**Institute for Applied Mathematics**
**SS 2020**
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# Stochastic Processes
# Sheet 10

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**Exercise 1** *[7 Pkt]*

Let $(X_n)_{n\in\mathbb{N}}$ be iid random variables with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = +1) = \frac{1}{2}$. Let $S_0 = 0$ and let $S_n = \sum_{i=1}^n X_i$ for all $n \geq 1$. Further, define for $a, b \in \mathbb{N}$ the following hitting times

$$\tau_{-a} = \inf\{n > 0 \mid S_n = -a\} \qquad \text{and} \qquad \tau_b = \inf\{n > 0 \mid S_n = b\}.$$

Set $\tau = \tau_{-a} \wedge \tau_b$. Prove that

1. $\mathbb{E}(\tau) < \infty$,
2. $(S_n^2 - n)_n$ is a martingale,
3. $\mathbb{E}(S_\tau) = 0$,
4. $\mathbb{P}(\tau_{-a} < \tau_b) = \frac{b}{a+b}$, (*Hint*: Use the Optional Stopping Theorem!)
5. $\mathbb{E}(\tau) = \mathbb{E}(S_\tau^2)$.

Finally, compute $\mathbb{E}(\tau)$.

**Exercise 2** *[5 Pkt]*

Suppose that $P(x, dy)$ is a transition kernel on a state space $(S, \mathcal{B})$. We say that a probability measure $\mu$ on $(S, \mathcal{B})$ satisfies the *detailed balance condition w.r.t. $P$* if and only if for all measurable $f : S \times S \to \mathbb{R}_+$,

$$\int\int \mu(dx)\, P(x, dy)\, f(x, y) \;=\; \int\int \mu(dy)\, P(y, dx)\, f(x, y).$$

a) Show that a measure that satisfies the detailed balance condition is invariant.

b) Suppose that $(X_n)$ is a stationary Markov chain with one step transition kernel $P$ and with initial distribution $\mu$. Show that $X_n \sim \mu$ for all $n \geq 0$.

c) Now let $p \in (0, 1)$, and consider a Markov chain with state space $\mathbb{Z}_+$ and transition probabilities $P(x, x+1) = p$ for $x \geq 0$, $P(x, x-1) = q := 1 - p$ for $x \geq 1$, and $P(0, 0) = q$.

(i) Find a nontrivial invariant measure.

(ii) Show that if $p < q$ then there is a unique invariant probability measure.

(iii) Show that if $p \geq q$ then an invariant probability measure does not exist.

**Exercise 3** *[3 Pkt]*

Show that a Markov chain with stationary probability kernels and initial distribution $P_0 = \pi$ is a stationary stochastic process if and only if $\pi$ is an invariant probability distribution.

**Exercise 4** *[5 Pkt]*

Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain taking values in $E = \{0, 1, \dots, N\}$ and with transition matrix $P$ given by

$$P(x,y) = \begin{cases} p & \text{if } y = x+1 \\ 1-p & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq x \leq N-1$ and for $0 < p < 1$, with the state $N$ being an absorption state. Define $\tau = \inf\{n \geq 0 : X_n = N\}$, i.e. the first time that $X$ reaches $N$.

1. Use the Markov property to prove that $u(x) = \mathbb{E}_x[\tau], x \in \{0, 1, \dots, N\}$ satisfies the following equation

$$u(x) = \begin{cases} 0 & \text{if } x = N \\ 1 + \sum_{y\in E} P(x,y)u(y) & \text{otherwise} \end{cases}$$

2. Compute $\mathbb{E}_x[\tau]$.

3. How many tosses of a fair coin are necessary on average to get six heads in a row?