Abstracts

Interacting particle systems and random matrices

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TASEP. We consider the totally asymmetric simple exclusion process on $\mathbb{Z}$ (in continuous time). Particles jump independently to their right neighboring site with rate 1, provided the site is empty. Denote by $\eta_t(j) \in \{0,1\}$ the occupation variable of site j at time t (with 1 meaning occupied). The standard representation as height function h (at position x and time t) is given by

$$h(x,t) = \begin{cases} 2N_t + \sum_{y=1}^{x}(1-2\eta_y(t)), & \text{for } x \geq 1, \\ 2N_t, & \text{for } x = 0, \\ 2N_t - \sum_{y=x+1}^{0}(1-2\eta_y(t)), & \text{for } x \leq -1, \end{cases} \tag{1}$$

where N_t is the number of particles which have crossed the bond 0 to 1 during the time span $[0,t]$. This model belongs to the Kardar-Parisi-Zhang [15] universality class of growth models in $1+1$ dimensions.

Universality is expected in the long time limit, i.e., the asymptotic height fluctuations should be independent of the particular model used to derive them. Unlike in the equilibrium statistical mechanics, the scaling exponents are not enough to single out the large time statistics: *initial conditions matter!* One still have to distinguish between (a) curved limit shape, (b) flat limit shape obtained from non-random initial fluctuations, (c) flat limit shape coming from stationary (random) initial conditions.

From KPZ scaling, the correlation length scales as $t^{2/3}$ and height fluctuations as $t^{1/3}$ [11, 20]. Therefore, given the limit shape

$$h_{\mathrm{ma}}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}, \tag{2}$$

the scaling limit to be considered is

$$h_t^{\mathrm{resc}}(u) = \frac{h(\xi t + u t^{2/3}, t) - t h_{\mathrm{ma}}(\xi + u t^{-1/3})}{t^{1/3}}, \tag{3}$$

with of course a freedom in the choice of scaling coefficients (independent of t) for horizontal and vertical scaling.

TASEP with step Initial Conditions. Consider first step initial condition, $\eta_j(0) = 1$ for $j \leq 0$ and $\eta_j(0) = 0$ for $j > 0$, i.e., $h(x,0) = |x|$. *The limit shape is curved*: $\frac{1}{2}(1+\xi^2)$ for $|\xi| \leq 1$. Let us focus around $\xi = 0$, i.e., consider

$$h_t^{\mathrm{resc}}(u) := \frac{h(2u(t/2)^{2/3}, t) - \left(t/2 + u^2 (t/2)^{1/3}\right)}{-(t/2)^{1/3}}. \tag{4}$$

For the one-point distribution it is proven [12] that

$$\lim_{t\to\infty} \mathbb{P}\left(h(0,t) \geq t/2 - s(t/2)^{1/3}\right) = F_2(s), \tag{5}$$

where F_2 is known as the GUE Tracy-Widom distribution, first discovered in random matrices [18]. Moreover, concerning the joint distributions, it is proven [6, 4, 13] that

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(u) = \mathcal{A}_2(u), \tag{6}$$

where $\mathcal{A}_2$ is called the Airy$_2$ process, first discovered in the PNG model by Prähofer and Spohn [16].

GUE matrices. The distribution function F_2 and the Airy$_2$ process describe also the statistics of the largest eigenvalue in the Gaussian Unitary Ensemble of random matrices. Consider $N \times N$ hermitian matrices distributed according to the probability measure

$$\text{const} \exp\left(-\mathrm{Tr}(H^2)/2N\right) dH, \tag{7}$$

where $dH = \prod_{i=1}^N dH_{i,i} \prod_{1\leq i<j\leq N} d\Re(H_{i,j}) d\Im(H_{i,j})$ is the reference measure. Denote by $\lambda_{N,\max}^{\mathrm{GUE}}$ the largest eigenvalue of a $N \times N$ GUE matrix. Then Tracy and Widom [18] showed that fluctuations of $\lambda_{N,\max}^{\mathrm{GUE}}$ are asymptotically F_2-distributed:

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{N,\max}^{\mathrm{GUE}} \leq 2N + sN^{1/3}\right) = F_2(s). \tag{8}$$

