Height fluctuations for the stationary KPZ equation

Patrik L. Ferrari

(joint work with Alexei Borodin, Ivan Corwin, and Bálint Vető)

In their seminal 1986 paper [13], Kardar, Parisi, and Zhang (KPZ) proposed the stochastic evolution equation for a height function $h(t,x) \in \mathbb{R}$ ($t \in \mathbb{R}_+$ is time and $x \in \mathbb{R}$ is space)

$$\partial_t h(t,x) = \tfrac{1}{2}\partial_x^2 h(t,x) + \tfrac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x).$$

The randomness ξ models the deposition mechanism and it is taken to be space-time Gaussian white noise, so that formally $\mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta(x-x')$. The Laplacian reflects the smoothing mechanism and the non-linearity reflects the slope-dependent growth velocity of the interface. Using earlier physical work of Forster, Nelson and Stephen [10] KPZ predicted that for large time t, the height function $h(t,x)$ has fluctuations around its mean of order $t^{1/3}$ with spatial correlation length of order $t^{2/3}$. For additional background, see the reviews [8, 9, 15, 17].

The physically relevant solution to the KPZ equation [17] is defined indirectly via the well-posed stochastic heat equation (SHE) with multiplicative noise [2, 5, 16],

$$\partial_t \mathcal{Z}(t,x) = \tfrac{1}{2}\partial_x^2 \mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x),$$

with initial condition $\mathcal{Z}(0,x) = \mathcal{Z}_0(x) = e^{h(0,x)}$. The SHE is well-posed and we defines $h(t,x) = \ln(\mathcal{Z}(t,x))$. This is called the Cole−Hopf solution of the KPZ equation.

By a version of the Feynman−Kac formula, the solution of the SHE can be written as

$$\mathcal{Z}(t,x) = \mathbb{E}_{t,x}\left[\mathcal{Z}_0(b(0)) \; : \exp : \left(-\int_0^t \xi(b(s),s)ds\right)\right]$$

where the expectation $\mathbb{E}_{t,x}$ is over a Brownian motion $b(\cdot)$ going backwards in time from $b(t) = x$, and where $: \exp :$ is the Wick ordered exponential. This provides an interpretation for $\mathcal{Z}(t,x)$ as the partition function of the continuum directed random polymer (CDRP) [1, 2].

Let $B(x)$ be a two-sided Brownian motion with $B(0) = 0$ and zero drift. Stationary (zero drift) initial data $h(0,x) = B(x)$ for the KPZ equation corresponds with SHE initial data $\mathcal{Z}(0,x) = e^{B(x)}$. This is called stationary because for any later time t, $h(t,\cdot)$ is marginally distributed as $\tilde{B}(\cdot) + h(t,0)$ where $\tilde{B}(\cdot)$ is a two-sided Brownian motion (though not independent of B or $h(t,0)$).

In our work [7] we provide an exact formula for the one-point probability distribution of the stationary solution to the KPZ equation, and a limit theorem for $h(t,x)$, after proper centering and scaling by $t^{1/3}$. This is made by analyzing a semidiscrete directed polymer model, which in its turn is obtained as a limit of the q-Whittaker process [6] with appropriate initial measure. A different expression using replica trick approach was obtained previously in [12]. The equivalence of the two formulas has not been shown so-far.

For simplicity, let us present the results in the case of zero-drift and position $x=0$ only.

Theorem 1. *Let $h(t,x)$ be the stationary (zero drift) solution to the KPZ equation and let K_0 denote the modified Bessel function. Then, for $t>0$, $\sigma=(2/t)^{1/3}$ and $S\in\mathbb{C}$ with positive real part,*

$$\mathbb{E}\left[2\sigma K_0\left(2\sqrt{S\,\exp\left[\tfrac{t}{24}+h(t,0)\right]}\right)\right]=f\left(S,\sigma\right),$$

where the function f is given below.

To define f, define on $\mathbb{R}_+$ the function

$$Q(x)=\frac{1}{2\pi\mathrm{i}}\int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}w\frac{\sigma\pi S^{-\sigma w}}{\sin(\pi-\sigma w)}e^{-w^3/3+wx}\frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)},$$

and the kernel

$$K(x,y)=\frac{1}{(2\pi\mathrm{i})^2}\int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}w\int_{\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}z\frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zy}}{e^{w^3/3-wx}}\frac{\Gamma(-\sigma z)}{\Gamma(\sigma z)}\frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)}.$$

Let $\gamma_{\mathrm{E}}=0.577\ldots$ be the Euler constant, $\gamma=\sigma(2\gamma_{\mathrm{E}}+\ln S)$ and define

$$f(S,\sigma)=-\det(\mathbb{1}-K)\Big[\gamma+\big\langle(\mathbb{1}-K)^{-1}(K1+Q),1\big\rangle+\big\langle(\mathbb{1}-K)^{-1}(1+Q),Q\big\rangle\Big].$$

where the determinants and scalar products are all meant in $L^2(\mathbb{R}_+)$.

