

Non-colliding Brownian bridges and the asymmetric tacnode process

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Abstract

We consider non-colliding Brownian bridges starting from two points and returning to the same position. These positions are chosen such that, in the limit of large number of bridges, the two families of bridges just touch each other forming a tacnode. We obtain the limiting process at the tacnode, the (*asymmetric*) *tacnode process*. It is a determinantal point process with correlation kernel given by two parameters: (1) the curvature's ratio $\lambda > 0$ of the limit shapes of the two families of bridges, (2) a parameter $\sigma \in \mathbb{R}$ controlling the interaction on the fluctuation scale. This generalizes the result for the symmetric tacnode process ($\lambda = 1$ case).

1 Introduction and results

Systems of non-colliding Brownian motions have been much studied recently. They arise in random matrix theory (see e.g. [17, 18, 20]), as limit processes of random walks, discrete growth models, and random tiling problems, see e.g. [9–13, 16, 21, 23, 24].

Considering non-colliding Brownian bridges (as well as discrete analogues), various kinds of determinantal processes appear naturally. Assume that the starting and ending points are chosen such that, in the limit of large number of bridges, the paths occupy a region bordered by a deterministic limit shape (see Figure 1 for an illustration). Then, inside the limit shape (in the bulk), one observes the process with the sine kernel, see e.g. [21]. At the edge of the limit shape, the last bridge is described asymptotically by the Airy₂ process [10, 12, 23]. Whenever there is a cusp in the limit shape, then the process around the cusp is the Pearcey process [4, 7, 22, 27]. All these processes are quite

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robust in the sense that by moving the initial and/or ending points of the bridges, the only changes are geometric (e.g., the position and direction of the edge/cusp changes and numerical coefficients in the scaling) but the processes are the same without free parameters.

The case of the tacnode is more delicate and the limit process is described by two parameters. Recently, three different approaches have been used to unravel the tacnode process. In the first work, Adler, Ferrari and van Moerbeke [1] derived the symmetric tacnode process from a limit of non-intersecting random walks. Meanwhile two other groups were after a solution for the Brownian bridge setting. Soon after [1], a solution appeared in term of a 4×4 Riemann-Hilbert problem by Delvaux, Kuijlaars and Zhang [8]. Their solution is for the generic tacnode process. The third approach, leading to, in our opinion, the simplest of the three formulations was posted more recently by Johansson [15]. In the latter, the asymptotic analysis was restricted to the symmetric tacnode. In the present paper, we analyse the general case starting with the result on two sets of Brownian bridges of [15].

The equivalence between the last two formulations follows from the fact that the starting model is the same. However it seems hard to compare the analytic formulas directly. The equivalence of the results between the random walk and Brownian bridge case is expected by universality, and it can be indirectly checked by analysing a discrete model with the two approaches, as it was made very recently for the double Aztec diamond in [2].

Now we introduce the model and state the result of this paper. We consider $(1 + \lambda)n$ non-colliding standard Brownian motions with two starting points and two endpoints where $\lambda > 0$ is a fixed parameter. More precisely, n of the Brownian motions start at a_1 at time 0 and arrive at a_1 at time 1, the remaining¹ λn Brownian particles have starting and ending points at a_2 at time 0 and 1 respectively with $a_1 < a_2$. For finite times t_1, \dots, t_k , the positions of the particles at these times form an extended determinantal point process (for more informations on determinantal point processes, see [3, 14, 19, 25, 26]). For a fixed integer n and a fixed $\lambda > 0$, let us denote by $\mathcal{L}_{n,\lambda n}(s, u, t, v)$ the kernel of this determinantal point process with $s, t \in (0, 1)$ and $u, v \in \mathbb{R}$. The kernel $\mathcal{L}_{n,\lambda n}(s, u, t, v)$ was obtained in [15], see Theorem 2.1 below for the formula.

In this paper, we take the $n \rightarrow \infty$ limit in the model described above. The global picture is that the two systems of non-colliding Brownian motions form two ellipses touching each other at a tacnode (see Figure 1 for an illustration). Under proper rescaling, we obtain a limiting determinantal point process in the neighborhood of the point of tangency.

Here we consider the general case when two parameters modulate the limit process. One of them measures the strength of interaction called σ , the other one is the asymmetry parameter called λ which we have chosen to be the ratio of curvatures of the two

¹We do not write here integer part of λn to keep the notation simple.

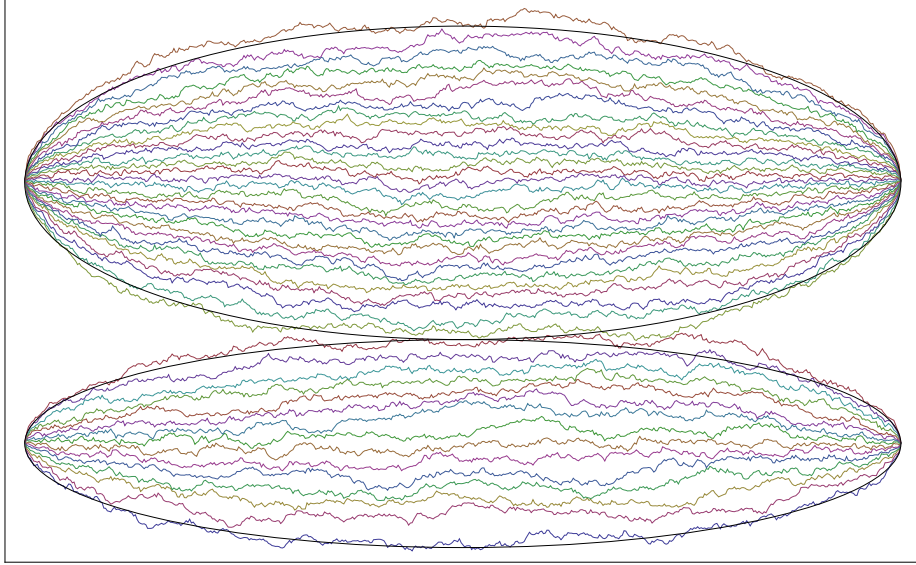


Figure 1: The asymmetric system of non-colliding Brownian motions with 15 respectively 30 paths in the two groups, i.e. $n = 15, \lambda = 2$.

ellipses at the point of tangency. For $\lambda = 1$, we get back to the symmetric case treated in [15].

