Institut für Angewandte Mathematik
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"Stochastic processes"

Mock Exam

In the following you find some exercises that were used in previous exams.

Exercise 1 (Measure theory and Stochastic processes) *[6 Pts]*

1. State the Radon-Nikodým theorem.
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let ν be a finite measure on $(\Omega, \mathcal{F})$ such that $\nu \ll \mathbb{P}$. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and for all $n \in \mathbb{N}$, let X_n be the Radon-Nikodým derivative of ν with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_n)$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.
3. State the Theorem of Daniell-Kolmogorov on the construction of stochastic processes for $S = \mathbb{R}$ and $I = \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Exercise 2 (Conditional expectation) *[6 Pts]*

1. Let $q : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ be such that

   (i) for each $x \in \mathbb{R}$, $q(x, \cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

   (ii) for each $B \in \mathcal{B}(\mathbb{R})$, $q(\cdot, B)$ is a Borel-measurable function.

   Let λ be a probability measure on $\mathbb{R}$. Define a probability measure $\mathbb{P}$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R})^{\otimes 2})$ by
   $$\mathbb{P}(A) = \int_{\mathbb{R}} d\lambda(x) \int_{\mathbb{R}} q(x, dy) \mathbb{1}_A(x, y), \qquad \text{for all } A \in \mathcal{B}(\mathbb{R})^{\otimes 2}. \tag{1}$$
   Let $\mathcal{F} \subset \mathcal{B}(\mathbb{R})^{\otimes 2}$ be the σ-algebra on $\mathbb{R}^2$ defined by
   $$\mathcal{F} = \{A \times \mathbb{R} \,|\, A \in \mathcal{B}(\mathbb{R})\}.$$
   Finally, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded measurable function. Show that $\mathbb{E}[\, f \,|\, \mathcal{F}\,]$, the conditional expectation of f given $\mathcal{F}$, is given by
   $$\mathbb{E}[\, f \,|\, \mathcal{F}\,](x, y) = \int_{\mathbb{R}} f(x, z) q(x, dz) \qquad \text{for almost all } (x, y) \in \mathbb{R}^2,$$
   (Hint: You need to show that the right-hand side of the equation above satisfies the defining properties of the conditional expectation given $\mathcal{F}$).
2. Let $Y_1, Y_2, \dots$ be independent and identically distributed random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathrm{Var}[Y_1] = \sigma^2 < \infty$. Let N be a non-negative integer valued random variable independent of the Y_n's with $\mathbb{E}[N^2] < \infty$. Compute the variance of the random variable $X := \sum_{k=1}^{N} Y_k$.

Exercise 3 (Martingales) [*6 Pts*]

1. State Doob's super-martingale convergence theorem.

2. Let $(Y_n)_{n\in\mathbb{N}}$ be independent and identically distributed random variables with

$$\mathbb{P}(Y_1 = 0) = \mathbb{P}(Y_1 = 2) = \frac{1}{2},$$

and set $X_n = \prod_{i=1}^n Y_i$ for $n \geq 1$. Prove that $(X_n)_{n\in\mathbb{N}}$ is a martingale with mean one, which converges almost surely to zero.

3. Let $(\xi_n)_{n\in\mathbb{N}}$ be iid random variables such that $\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}$. Let for all $n \in \mathbb{N}$, $S_n = \sum_{k=1}^n \xi_k \xi_{k-1}$. Decide whether $(S_n)_{n\in\mathbb{N}}$ is a submartingale or a supermartingale or a martingale.

Exercise 4 (Stopping Times) [*6 Pts*]

1. State Doob's optional stopping theorem.

2. Let $(X_n)_{n\in\mathbb{N}}$ be independent and identically distributed random variables with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = +1) = \frac{1}{2}$. Let $S_0 = 0$ and let $S_n = \sum_{i=1}^n X_i$ for all $n \geq 1$. Define for $a, b \in \mathbb{N}$ the following hitting times

$$\tau_{-a} = \inf\{n > 0 \mid S_n = -a\} \qquad \text{and} \qquad \tau_b = \inf\{n > 0 \mid S_n = b\}.$$

Set $\tau = \tau_{-a} \wedge \tau_b$. Compute $\mathbb{E}(\tau)$.

Exercise 5 (Markov processes) [*6 Pts*]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X be a Markov process with state space S and generator L, and let $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ be the corresponding natural filtration.

1. State the discrete time martingale problem.

2. Let $D \subset S$ be non-empty and open and let g be a measurable function on D. Under which assumptions does the problem

$$\begin{aligned} -(Lf)(x) &= g(x), \quad x \in D, \\ f(x) &= 0, \quad x \in D^c. \end{aligned}$$

have a unique solution? Give an explicit representation of this solution.

3. Let $h : S \to \mathbb{R}_+$ be a positive harmonic function.

   (i) Give the definition of the h-transformed measure $\mathbb{P}^h$.

   (ii) Let Y be $\mathcal{F}_s$-measurable and bounded. Show that for any $t \geq s$,

$$\mathbb{E}^h[Y|\mathcal{F}_0] = \frac{1}{h(X_0)}\mathbb{E}[h(X_t)Y|\mathcal{F}_0].$$

Exercise 6 (Brownian motion) [*6 Pts*]

1. Give the definition of the one-dimensional Brownian motion starting in 0.
2. Let $(B_t)_{t\in\mathbb{R}_+}$ be the one-dimensional Brownian motion starting in 0. Show that $(B_t^2 - t)_{t\in\mathbb{R}_+}$ is a martingale.
3. Let $(B_t)_{t\in\mathbb{R}_+}$ be the one-dimensional Brownian motion starting in 0. Show that $(B_t^3 - 3tB_t)_{t\in[0,\infty)}$ is a martingale.