Institute for Applied Mathematics
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Stochastic Processes
Sheet 6

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Remark. In this exercise you may use the following facts.

1. The law of an $\mathbb{R}^n$-valued random vector X is uniquely determined by its multi-dimensional characteristic function given by $\mathbb{R}^n \ni u \mapsto \mathbb{E}[e^{i\langle u,X\rangle}]$.

2. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^n$-valued random vectors with multi-dimensional characteristic functions given by ϕ_k. Let X be an $\mathbb{R}^n$-valued random vector with multi-dimensional characteristic function given by ϕ. If $\lim_{k\to\infty}\phi_k(u) = \phi(u)$ for all $u \in \mathbb{R}^n$, then the sequence $(X_k)_{k\in\mathbb{N}}$ converges in law to X.

Exercise 1 [*6 Pkt*]

1. Let X be an $\mathbb{R}^n$-valued Gaussian vector with mean zero and covariance matrix C. Show that the components $X_1,\dots,X_n$ of X are independent if and only if the covariance matrix C of X is diagonal.

2. Let N be a real valued Gaussian random variable with mean zero and variance 1 and let Z be another random variable, independent of N, with $\mathbb{P}(Z=1) = \mathbb{P}(Z=-1) = 1/2$. Let $Y = NZ$. Show that
   (a) Y is also a Gaussian random variable;
   (b) $\mathrm{Cov}(N,Y) = 0$, but N and Y are not independent;
   (c) (N,Y) is not jointly Gaussian.

Exercise 2 [*2 Pkt*]

Let X be an $\mathbb{R}^n$-valued Gaussian vector with mean zero and covariance matrix C. Let A be an invertible $n\times n$ matrix. Show that AX is a Gaussian vector as well, and compute its covariance matrix.

Exercise 3 *[6 Pkt]*

Let $X_k, k \in \mathbb{N}$ be independent Gaussian random variables with mean 0 and variance 1. For $n \in \mathbb{N}$ and $t \in [0,1]$ let

$$Z_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k,$$

where $[x]$ represents the largest integer smaller than x (i.e. $[\pi] = 3$). Hence, for all n, $(Z_n(t))_{t\in[0,1]}$ is a stochastic process with state space $\mathbb{R}$.

1. Compute for every partition $0 \le t_1 < \ldots < t_N \le 1$, $N \in \mathbb{N}$, the covariance matrix $(\mathrm{Cov}(Z_n(t_i), Z_n(t_j)))_{ij}$ and show that for $n \to \infty$ this matrix converges to a matrix C with $C_{ij} = t_i \wedge t_j$.

2. Show that the finite dimensional distributions of Z_n converge in law to the "Brownian motion" as $n \to \infty$ (see Section 3.3.2 from the Lecture Notes). This means that for every partition $0 \le t_1 < \ldots < t_N \le 1$, $N \in \mathbb{N}$, the vectors $(Z_n(t_1), \ldots Z_n(t_N))$ converge in law to an N-dimensional Gaussian random vector with mean zero and covariance matrix C where $C_{ij} = t_i \wedge t_j$.

Exercise 4 *[4 Pkt]*

Show that the results of exercise 3.2. remain true, if instead of requiring that the $X_k, k \in \mathbb{N}$ are Gaussian we just assume that the $X_k, k \in \mathbb{N}$ are independent and identically distributed with $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}[X_1] = 1$.

Hint: The following identity might be useful:

$$(Z_n(t_1), \ldots Z_n(t_N))^T = \begin{pmatrix} Z_n(t_1) \\ Z_n(t_1) + (Z_n(t_2) - Z_n(t_1)) \\ \vdots \\ Z_n(t_1) + \sum_{i=1}^{N-1}(Z_n(t_{i+1}) - Z_n(t_i)) \end{pmatrix}.$$

Exercise 5 *[2 Pkt]*

Prove the Chapman-Kolmogorov equations (Lemma 3.17 in the lecture notes).