The scaling of the starting and ending points is

$$a_1 = -\left(\sqrt{n} + \frac{\sigma}{2}n^{-1/6}\right), \quad (1)$$

$$a_2 = \sqrt{\lambda}\left(\sqrt{n} + \frac{\sigma}{2}n^{-1/6}\right). \quad (2)$$

We denote by a the distance of the two endpoints:

$$a = a_2 - a_1 = \left(1 + \sqrt{\lambda}\right)\left(\sqrt{n} + \frac{\sigma}{2}n^{-1/6}\right). \quad (3)$$

In this setting, the tacnode is at $(1/2, 0)$, so that the space-time scaling that we need to consider is

$$\begin{aligned} s &= \frac{1}{2}\left(1 + \tau_1 n^{-1/3}\right), & t &= \frac{1}{2}\left(1 + \tau_2 n^{-1/3}\right), \\ u &= \frac{1}{2}\xi_1 n^{-1/6}, & v &= \frac{1}{2}\xi_2 n^{-1/6}. \end{aligned} \quad (4)$$

The limiting kernel takes the form

$$\mathcal{L}_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) = -\mathbb{1}(\tau_1 < \tau_2)p(\tau_2 - \tau_1; \xi_1, \xi_2)$$

$$+ L_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) + \lambda^{1/6} L_{\text{tac}}^{\lambda^{-1}, \lambda^{2/3} \sigma}(\lambda^{1/3} \tau_1, -\lambda^{1/6} \xi_1, \lambda^{1/3} \tau_2, -\lambda^{1/6} \xi_2) \quad (5)$$

where

$$p(t; x, y) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(y-x)^2}{4t}\right) \quad (6)$$

is the Gaussian kernel. To describe L_{tac} , we need to introduce some notations. For a parameter s , let

$$\text{Ai}^{(s)}(x) = e^{\frac{2}{3}s^3 + xs} \text{Ai}(s^2 + x) \quad (7)$$

be the extended Airy function where $\text{Ai}^{(0)} = \text{Ai}$ is the standard Airy function. The extended Airy kernel is given by

$$K_{\text{Ai}}^{(\alpha, \beta)}(x, y) = \int_0^\infty \text{Ai}^{(\alpha)}(x+u) \text{Ai}^{(\beta)}(y+u) du \quad (8)$$

where $K_{\text{Ai}}^{(0,0)} = K_{\text{Ai}}$ is the standard Airy kernel. Let us denote the function

$$B_{\tau, \xi}^\lambda(x) = \int_0^\infty \text{Ai}^{(\tau)}\left(\xi + \left(1 + \sqrt{\lambda^{-1}}\right)^{1/3} \mu\right) \text{Ai}(x + \mu) d\mu \quad (9)$$

which is reminiscent of the definition of the Airy kernel. Let also

$$b_{\tau, \xi}^\lambda(x) = \lambda^{1/6} \text{Ai}^{(\lambda^{1/3} \tau)}(-\lambda^{1/6} \xi + (1 + \sqrt{\lambda})^{1/3} x), \quad (10)$$

and define

$$\begin{aligned} L_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) &= K_{\text{Ai}}^{(-\tau_1, \tau_2)}(\sigma + \xi_1, \sigma + \xi_2) \\ &+ (1 + \sqrt{\lambda^{-1}})^{1/3} \langle B_{\tau_2, \sigma + \xi_2}^\lambda - b_{\tau_2, \sigma + \xi_2}^\lambda, (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} B_{-\tau_1, \sigma + \xi_1}^\lambda \rangle_{L^2((\tilde{\sigma}, \infty))} \end{aligned} \quad (11)$$

where $\chi_a(x) = \mathbb{1}(x > a)$ and

$$\tilde{\sigma} = \lambda^{1/6} (1 + \sqrt{\lambda})^{2/3} \sigma. \quad (12)$$

Now we can state our main result.

Theorem 1.1. *The (asymmetric) tacnode process $\mathcal{T}^{\sigma, \lambda}$ obtained by the limit of the two non-colliding families of n respectively λn Brownian motions under the scaling (1)–(4) in the neighborhood of the tacnode is given by the following gap probabilities. For any k and $t_1, \dots, t_k \in (0, 1)$ and for any compact set $E \subset \{t_1, \dots, t_k\} \times \mathbb{R}$,*

$$\mathbf{P}(\mathcal{T}^{\sigma, \lambda}(\mathbb{1}_E) = \emptyset) = \det(\mathbb{1} - \mathcal{L}_{\text{tac}}^{\lambda, \sigma})_{L^2(E)} \quad (13)$$

where $\mathcal{L}_{\text{tac}}^{\lambda, \sigma}$ is the extended kernel given by (5).

Remark 1. The (asymmetric) tacnode process has an intrinsic symmetry under the reflection on the horizontal axis that is inherited from the finite system of Brownian motions. This corresponds to the following transformation of the variables:

$$\lambda \longrightarrow \lambda^{-1}, \quad (14)$$

$$n \longrightarrow \lambda n, \quad (15)$$

$$\tau_i \longrightarrow \lambda^{1/3} \tau_i, \quad (16)$$

$$\xi_i \longrightarrow -\lambda^{1/6} \xi_i, \quad (17)$$

$$\sigma \longrightarrow \lambda^{2/3} \sigma. \quad (18)$$

The different powers of λ in the change of parameters (16)–(18) is necessary for observing the process on the same scale. Note that $\tilde{\sigma}$ given in (12) is left invariant under the above transformation.

Next we present an alternative formulation of $\mathcal{L}_{\text{tac}}^{\lambda, \sigma}$, inspired by the analogous reformulation of the kernel in [2]. Let us introduce the function

$$\begin{aligned} C_{\tau, \xi}^{\lambda}(x) &= b_{\tau, \xi}^{\lambda}(x) - B_{\tau, \xi}^{\lambda}(x) \\ &= \lambda^{1/6} \text{Ai}(\lambda^{1/3} \tau) (-\lambda^{1/6} \xi + (1 + \sqrt{\lambda})^{1/3} x) \\ &\quad - \int_0^{\infty} \text{Ai}(\tau) (\xi + (1 + \sqrt{\lambda^{-1}})^{1/3} \mu) \text{Ai}(x + \mu) d\mu. \end{aligned} \quad (19)$$

Using this definition, we can give another expression for the kernel $\mathcal{L}_{\text{tac}}^{\lambda, \sigma}$ which is formally similar to (5), but the ingredients can be given by a single integral as follows.