There is an explicit inversion formula, although this is not needed to get the large time limit.

Corollary 2. *For any $r\in\mathbb{R}$, we have*

$$\begin{aligned}&\mathbb{P}\left(h(t,0)\leq-\frac{t}{24}+r\left(\frac{t}{2}\right)^{1/3}\right)\\&=\frac{1}{\sigma^2}\frac{1}{2\pi\mathrm{i}}\int_{-\delta+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi}{\Gamma(-\xi)\Gamma(-\xi+1)}\int_{\mathbb{R}}\mathrm{d}x\,e^{x\xi/\sigma}f\left(e^{-\frac{x+r}{\sigma}},\sigma\right)\end{aligned}$$

for any $\delta>0$ and where $\sigma=(2/t)^{1/3}$.

Universality arises in the large time limit. Indeed, we recover the distribution formula obtained before for the totally asymmetric simple exclusion process and for the polynuclear growth model [3,4,11,14].

Corollary 3. *For any $r\in\mathbb{R}$,*

$$\lim_{t\to\infty}\mathbb{P}\left(h(t,0)\leq-\frac{t}{24}+r\left(\frac{t}{2}\right)^{1/3}\right)=F_0(r),$$

where F_0 is given by

$$F_0(r)=\frac{\partial}{\partial r}\left(g(r)\det\left(\mathbb{1}-P_rK_{\mathrm{Ai}}P_r\right)_{L^2(\mathbb{R})}\right),$$

where $P_s(x) = \mathbb{1}_{\{x>s\}}$, $K_{\rm Ai}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$ *is the Airy kernel, and* $g(r)$ *is given below.*

To define the function g we need some notations. For $s \in \mathbb{R}$, define

$$\mathcal{R} = s + \int_s^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \mathrm{Ai}(x+y), \quad \Psi(y) = 1 - \int_0^\infty \mathrm{d}x \mathrm{Ai}(x+y),$$

$$\Phi(x) = \int_0^\infty \mathrm{d}\lambda \int_s^\infty dy \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \int_0^\infty \mathrm{d}y \mathrm{Ai}(y+x).$$

Then the function g is defined by

$$g(s) = \mathcal{R} - \big\langle (\mathbb{1} - P_s K_{\rm Ai} P_s)^{-1} P_s \Phi, P_s \Psi \big\rangle.$$

References

[1] T. Alberts, K. Khanin, and J. Quastel, *The continuum directed random polymer*, J. Stat. Phys. **154** (2014), 305–326.

[2] G. Amir, I. Corwin, and J. Quastel, *Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions*, Comm. Pure Appl. Math. **64** (2011), 466–537.

[3] J. Baik, P.L. Ferrari, and S. Péché, *Limit process of stationary TASEP near the characteristic line*, Comm. Pure Appl. Math. **63** (2010), 1017–1070.

[4] J. Baik and E.M. Rains, *Limiting distributions for a polynuclear growth model with external sources*, J. Stat. Phys. **100** (2000), 523–542.

[5] L. Bertini and G. Giacomin, *Stochastic Burgers and KPZ equations from particle system*, Comm. Math. Phys. **183** (1997), 571–607.

[6] A. Borodin and I. Corwin, *Macdonald processes*, Probab. Theory Relat. Fields **158** (2014), 225–400.

[7] A. Borodin, I. Corwin, P.L. Ferrari, and B. Vető, *Height fluctuations for the stationary KPZ equation*, In preparation (2014).

[8] I. Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matices: Theory Appl. **1** (2012), 1130001.

[9] P.L. Ferrari, *From interacting particle systems to random matrices*, J. Stat. Mech. (2010), P10016.

[10] D. Forster, D.R. Nelson, and M.J. Stephen, *Large-distance and long-time properties of a randomly stirred fluid*, Phys. Rev. A **16** (1977), 732–749.

[11] T. Imamura and T. Sasamoto, *Fluctuations of the one-dimensional polynuclear growth model with external sources*, Nucl. Phys. B **699** (2004), 503–544.

[12] T. Imamura and T. Sasamoto, *Stationary correlations for the 1D KPZ equation*, J. Stat. Phys. **150** (2013), 908–939.

[13] M. Kardar, G. Parisi, and Y.Z. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.

[14] M. Prähofer and H. Spohn, *Exact scaling function for one-dimensional stationary KPZ growth*, J. Stat. Phys. **115** (2004), 255–279.

[15] J. Quastel, *Introduction to KPZ*, `http://www.math.toronto.edu/quastel/survey.pdf` (2014).

[16] T. Sasamoto and H. Spohn, *Exact height distributions for the KPZ equation with narrow wedge initial condition*, Nucl. Phys. B **834** (2010), 523–542.

[17] T. Sasamoto and H. Spohn, *The 1+1-dimensional Kardar-Parisi-Zhang equation and its universality class*, J. Stat. Mech. **P11013** (2010).