Institute for Applied Mathematics
SS 2020
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Stochastic Processes
Sheet 4

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Exercise 1 *[6 Pkt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let X and Y be absolutely integrable random variables.

1. Show that the map $X \to \mathbb{E}(X|\mathcal{G})$ is linear.
2. Show that if $\mathcal{B} \subset \mathcal{G}$ is a σ-algebra, then $\mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{B}] = \mathbb{E}(X|\mathcal{B})$ a.s.
3. Show that if $X \leq Y$ a.s., then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$ a.s.
4. Show that $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X||\mathcal{G})$ a.s.;
5. Assume that there exists $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathbb{R}$ such that $\mathbb{P}(Y \in \{p_1, \dots, p_n\}) = 1$ and $\mathbb{P}(Y = p_i) > 0$ for $i = 1, \dots, n$. Compute $\mathbb{E}\left(X|\sigma(Y)\right)$.

Exercise 2 *[4 Pkt]*

Let $\Omega = \{\omega_1, \dots, \omega_5\}$ and let $\mathcal{F} = 2^\Omega$ be the power set Ω. Let $\mathbb{P}$ be the unique probability measure such that

$$\mathbb{P}\big[\{\omega_1\}\big] = \tfrac{1}{10}, \quad \mathbb{P}\big[\{\omega_2\}\big] = \mathbb{P}\big[\{\omega_3\}\big] = \mathbb{P}\big[\{\omega_4\}\big] = \tfrac{1}{5}, \quad \mathbb{P}\big[\{\omega_5\}\big] = \tfrac{3}{10}.$$

Consider the σ-algebra $\mathcal{F}_1 = \sigma\big(\{\omega_1, \omega_4\}, \{\omega_5\}, \{\omega_2, \omega_3\}\big)$ and the random variable X defined by the following: $X(\omega_1) = 1$, $X(\omega_2) = 2$, $X(\omega_3) = 4$, $X(\omega_4) = 7$ and $X(\omega_5) = 12$. Compute $\mathbb{E}[X|\mathcal{F}_1]$.

Exercise 3 *[5 Pkt]*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.

1. Prove the conditional Markov inequality, i.e. show that, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and such that $f(|X|)$ is integrable, then

$$\mathbb{P}[|X| \geq \alpha | \mathcal{G}] \leq \frac{1}{f(\alpha)} \mathbb{E}[f(|X|)|\mathcal{G}] \quad \mathbb{P}\text{-a.s.}$$

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function, X and $\phi(X)$ be integrable random variables. Prove the conditional Jensen inequality

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

   Hint: You can use that for $x, y \in \mathbb{R}$ there exists a measurable function $c : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) \geq \phi(y) + c(y)(x - y).$$

Exercise 4 *[5 Pkt]*

Let X be integrable and let Y be bounded and $\mathcal{G}$-measurable. By using the monotone class theorem, show that

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}$$