Proposition 1.2. *With*

$$\tilde{L}_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) = (1 + \sqrt{\lambda^{-1}})^{1/3} \int_{\tilde{\sigma}}^{\infty} (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} C_{-\tau_1, \sigma + \xi_1}^{\lambda}(x) b_{\tau_2, \sigma + \xi_2}^{\lambda}(x) dx, \quad (20)$$

we have

$$\begin{aligned} \mathcal{L}_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) &= -\mathbb{1}(\tau_1 < \tau_2) p(\tau_2 - \tau_1; \xi_1, \xi_2) \\ &\quad + \tilde{L}_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) + \lambda^{1/6} \tilde{L}_{\text{tac}}^{\lambda^{-1}, \lambda^{2/3} \sigma}(\lambda^{1/3} \tau_1, -\lambda^{1/6} \xi_1, \lambda^{1/3} \tau_2, -\lambda^{1/6} \xi_2). \end{aligned} \quad (21)$$

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2 Johansson's formula

In this section, we recall Theorem 1.4 of Johansson in [15], the starting point for our analysis. He obtains a formula for the correlation kernel of two non-colliding families of Brownian particles with the following properties. The first family consists of n particles, and they start at position a_1 at time 0 and end at a_1 at time 1. The other family has m Brownian motions which start at position a_2 at time 0 and end at position a_2 at time 1 with $a_2 > a_1$. This system of Brownian motions conditioned on no intersection in the time interval $(0, 1)$ forms an extended determinantal point process with kernel $\mathcal{L}_{n,m}(s, u, t, v)$ given as follows.

Let $a = a_2 - a_1$ and $d > 0$ a parameter which can be chosen freely in Theorem 2.1. We use the notation

$$\mathcal{A}_{s,u,t,v}^1 = \frac{d^2}{(2\pi i)^2 \sqrt{(1-s)(1-t)}} \int_{i\mathbb{R}} dw \int_{D_{a_1}} dz \left(\frac{1-w/a_1}{1-z/a_1} \right)^n \frac{1}{w-z} \times \exp \left(-\frac{sz^2}{2(1-s)} - a_1 z + \frac{uz}{1-s} + \frac{tw^2}{2(1-t)} + a_1 w - \frac{vw}{1-t} \right), \quad (22)$$

$$\mathcal{B}_{t,v}^1(x) = \frac{d\sqrt{a}}{(2\pi i)^2 \sqrt{1-t}} \int_{i\mathbb{R}} dw \int_{D_{a_1}} dz \left(\frac{1-w/a_1}{1-z/a_1} \right)^n (1-z/a_2)^m \frac{1}{z-w} \times \exp \left(\frac{tw^2}{2(1-t)} + a_1 w - \frac{vw}{1-t} + axz \right), \quad (23)$$

$$\beta_{t,v}^1(x) = \frac{d\sqrt{a}}{2\pi i \sqrt{1-t}} \int_{i\mathbb{R}} dw \left(1 - \frac{w}{a_2} \right)^m \exp \left(\frac{tw^2}{2(1-t)} + a_1 w - \frac{vw}{1-t} + axw \right), \quad (24)$$

$$\mathcal{C}_{s,u}^1(y) = \frac{d\sqrt{a}}{(2\pi i)^2 \sqrt{1-s}} \int_{D_{a_1}} dz \int_{D_{a_2}} dw \left(\frac{1-w/a_1}{1-z/a_1} \right)^n \left(\frac{1}{1-w/a_2} \right)^m \frac{1}{w-z} \times \exp \left(-\frac{sz^2}{2(1-s)} - a_1 z + \frac{uz}{1-s} - ayw \right), \quad (25)$$

$$M_0^1(x, y) = \frac{a}{(2\pi i)^2} \int_{D_{a_1}} dz \int_{D_{a_2}} dw \left(\frac{1-w/a_1}{1-z/a_1} \right)^n \left(\frac{1-z/a_2}{1-w/a_2} \right)^m \frac{1}{z-w} e^{axz-ayw} \quad (26)$$

where D_{a_1} and D_{a_2} are counterclockwise oriented circles around a_1 and a_2 respectively with small radii.

Very similarly, let

$$\mathcal{A}_{s,u,t,v}^2 = \frac{d^2}{(2\pi i)^2 \sqrt{(1-s)(1-t)}} \int_{i\mathbb{R}} dw \int_{D_{a_2}} dz \left(\frac{1-w/a_2}{1-z/a_2} \right)^m \frac{1}{w-z} \times \exp \left(-\frac{sz^2}{2(1-s)} - a_2 z + \frac{uz}{1-s} + \frac{tw^2}{2(1-t)} + a_2 w - \frac{vw}{1-t} \right), \quad (27)$$

$$\mathcal{B}_{t,v}^2(x) = \frac{d\sqrt{a}}{(2\pi i)^2 \sqrt{1-t}} \int_{i\mathbb{R}} dw \int_{D_{a_2}} dz (1-z/a_1)^n \left(\frac{1-w/a_2}{1-z/a_2} \right)^m \frac{1}{z-w} \quad (28)$$

$$\begin{aligned} & \times \exp\left(\frac{tw^2}{2(1-t)} + a_2w - \frac{vw}{1-t} - axz\right), \\ \beta_{t,v}^2(x) &= \frac{d\sqrt{a}}{2\pi i\sqrt{1-t}} \int_{i\mathbb{R}} dw \left(1 - \frac{w}{a_1}\right)^n \exp\left(\frac{tw^2}{2(1-t)} + a_2w - \frac{vw}{1-t} - axw\right), \end{aligned} \quad (29)$$

$$\begin{aligned} \mathcal{C}_{s,u}^2(y) &= \frac{d\sqrt{a}}{(2\pi i)^2\sqrt{1-s}} \int_{D_{a_2}} dz \int_{D_{a_1}} dw \left(\frac{1}{1-w/a_1}\right)^n \left(\frac{1-w/a_2}{1-z/a_2}\right)^m \frac{1}{z-w} \\ & \times \exp\left(-\frac{sz^2}{2(1-s)} - a_2z + \frac{uz}{1-s} + ayw\right), \end{aligned} \quad (30)$$

$$M_0^2(x, y) = \frac{a}{(2\pi i)^2} \int_{D_{a_2}} dz \int_{D_{a_1}} dw \left(\frac{1-z/a_1}{1-w/a_1}\right)^n \left(\frac{1-w/a_2}{1-z/a_2}\right)^m \frac{1}{w-z} e^{-axz+ayw}. \quad (31)$$

Furthermore, let

$$q(s, u, t, v) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(u-v)^2}{2(t-s)} + \frac{u^2}{2(1-s)} - \frac{v^2}{2(1-t)}\right) \mathbb{1}(s < t) \quad (32)$$

denote the conjugated Brownian kernel.