The parallel between GUE and TASEP with step initial condition goes even further. Dyson's Brownian Motion (DBM) is a matrix-valued Ornstein-Uhlenbeck process introduced by Dyson in 1962 [7]. More precisely, the GUE DBM is the stationary process on matrices $H(t)$ whose evolution is governed by

$$dH(t) = -\frac{1}{2N} H(t)dt + dB(t) \tag{9}$$

where $dB(t)$ is a (hermitian) matrix-valued Brownian motion. More precisely, the entries $B_{i,i}(t)$, $1 \leq i \leq N$, $\Re(B_{i,j})(t)$ and $\Im(B_{i,j})(t)$, $1 \leq i < j \leq N$, perform independent Brownian motions with variance t for diagonal terms and $t/2$ for the remaining entries. Denote by $\lambda_{N,\max}^{\mathrm{GUE}}(t)$ the largest eigenvalue at time t (when started from the stationary measure (7)). Its evolution is, in the large N limit, governed by the Airy$_2$ process:

$$\lim_{N\to\infty} \frac{\lambda_{N,\max}^{\mathrm{GUE}}(2uN^{2/3}) - 2N}{N^{1/3}} = \mathcal{A}_2(u). \tag{10}$$

TASEP with step Alternating Conditions. The alternating initial condition is the following: $\eta_j(0) = 0$ for odd j and $\eta_j(0) = 1$ for even j. *The limit shape flat, not curved:* $h_{\mathrm{ma}}(\xi) = 1/2$. Thus, the rescaled height function becomes

$$h_t^{\mathrm{resc}}(u) := \frac{h(2ut^{2/3}, t) - t/2}{-t^{1/3}}. \tag{11}$$

In the large time limit

$$\lim_{t\to\infty} \mathbb{P}\left(h(0,t) \geq t/2 - st^{1/3}\right) = F_1(2s), \tag{12}$$

where F_1 is known as the GOE Tracy-Widom distribution, first discovered in random matrices [19]. Moreover, as a process, it was discovered by Sasamoto, see [17, 5], it holds

$$\lim_{t\to\infty} h_t^{\rm resc}(u) = \mathcal{A}_1(u), \tag{13}$$

where $\mathcal{A}_1$ is called the Airy$_1$ process.

GOE matrices. The Gaussian Orthogonal Ensemble (GOE) of random matrices has density on $N \times N$ symmetric matrices

$$\text{const} \exp\left(-\mathrm{Tr}(H^2)/4N\right) dH, \tag{14}$$

where $dH = \prod_{1\leq i\leq j\leq N} dH_{i,j}$ is the reference measure. Denote by $\lambda_{N,\max}^{\rm GOE}$ its largest eigenvalue. The asymptotic distribution of the largest eigenvalue is F_1 [19]:

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{N,\max}^{\rm GOE} - 2N \leq sN^{1/3}\right) = F_1(s). \tag{15}$$

Also DBM is defined for symmetric matrices by

$$dH(t) = -\frac{1}{4N} H(t) dt + dB(t) \tag{16}$$

where $dB(t)$ is a symmetric matrix-valued Brownian motion (as before without imaginary parts). However, numerical evidence shows [2] that, the limit process of a properly rescaled $\lambda_{N,\max}^{\rm GOE}(t)$ is not the Airy$_1$ process:

$$\lim_{N\to\infty} \frac{\lambda_{N,\max}^{\rm GOE}(8uN^{2/3}) - 2N}{2N^{1/3}} =: \mathcal{B}_1(u) \neq \mathcal{A}_1(u). \tag{17}$$

The process $\mathcal{B}_1$ is yet unknown.

Conclusion. First of all, *initial conditions matter* for the long time statistics of interfaces in the KPZ class. Secondly, there are partial connections with random matrices. The parallel between TASEP with step initial conditions and GUE random matrices holds for joint distributions. To understand this connection, one way is to compare the interlacing structure on the GUE minors [14] and the interlacing structure on an extension of TASEP to a $2+1$ interacting particle system introduced in [3], see the lecture notes [8] for details. Note that the connection extends *partially* to the evolution of minors as shown in [10, 1]. On the other hand, the parallel between TASEP with alternating initial conditions and GOE random matrices stops at the level of one-point distributions. For a more extended discussion, see the recent review paper [9].

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