Let $x = (x(j))_{j=1,\dots,m}$ be the left-eigenvector of the transition matrix, i.e. there exists some $\lambda \in \mathbb{R}$ such that $\sum_j p_{ji}x(j) = \lambda x(i)$ for all i. Let $Z_n = \lambda^{-n}x(Y_n)$. Show that $(Z_n)_{n\in\mathbb{N}_0}$ is a martingale.

Exercise 2 [*5 Pkt*]

Let $(Y_n)_{n\in\mathbb{N}_0}$ be independent standard normal random variables. Let $S_n = \sum_{i=1}^n Y_i$ and $X_n = e^{S_n - \frac{n}{2}}$ for $n \geq 1$. Prove that

1. $(X_n)_{n\in\mathbb{N}_0}$ is a martingale,
2. $\lim_{n\to\infty} X_n = 0$ a.s,
3. $\lim_{n\to\infty} \mathbb{E}[X_n^p] = 0$, if and only if $p < 1$. (Hence, although the limit of $(X_n)_n$ is in L^1, $(X_n)_n$ does not converge to zero in L^1.)

Exercise 3 [*5 Pkt*]

Let X_n, $n \in \mathbb{N}$ be a sequence of random variables. Define $\mathcal{T}_n = \sigma(X_k, k \geq n)$ and $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathcal{T}_n$. The σ-algebra $\mathcal{T}$ is called the *tail σ-algebra*, and elements from $\mathcal{T}$ are called *tail events*. Furthermore, call a $\mathbb{R} \cup \{\pm\infty\}$-valued random variable η *degenerate*, if there exists a $c \in \mathbb{R} \cup \{\pm\infty\}$, such that $\eta = c$ a.s.

1. Which of the following events are $\mathcal{T}$ measurable?

$$\Big\{\lim_{n\to\infty} X_n \text{ exists}\Big\}, \qquad \Big\{\sup_n X_n < c\Big\}, \qquad \Big\{\limsup_{n\to\infty} X_n < c\Big\}$$

$$\Big\{\sum_{n=1}^{\infty} X_n \text{ converges}\Big\}, \qquad \Big\{\sum_{n=1}^{\infty} |X_n| < c\Big\}.$$

2. Now suppose that X_n, $n \in \mathbb{N}$ is a sequence of independent random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Show that $\limsup_{n\to\infty} X_n$ and $\limsup_{n\to\infty} \frac{S_n}{n}$ are degenerate random variables.

Exercise 4 *[5 Pkt]*

Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, P)$, where Y_1 is not degenerate. Let $X_n := \sum_{k=1}^n Y_k$ and

$$\begin{aligned} &\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \quad \phi(u) = \log E[\exp(uY_1)], \\ &\mathcal{U} := \{u \in \mathbb{R} | \, \phi(u) \in \mathbb{R}\}. \end{aligned}$$

1. Find a function $g : \mathbb{N}\times\mathcal{U} \to \mathbb{R}$, such that $M_n(u) := \exp(uX_n - g(n,u))$ is a martingale for all $u \in \mathcal{U}$.
2. Explain why $(M_n(u))_n$ with $u \in \mathcal{U}$ converges almost surely and verify that $0 \in \mathcal{U}$. Show that for $u \neq 0$, $\phi(tu) < t\phi(u)$ for all $t \in (0,1)$ and that the martingale $(M_n(u))_n$ converges almost surely to zero.

   Hint: To show $\phi(tu) < t\phi(u)$, ask yourself which are the only cases, where equality holds in Jensen's inequality!