The theorem below is a consequence of Theorem 1.4 in [15].

Theorem 2.1 (Johansson 2011). *The extended determinantal kernel for $n + m$ non-colliding Brownian motions with two starting points and two endpoints described above can be written as*

$$\begin{aligned} \mathcal{L}_{n,m}(s, u, t, v) &= -q(s, u, t, v) \\ & + d^{-2} \mathcal{A}_{s,u,t,v}^1 + d^{-2} \langle (\mathcal{B}_{t,v}^1 + \beta_{t,v}^1), (\mathbb{1} - M_0^1)^{-1} \mathcal{C}_{s,u}^1 \rangle_{L^2((1,\infty))} \\ & + d^{-2} \mathcal{A}_{s,u,t,v}^2 + d^{-2} \langle (\mathcal{B}_{t,v}^2 + \beta_{t,v}^2), (\mathbb{1} - M_0^2)^{-1} \mathcal{C}_{s,u}^2 \rangle_{L^2((1,\infty))}. \end{aligned} \quad (33)$$

Remark 2. It is proved in Lemma 1.2 of [15] that $\det(\mathbb{1} - M_0^1)_{L^2((1,\infty))} > 0$ and $\det(\mathbb{1} - M_0^2)_{L^2((1,\infty))} > 0$ so that (33) makes sense.

Remark 3. This configuration has a natural symmetry. By reflecting the vertical direction, one observes the same process with parameters modified as follows:

$$n \leftrightarrow m, \quad a_1 \rightarrow -a_2, \quad a_2 \rightarrow -a_1, \quad u \rightarrow -u, \quad v \rightarrow -v, \quad (34)$$

s and t are unchanged. It is easy to check that the ingredients $\mathcal{A}_{s,u,t,v}^1, \mathcal{B}_{t,v}^1, \beta_{t,v}^1, \mathcal{C}_{s,u}^1$ and M_0^1 of the kernel of the finite system in (22)–(26) transform to their counterparts with upper index 2 after taking the change of parameters (34) and that $q(s, u, t, v)$ is invariant under this action.

The symmetry of the limiting tacnode process established in Remark 1 is a consequence of the discrete symmetry.

3 Proof of the main results

First, we give the proof of Theorem 1.1 using two lemmas which are proved in Section 4. We start with formula (33), and we apply it to our present setting. Then, we perform asymptotic analysis for the functions obtained in this way.

The appropriate order of the parameter d is

$$d = \frac{n^{-1/12}}{\sqrt{2}}, \quad (35)$$

since we want $d^2 \mathcal{L}_{n,\lambda n}$ to converge, so d^2 is the scaling of the space variables, see (4).

The strategy of the proof is that first, we establish pointwise convergence of the elements of the kernel to the appropriate functions in Lemma 3.1. Then, we give uniform bounds on the functions in Lemma 3.2. This gives, using dominated convergence, that the kernel $\mathcal{L}_{n,\lambda n}$ under the scaling (1)–(4) converges pointwise to $\mathcal{L}_{\text{tac}}^{\sigma,\lambda}$. It turns out that this convergence is uniform on compact sets. Dominated convergence ensures that also the gap probabilities expressed by Fredholm determinants converge.

Lemma 3.1. *Under the scaling given by (1)–(4), the following pointwise limits hold as $n \rightarrow \infty$.*

$$\mathcal{A}_{s,u,t,v}^1 \rightarrow K_{\text{Ai}}^{(-\tau_1, \tau_2)}(\sigma + \xi_1, \sigma + \xi_2), \quad (36)$$

$$n^{-1/3} \mathcal{B}_{t,v}^1(1 + xn^{-2/3}) \rightarrow -(1 + \sqrt{\lambda})^{1/2} B_{\tau_2, \sigma + \xi_2}^\lambda(\tilde{\sigma} + \tilde{x}), \quad (37)$$

$$n^{-1/3} \beta_{t,v}^1(1 + xn^{-2/3}) \rightarrow (1 + \sqrt{\lambda})^{1/2} b_{\tau_2, \sigma + \xi_2}^\lambda(\tilde{\sigma} + \tilde{x}), \quad (38)$$

$$n^{-1/3} \mathcal{C}_{s,u}^1(1 + yn^{-2/3}) \rightarrow -(1 + \sqrt{\lambda})^{1/2} B_{-\tau_1, \sigma + \xi_1}^\lambda(\tilde{\sigma} + \tilde{y}), \quad (39)$$

$$n^{-2/3} M_0^1(1 + xn^{-2/3}, 1 + yn^{-2/3}) \rightarrow \lambda^{1/6} (1 + \sqrt{\lambda})^{2/3} K_{\text{Ai}}(\tilde{\sigma} + \tilde{x}, \tilde{\sigma} + \tilde{y}) \quad (40)$$

where $\tilde{x} = \lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}x$ and $\tilde{y} = \lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}y$ and see (12).

