Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 7

Hand in Friday, 12th of June 2020 before the lecture

Exercise 1 *[6 Pkt]*

Consider the "Brownian motion" process X defined in Section 3.3.2 from the Lecture Notes. Show that this process is Markov in the sense that all finite dimensional distributions satisfy the Markov property. That is, show that for all $N \in \mathbb{N}$, $0 \le t_1 < \ldots < t_N < \infty$ and all measurable and bounded functions $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[f(X_{t_N})|\sigma(X_{t_{N-1}}, \ldots, X_{t_1})] \;=\; \mathbb{E}[f(X_{t_N})|X_{t_{N-1}}] \quad \text{a.s.}$$

Hint: Let $Y_N = X_{t_N} - X_{t_{N-1}}$. Show first that the random vector $Y_N, X_{t_{N-1}}, \ldots X_{t_1}$ is Gaussian and that Y_N is independent of $\sigma(X_{t_{N-1}}, \ldots X_{t_1})$.

Exercise 2 *[3 Pkt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X = (X_n)_{n\in\mathbb{N}_0}$ be a martingale with respect to the filtration $\{\mathcal{G}_n : n \in \mathbb{N}_0\}$. Let $\mathcal{F}_n = \sigma(\{X_k : 0 \le k \le n\})$. Show that $\mathcal{F}_n \subset \mathcal{G}_n$ for all $n \in \mathbb{N}_0$ and that X is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Exercise 3 *[7 Pkt]*

Let X, Y be martingales with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that

(a) For $a, b \in \mathbb{R}$, $aX + bY$ is a martingale,

(b) $\max\{X, Y\}$ is a submartingale,

(c) $\min\{X, Y\}$ is a supermartingale.

(d) Let ϕ a convex function such that $\mathbb{E}|\phi(X_n)| < \infty$ for all n. Then $\big(\phi(X_n)\big)_{n\in\mathbb{N}_0}$ is a submartingale.

(e) Let Z be a submartingale and ϕ a convex non-decreasing function with $\mathbb{E}|\phi(Z_n)| < \infty$ for all n. Then $\big(\phi(Z_n)\big)_{n\in\mathbb{N}_0}$ is a submartingale.

Exercise 4 *[4 Pkt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let ν be a finite measure on $(\Omega, \mathcal{F})$ such that $\nu \ll \mathbb{P}$. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and for all $n \in \mathbb{N}$, let X_n be the Radon-Nikodým derivative of ν with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_n)$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.