Anomalous shock fluctuations in TASEP

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(joint work with Peter Nejjar)

We considers the simplest non-reversible interacting stochastic particle system, namely the totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$. Despite its simplicity, this model is full of interesting features. In TASEP, particles independently try to jump to their right neighbor site at a constant rate and jumps occur if the exclusion constraint is satisfied: no site can be occupied by more than one particle. Under hydrodynamic scaling, the particle density solves the deterministic Burgers equation (see e.g. [18, 1]). This model belongs to the Kardar-Parisi-Zhang (KPZ) universality class [16] (see [5] for a recent review).

We are interested in the fluctuations around the macroscopic behavior given in terms of the solution of the Burgers equation and we focus on the fluctuations of particles' positions. Depending on the initial condition, the deterministic solution may have parts of constant and decreasing density, as well as a discontinuity, also referred to as shock. The fluctuations of the shock location have attracted a lot of attention.

For TASEP product Bernoulli measures are the only translation invariant stationary measures [17]. In the first works one considered initial configurations to have a shock at the origin, with Bernoulli measures with density ρ (resp. λ) at its left (resp. right), with $\rho < \lambda$. The shock location is often identified by the position of a second class particle. In this case, the shock fluctuations are Gaussian in the scale $t^{1/2}$ [9, 10, 14]. Microscopic information on the shock are available too [7, 11, 8, 3]. The origin of the $t^{1/2}$ fluctuations lies in the randomness of the initial conditions, since fluctuations coming from the dynamics grow only as $t^{1/3}$. If the initial randomness is only at one side of the shock, a similar picture still holds. For example, in [4] one considers the initial condition is Bernoulli-ρ to the right and periodic with density $1/2$ to the left of the origin. When $\rho > 1/2$ there is a shock with Gaussian fluctuations in the scale $t^{1/2}$. In that work, the fluctuations of the shock position are derived from the ones of the particle positions. The result fits in with the heuristic argument in [19] (Section 5). The Gaussian form of the distribution function is not robust (see for instance Remark 17 in [4]).

In the paper [12] we study the fluctuation laws around a shock occurring without initial randomness are analyzed. In that case, one heuristically expects that the shock fluctuations, but also tagged particles fluctuations, live only on a scale of order $t^{1/3}$, see [2] for a physical argument. We find that the distribution function of a particle position (and also of tagged particles) is a product of two other distribution functions. The reason of the product form of the distribution function is that (1) at the shock two characteristics merge and (2) along the characteristics decorrelation is slow [13, 6].

More precisely, if we look at the history of a particle close to the shock at time t, it has non-trivial correlations with a region of width $\mathcal{O}(t^{2/3})$ around the characteristics, see Figure 1. At the shock the two characteristics come together

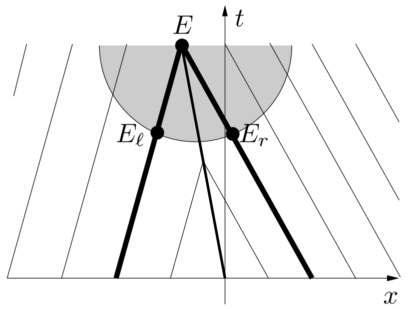

FIGURE 1. Illustration of the characteristics for TASEP. E is the shock location, where two characteristics merges (the thick lines). The gray region is of order t^ν for some $2/3 < \nu < 1$. Due to the slow decorrelation along characteritics, at large time t the fluctuations at E originates from the ones at E_ℓ and E_r.

with a positive angle so that at time $t - t^\nu$, $2/3 < \nu < 1$, their distance will be farther away than $\mathcal{O}(t^{2/3})$ (as proven for the step-initial condition situation by Johansson in [15]). This implies that the fluctuations built up along the two characteristics before time $t - t^\nu$ will be (asymptotically) independent. But if we stay on a characteristic, then the dynamical fluctuations created between time $t - t^\nu$ and time t are only $o(t^{1/3})$, which are irrelevant with respect to the total fluctuations present at time $t - t^\nu$ that are of order $t^{1/3}$ (this is also known as the slow-decorrelation phenomenon [13, 6]).

To generate a shock between two regions of constant density, we consider the initial condition where $2\mathbb{Z}$ is fully occupied and where the jump rates of particles starting to the left (resp. right) of the origin is equal to 1 (resp. $\alpha < 1$). We prove in Corollary 1.5 of [12] the following result.

Theorem. *Let $x_n(0) = -2n$ for $n \in \mathbb{Z}$. For $\alpha < 1$ let $\mu = \frac{4}{2-\alpha}$ and $v = -\frac{1-\alpha}{2}$. Then it holds*

$$\lim_{t\to\infty} \mathbb{P}\left(x_{t/\mu+\xi t^{1/3}}(t) \geq vt - st^{1/3}\right) = F_1\left(\frac{s-2\xi}{\sigma_1}\right) F_1\left(\frac{s-2\xi/(2-\alpha)}{\sigma_2}\right), \tag{1}$$

with $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = \frac{\alpha^{1/3}(2-2\alpha+\alpha^2)^{1/3}}{2(2-\alpha)^{2/3}}$. F_1 is the GOE Tracy-Widom distribution function [20].

As one can see from (1) the shock moves with speed v. When ξ is very large we are in the region before the shock, where the density of particle is $1/2$. Indeed, by replacing $s \to s + 2\xi$ and taking the $\xi \to \infty$ limit, then (1) converges to $F_1(s/\sigma_1)$. Similarly, when $-\xi$ is very large we are already in the shock, where the density of particles in $(2-\alpha)/2$. Indeed, by replacing $s \to s+2\xi/(2-\alpha)$ and taking $\xi \to -\infty$, then (1) converges to $F_1(s/\sigma_2)$. This is the reason why we call this situation a F_1–F_1 shock.

Actually, in [12] we describe also other shock situations (see Corollaries 1.6 and 1.7 therein). Further, for the proof it is convenient (not strictly necessary) to look at the problem from a last passage percolation point of view. In [12] we first determine the analogue results for that model and in a second time relate this to the TASEP picture.

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