The analogue for the second set of terms is

$$\mathcal{A}_{s,u,t,v}^2 \rightarrow \lambda^{1/6} K_{\text{Ai}}^{(-\lambda^{1/3}\tau_1, \lambda^{1/3}\tau_2)}(\lambda^{2/3}\sigma - \lambda^{1/6}\xi_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2), \quad (41)$$

$$n^{-1/3} \mathcal{B}_{t,v}^2(1 + xn^{-2/3}) \rightarrow -\lambda^{1/6} (1 + \sqrt{\lambda})^{1/2} B_{\lambda^{1/3}\tau_2, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2}^{\lambda^{-1}}(\tilde{\sigma} + \tilde{x}), \quad (42)$$

$$n^{-1/3} \beta_{t,v}^2(1 + xn^{-2/3}) \rightarrow \lambda^{1/6} (1 + \sqrt{\lambda})^{1/2} b_{\lambda^{1/3}\tau_2, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2}^{\lambda^{-1}}(\tilde{\sigma} + \tilde{x}), \quad (43)$$

$$n^{-1/3} \mathcal{C}_{s,u}^2(1 + yn^{-2/3}) \rightarrow -\lambda^{1/6} (1 + \sqrt{\lambda})^{1/2} B_{-\lambda^{1/3}\tau_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_1}^{\lambda^{-1}}(\tilde{\sigma} + \tilde{y}), \quad (44)$$

$$n^{-2/3} M_0^2(1 + xn^{-2/3}, 1 + yn^{-2/3}) \rightarrow \lambda^{1/6} (1 + \sqrt{\lambda})^{2/3} K_{\text{Ai}}(\tilde{\sigma} + \tilde{x}, \tilde{\sigma} + \tilde{y}). \quad (45)$$

The convergence is uniform for ξ_1 and ξ_2 in a compact subset of \mathbb{R} .

Lemma 3.2. *There are constants $c, C > 0$ such that for all $x, y \geq 0$, we have the following bounds*

$$|\mathcal{A}_{s,u,t,v}^i| \leq C, \quad (46)$$

$$|n^{-1/3} \mathcal{B}_{t,v}^i(1 + xn^{-2/3})| \leq Ce^{-cx}, \quad (47)$$

$$|n^{-1/3} \beta_{t,v}^i(1 + xn^{-2/3})| \leq Ce^{-cx}, \quad (48)$$

$$|n^{-1/3} \mathcal{C}_{s,u}^i(1 + yn^{-2/3})| \leq Ce^{-cy}, \quad (49)$$

$$|n^{-2/3} M_0^i(1 + xn^{-2/3}, 1 + yn^{-2/3})| \leq Ce^{-c(x+y)} \quad (50)$$

for $i = 1, 2$. These bounds are uniform for ξ_1 and ξ_2 in a compact subset of \mathbb{R} .

Proof of Theorem 1.1. First, we show that, with the scaling (1)–(4) and (35), we have

$$d^2 \mathcal{L}_{n,\lambda n}(s, u, t, v) \longrightarrow \mathcal{L}_{\text{tac}}^{\lambda,\sigma}(\tau_1, \xi_1, \tau_2, \xi_2). \quad (51)$$

The convergence of $d^2 q(s, t, u, v)$ to the first term on the right-hand side of (5) is obvious. By (36) and (41), it is enough to work with the scalar products in (33).

As in the original formulation in [15], we write

$$\langle (\mathcal{B}_{t,v}^1 + \beta_{t,v}^1), (\mathbb{1} - M_0^1)^{-1} \mathcal{C}_{s,u}^1 \rangle_{L^2((1,\infty))} = \frac{\det(\mathbb{1} - M_0^1 + (\mathcal{B}_{t,v}^1 + \beta_{t,v}^1) \otimes \mathcal{C}_{s,u}^1)_{L^2((1,\infty))}}{\det(\mathbb{1} - M_0^1)_{L^2((1,\infty))}}. \quad (52)$$

For this proof, let

$$\mathcal{D}(x, y) = M_0^1(x, y) - (\mathcal{B}_{t,v}^1(x) + \beta_{t,v}^1(x)) \mathcal{C}_{s,u}^1(y). \quad (53)$$

Then the Fredholm determinant in the numerator of (52) can be expressed as

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[1,\infty)^m} \det(\mathcal{D}(\rho_i, \rho_j))_{1 \leq i, j \leq m} d^m \rho. \quad (54)$$

This is equal to

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[0,\infty)^m} \det \left(\frac{n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}} \mathcal{D} \left(1 + \frac{x_i n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}}, 1 + \frac{x_j n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}} \right) \right) d^m x \quad (55)$$

after the change of variables $\rho_i = 1 + x_i n^{-2/3} \lambda^{-1/6} (1 + \sqrt{\lambda})^{-2/3}$.

Using the pointwise convergence in (37)–(40) along with the bounds (47)–(50) and Hadamard bound on the determinant², the dominated convergence theorem implies that the numerator of (52) converges to

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[0,\infty)^m} \det(D(x_i, x_j))_{1 \leq i, j \leq m} d^m x \quad (56)$$

²Hadamard bound: the absolute value of a determinant of an $n \times n$ matrix with entries of absolute value not exceeding 1 is bounded by $n^{n/2}$.

with

$$D(x, y) = K_{\text{Ai}}(\tilde{\sigma} + x, \tilde{\sigma} + y) - (1 + \sqrt{\lambda^{-1}})^{-1/3} (B_{\tau_2, \sigma + \xi_2}^\lambda(\tilde{\sigma} + x) - b_{\tau_2, \sigma + \xi_2}^\lambda(\tilde{\sigma} + x)) B_{-\tau_1, \sigma + \xi_1}^\lambda(\tilde{\sigma} + y). \quad (57)$$

A similar argument can be used for the denominator of (52), which gives that the expression in (52) converges to the second term on the right-hand side of (11).

In the same way, one shows that

$$\langle (\mathcal{B}_{t,v}^2 + \beta_{t,v}^2), (\mathbb{1} - M_0^2)^{-1} \mathcal{C}_{s,u}^2 \rangle_{L^2((1,\infty))} \quad (58)$$

converges to the scalar product appearing in the last term of (5) using the remaining set of assertions in Lemma 3.1 and also Lemma 3.2. Alternatively, one can refer to the symmetry established in Remark 3 to get the limit of (58). This verifies (51).

It remains to argue that the process $\mathcal{T}^{\sigma,\lambda}$ exists as a determinantal point process. For this, we give a uniform bound on $d^2 \mathcal{L}_{n,\lambda n}(s, u, t, v)$ as ξ_1 and ξ_2 are from a compact subset of \mathbb{R} .

