Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 5

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Exercise 1 *[8 Pkt]*

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_0 \subset \mathcal{F}$ be a σ-Algebra. Let $F : \mathbb{R}^2 \to [0, \infty)$ be measurable, X be a random variable independent of $\mathcal{F}_0$, and Y_0 be an $\mathcal{F}_0$-measurable random variable. Show that
$$\mathbb{E}\big[F(X, Y_0)|\mathcal{F}_0\big](\omega) = \int_\Omega F(X(\omega'), Y_0(\omega))\, \mathbb{P}(d\omega') \qquad \text{a.s.}$$

2. Let T_1 and T_2 be independent exponential random variables with parameter $\alpha > 0$ and let $X = \min(T_1, T_2)$. Compute $\mathbb{E}[X|T_1]$.

Exercise 2 *[4 Pkt]*

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $Y_1 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Further, let N be a non-negative integer valued random variable, independent of the Y_n's with $\mathbb{E}[N] < \infty$. Define the random variable $X = \sum_{k=1}^N Y_k$ and compute $\mathbb{E}[X]$.

Exercise 3 *[8 Pkt]*

Let X_1, X_2 be two jointly Gaussian random variables, i.e. two real valued random variables whose joint distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ with density
$$p(x_1, x_2) = \frac{\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

1. Show that $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Var}[X_i] = \sigma_i^2$ for $i = 1, 2$ and $\mathrm{Cov}(X_1, X_2)/(\sigma_1\sigma_2) = \rho$[1]. For the computations, you can use all known facts about one-dimensional Gaussian random variables.

2. Find the conditional expectation $\mathbb{E}[X_1|X_2]$ and the conditional density $f_{X_1|X_2}$ for X_1 given X_2.

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[1] For random variables X and Y, $\rho(X, Y) \equiv \mathrm{Cov}(X, Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$ is called the *correlation between X and Y*.