For $d^2 q(s, u, t, v)$, the assertion is clear, for $\mathcal{A}_{s,u,t,v}^i$, it follows from (46). The two scalar products in (33) are bounded as follows. We again consider the right-hand side of (52). In the Fredholm expansion of the numerator (55), we get the bound

$$\left| \frac{n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}} \mathcal{D} \left(1 + \frac{x_i n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}}, 1 + \frac{x_j n^{-2/3}}{\lambda^{1/6}(1 + \sqrt{\lambda})^{2/3}} \right) \right| \leq C e^{-c(x_i + x_j)} \quad (59)$$

with uniform $C, c > 0$ on the compact subsets based on (47)–(50). Hence the convergence of (55) to (56) is uniform as ξ_1 and ξ_2 are in a compact set. The denominator of (52) is strictly positive by Lemma 1.2 of [15] and it converges to

$$\det(\mathbb{1} - M_0^1)_{L^2((1,\infty))} \longrightarrow \det(\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})_{L^2((\tilde{\sigma},\infty))} = F_2(\tilde{\sigma}) > 0 \quad (60)$$

where F_2 is the *Tracy-Widom distribution* function [28]. This shows that the first scalar product in (33) remains uniformly bounded on compact sets. One can proceed similarly with the second one. Therefore, the existence of the gap probabilities of the process $\mathcal{T}^{\sigma,\lambda}$ follows by expanding the Fredholm determinant of the finite size kernel and from the Hadamard bound on the determinant. This completes the proof of Theorem 1.1. \square

The following proof is similar to that of Theorem 1.3 in [2], but the idea is adapted to the asymmetric case, so we give it completely.

Proof of Proposition 1.2. In order to rewrite the kernel in (5), we define the function

$$S_{\tau,\xi}^{\lambda,\sigma}(x) = \text{Ai}^{(\tau)} \left(\lambda^{-1/6} (\lambda^{1/6} \xi - \lambda^{2/3} \sigma) + (1 + \sqrt{\lambda^{-1}})^{1/3} x \right) = \lambda^{1/6} b_{\lambda^{1/3} \tau, \lambda^{2/3} \sigma - \lambda^{1/6} \xi}^{\lambda^{-1}}(x) \quad (61)$$

and the operator T on $L^2((\tilde{\sigma}, \infty))$ with kernel function

$$T(x, y) = \text{Ai}(x + y - \tilde{\sigma}). \quad (62)$$

One observes that, since we have

$$K_{\text{Ai}}(x, y) = \int_{\tilde{\sigma}}^{\infty} \text{Ai}(x + u - \tilde{\sigma}) \text{Ai}(y + u - \tilde{\sigma}) du \quad (63)$$

on $L^2((\tilde{\sigma}, \infty))$, one can write

$$\chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}} = T^2 \quad (64)$$

and also

$$(\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} = \sum_{r=0}^{\infty} T^{2r}. \quad (65)$$

Note that, using the notations (61) and (62), we have

$$C_{\tau, \sigma + \xi}^{\lambda}(x) = \lambda^{1/6} S_{\lambda^{1/3}\tau, -\lambda^{1/6}\xi}^{\lambda^{-1}, \lambda^{2/3}\sigma}(x) - T S_{\tau, \xi}^{\lambda, \sigma}(x). \quad (66)$$

Similarly,

$$C_{\lambda^{1/3}\tau, \lambda^{2/3}\sigma - \lambda^{1/6}\xi}^{\lambda^{-1}}(x) = \lambda^{-1/6} S_{\tau, \xi}^{\lambda, \sigma}(x) - T S_{\lambda^{1/3}\tau, -\lambda^{1/6}\xi}^{\lambda^{-1}, \lambda^{2/3}\sigma}(x). \quad (67)$$

One can also see easily that

$$K_{\text{Ai}}^{(-\tau_1, \tau_2)}(\sigma + \xi_1, \sigma + \xi_2) = (1 + \sqrt{\lambda^{-1}})^{1/3} \left\langle S_{-\tau_1, \xi_1}^{\lambda, \sigma}, S_{\tau_2, \xi_2}^{\lambda, \sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))}, \quad (68)$$

and

$$\begin{aligned} K_{\text{Ai}}^{(-\lambda^{1/3}\tau_1, \lambda^{1/3}\tau_2)}(\lambda^{2/3}\sigma - \lambda^{1/6}\xi_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2) \\ = (1 + \sqrt{\lambda})^{1/3} \left\langle S_{-\lambda^{1/3}\tau_1, -\lambda^{1/6}\xi_1}^{\lambda^{-1}, \lambda^{2/3}\sigma}, S_{\lambda^{1/3}\tau_2, -\lambda^{1/6}\xi_2}^{\lambda^{-1}, \lambda^{2/3}\sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))}. \end{aligned} \quad (69)$$

Starting from (5) and (11), we can rewrite the kernel $\mathcal{L}_{\text{tac}}^{\lambda, \sigma}$ using (19) as follows.

$$\begin{aligned} \mathcal{L}_{\text{tac}}^{\lambda, \sigma}(\tau_1, \xi_1, \tau_2, \xi_2) \\ = -\mathbb{1}(\tau_1 < \tau_2) p(\tau_2 - \tau_1; \xi_1, \xi_2) \\ + K_{\text{Ai}}^{(-\tau_1, \tau_2)}(\sigma + \xi_1, \sigma + \xi_2) + \lambda^{1/6} K_{\text{Ai}}^{(-\lambda^{1/3}\tau_1, \lambda^{1/3}\tau_2)}(\lambda^{2/3}\sigma - \lambda^{1/6}\xi_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2) \\ + (1 + \sqrt{\lambda^{-1}})^{1/3} \int_{\tilde{\sigma}}^{\infty} (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} C_{\tau_2, \sigma + \xi_2}^{\lambda}(x) (C_{-\tau_1, \sigma + \xi_1}^{\lambda}(x) - b_{-\tau_1, \sigma + \xi_1}^{\lambda}(x)) dx \\ + \lambda^{1/6} (1 + \sqrt{\lambda})^{1/3} \int_{\tilde{\sigma}}^{\infty} (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} C_{\lambda^{1/3}\tau_2, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_2}^{\lambda^{-1}}(x) \\ \times (C_{-\lambda^{1/3}\tau_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_1}^{\lambda^{-1}}(x) - b_{-\lambda^{1/3}\tau_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_1}^{\lambda^{-1}}(x)) dx. \end{aligned} \quad (70)$$

If we use (68) and (69) for the first two Airy kernels, (66), (67) and (61) for the two integrals and (65), then we get

$$\begin{aligned}
\mathcal{L}_{\text{tac}}^{\lambda,\sigma}(\tau_1, \xi_1, \tau_2, \xi_2) &= -\mathbb{1}(\tau_1 < \tau_2)p(\tau_2 - \tau_1; \xi_1, \xi_2) \\
&+ (1 + \sqrt{\lambda^{-1}})^{1/3} \left\langle \sum_{r=0}^{\infty} T^{2r} S_{-\tau_1, \xi_1}^{\lambda, \sigma}, S_{\tau_2, \xi_2}^{\lambda, \sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))} \\
&+ \lambda^{1/6} (1 + \sqrt{\lambda})^{1/3} \left\langle \sum_{r=0}^{\infty} T^{2r} S_{-\lambda^{1/3}\tau_1, -\lambda^{1/6}\xi_1}^{\lambda^{-1}, \lambda^{2/3}\sigma}, S_{\lambda^{1/3}\tau_2, -\lambda^{1/6}\xi_2}^{\lambda^{-1}, \lambda^{2/3}\sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))} \\
&- (1 + \sqrt{\lambda})^{1/3} \left\langle \sum_{r=0}^{\infty} T^{2r+1} S_{-\tau_1, \xi_1}^{\lambda, \sigma}, S_{\lambda^{1/3}\tau_2, -\lambda^{1/6}\xi_2}^{\lambda^{-1}, \lambda^{2/3}\sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))} \\
&- (1 + \sqrt{\lambda})^{1/3} \left\langle \sum_{r=0}^{\infty} T^{2r+1} S_{-\lambda^{1/3}\tau_1, -\lambda^{1/6}\xi_1}^{\lambda^{-1}, \lambda^{2/3}\sigma}, S_{\tau_2, \xi_2}^{\lambda, \sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))}.
\end{aligned} \tag{71}$$

Note that the scalar products in the third and the fourth terms on the right-hand side of (71) can be combined using (66), respectively, the second and the fifth terms can be joined by (67) yielding

$$\begin{aligned}
\mathcal{L}_{\text{tac}}^{\lambda,\sigma}(\tau_1, \xi_1, \tau_2, \xi_2) &= -\mathbb{1}(\tau_1 < \tau_2)p(\tau_2 - \tau_1; \xi_1, \xi_2) \\
&+ (1 + \sqrt{\lambda})^{1/3} \left\langle (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} C_{-\tau_1, \sigma + \xi_1}^{\lambda}, S_{\lambda^{1/3}\tau_2, -\lambda^{1/6}\xi_2}^{\lambda^{-1}, \lambda^{2/3}\sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))} \\
&+ (1 + \sqrt{\lambda})^{1/3} \left\langle (\mathbb{1} - \chi_{\tilde{\sigma}} K_{\text{Ai}} \chi_{\tilde{\sigma}})^{-1} C_{-\lambda^{1/3}\tau_1, \lambda^{2/3}\sigma - \lambda^{1/6}\xi_1}^{\lambda^{-1}}, S_{\tau_2, \xi_2}^{\lambda, \sigma} \right\rangle_{L^2((\tilde{\sigma}, \infty))}
\end{aligned} \tag{72}$$

which proves the proposition by (61). \square

4 Asymptotic analysis

Proof of Lemma 3.1. By Remark 3, it is enough to prove the first set of statements (36)–(40). We write down the ingredients of the kernel in Theorem 2.1 with the values given by (1)–(2). Then, we use the method of saddle point analysis to get the limits in Lemma 3.1. For the asymptotic analysis, we use a standard pattern explained e.g. in Section 6 of [5].

For (22), after change of variables $w \rightarrow aw/(1 + \sqrt{\lambda})$ and $z \rightarrow -az/(1 + \sqrt{\lambda})$, we get

$$\begin{aligned} \mathcal{A}_{s,u,t,v}^1 &= \frac{d^2 a}{(1 + \sqrt{\lambda})\sqrt{(1-s)(1-t)}} \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} dw \int_{D_1} dz \left(\frac{1+w}{1-z} \right)^n \frac{1}{w+z} \\ &\quad \times \exp \left(-\frac{a^2 s z^2}{(1 + \sqrt{\lambda})^2 2(1-s)} - \frac{a^2 z}{(1 + \sqrt{\lambda})^2} - \frac{a u z}{(1 + \sqrt{\lambda})(1-s)} \right) \\ &\quad \times \exp \left(\frac{a^2 t w^2}{(1 + \sqrt{\lambda})^2 2(1-t)} - \frac{a^2 w}{(1 + \sqrt{\lambda})^2} - \frac{a v w}{(1 + \sqrt{\lambda})(1-t)} \right). \end{aligned} \quad (73)$$

First we observe that $\text{Re}(z+w) > 0$, hence we can write

$$\frac{1}{z+w} = n^{1/3} \int_0^\infty e^{-n^{1/3}\mu(z+w)} d\mu. \quad (74)$$

By substituting this and (3)–(4) and by Taylor expansion, we obtain

$$\begin{aligned} \mathcal{A}_{s,u,t,v}^1 &= (n^{2/3} + o(1)) \int_0^\infty d\mu \\ &\quad \times \frac{1}{2\pi i} \int_{i\mathbb{R}} dw \exp \left(n \left[\log(1+w) + \frac{w^2}{2} - w \right] + n^{2/3} \tau_2 w^2 - n^{1/3} [\sigma + \xi_2 + \mu] w \right) \\ &\quad \times \frac{1}{2\pi i} \int_{D_1} dz \exp \left(n \left[-\log(1-z) - \frac{z^2}{2} - z \right] - n^{2/3} \tau_1 z^2 - n^{1/3} [\sigma + \xi_1 + \mu] z \right) \\ &\quad \times \exp \left(\mathcal{O} \left(n^{1/3} w^2 + w + n^{1/3} z^2 + z \right) \right). \end{aligned} \quad (75)$$

A change of variables $w \rightarrow aw/(1 + \sqrt{\lambda})$ and $z \rightarrow -az/(1 + \sqrt{\lambda})$ in (23) gives

$$\begin{aligned} \mathcal{B}_{t,v}^1(x) &= \frac{d a^{3/2}}{(1 + \sqrt{\lambda})\sqrt{1-t}} \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} dw \int_{D_1} dz \left(\frac{1+w}{1-z} \right)^n \left(1 + \frac{z}{\sqrt{\lambda}} \right)^{\lambda n} \frac{1}{z+w} \\ &\quad \times \exp \left(\frac{a^2 t w^2}{(1 + \sqrt{\lambda})^2 2(1-t)} - \frac{a^2 w}{(1 + \sqrt{\lambda})^2} - \frac{a v w}{(1 + \sqrt{\lambda})(1-t)} - \frac{a^2 x z}{1 + \sqrt{\lambda}} \right) \end{aligned} \quad (76)$$

which yields

$$\begin{aligned} n^{-1/3} \mathcal{B}_{t,v}^1(1 + x n^{-2/3}) &= (n^{2/3} + o(1)) (1 + \sqrt{\lambda})^{1/2} \int_0^\infty d\mu \\ &\quad \times \frac{1}{2\pi i} \int_{i\mathbb{R}} dw \exp \left(n \left[\log(1+w) + \frac{w^2}{2} - w \right] - n^{2/3} \tau_2 w^2 - n^{1/3} [\sigma + \xi_2 + \mu] w \right) \\ &\quad \times \frac{1}{2\pi i} \int_{D_1} dz \exp \left(n \left[\lambda \log \left(1 + \frac{z}{\sqrt{\lambda}} \right) - \log(1-z) - (1 + \sqrt{\lambda}) z \right] \right) \\ &\quad \times \exp \left(-n^{1/3} [(1 + \sqrt{\lambda})(x + \sigma) + \mu] z + \mathcal{O} \left(n^{1/3} w^2 + w + n^{1/3} z^2 + z \right) \right). \end{aligned} \quad (77)$$

In the definition of $\beta_{t,v}^1$, after $w \rightarrow -aw/(1 + \sqrt{\lambda})$, one obtains

$$\begin{aligned} \beta_{t,v}^1(x) &= \frac{da^{3/2}}{(1 + \sqrt{\lambda})\sqrt{1-t}} \frac{1}{2\pi i} \int_{i\mathbb{R}} dw \left(1 + \frac{w}{\sqrt{\lambda}}\right)^{\lambda n} \\ &\quad \times \exp\left(\frac{a^2tw^2}{(1 + \sqrt{\lambda})^2 2(1-t)} + \frac{a^2w}{(1 + \sqrt{\lambda})^2} + \frac{avw}{(1 + \sqrt{\lambda})(1-t)} - \frac{a^2xw}{1 + \sqrt{\lambda}}\right). \end{aligned} \quad (78)$$

Hence,

$$\begin{aligned} n^{-1/3}\beta_{t,v}^1(1 + xn^{-2/3}) &= (n^{2/3} + o(1))(1 + \sqrt{\lambda})^{1/2} \\ &\quad \times \frac{1}{2\pi i} \int_{i\mathbb{R}} dw \exp\left(n \left[\lambda \log\left(1 + \frac{w}{\sqrt{\lambda}}\right) + \frac{w^2}{2} - w\right]\right) \\ &\quad \times \exp\left(n^{2/3}\tau_2w^2 - n^{1/3}[\sqrt{\lambda}\sigma - \xi_2 + (1 + \sqrt{\lambda})x]w + \mathcal{O}(n^{1/3}w^2 + w)\right). \end{aligned} \quad (79)$$

Similarly in (25) with $w \rightarrow aw/(1 + \sqrt{\lambda})$ and $z \rightarrow -az/(1 + \sqrt{\lambda})$, we get

$$\begin{aligned} \mathcal{C}_{s,u}^1(y) &= -\frac{da^{3/2}}{(1 + \sqrt{\lambda})\sqrt{1-s}} \frac{1}{(2\pi i)^2} \int_{D_1} dz \int_{D_{\sqrt{\lambda}}} dw \left(\frac{1+w}{1-z}\right)^n \left(\frac{1}{1-\frac{w}{\sqrt{\lambda}}}\right)^{\lambda n} \frac{1}{w+z} \\ &\quad \times \exp\left(-\frac{a^2sz^2}{(1 + \sqrt{\lambda})^2 2(1-s)} - \frac{a^2z}{(1 + \sqrt{\lambda})^2} - \frac{auz}{(1 + \sqrt{\lambda})(1-s)} - \frac{a^2yw}{1 + \sqrt{\lambda}}\right). \end{aligned} \quad (80)$$

After substituting (3)–(4), it becomes

$$\begin{aligned} n^{-1/3}\mathcal{C}_{s,u}^1(1 + yn^{-2/3}) &= -(n^{2/3} + o(1))(1 + \sqrt{\lambda})^{1/2} \int_0^\infty d\mu \\ &\quad \times \frac{1}{2\pi i} \int_{D_1} dz \exp\left(n \left[-\log(1-z) - \frac{z^2}{2} - z\right] - n^{2/3}\tau_1z^2 - n^{1/3}[\sigma + \xi_1 + \mu]z\right) \\ &\quad \times \frac{1}{2\pi i} \int_{D_{\sqrt{\lambda}}} dw \exp\left(n \left[\log(1+w) - \lambda \log\left(1 - \frac{w}{\sqrt{\lambda}}\right) - (1 + \sqrt{\lambda})w\right]\right) \\ &\quad \times \exp\left(-n^{1/3}[(1 + \sqrt{\lambda})(\sigma + y) + \mu]w + \mathcal{O}(n^{1/3}w^2 + w + n^{1/3}z^2 + z)\right). \end{aligned} \quad (81)$$

Using the same change of variables $w \rightarrow aw/(1 + \sqrt{\lambda})$ and $z \rightarrow -az/(1 + \sqrt{\lambda})$ in M_0^1 , we have

$$\begin{aligned} M_0^1(x, y) &= \frac{a^2}{(1 + \sqrt{\lambda})} \frac{1}{(2\pi i)^2} \int_{D_1} dz \int_{D_{\sqrt{\lambda}}} dw \left(\frac{1+w}{1-z}\right)^n \left(\frac{1+\frac{z}{\sqrt{\lambda}}}{1-\frac{w}{\sqrt{\lambda}}}\right)^{\lambda n} \frac{1}{z+w} \\ &\quad \times \exp\left(-\frac{a^2xz}{1 + \sqrt{\lambda}} - \frac{a^2yw}{1 + \sqrt{\lambda}}\right). \end{aligned} \quad (82)$$

That is,

$$\begin{aligned}
n^{-2/3} M_0^1(1 + xn^{-2/3}, 1 + yn^{-2/3}) &= (n^{2/3} + o(1))(1 + \sqrt{\lambda}) \int_0^\infty d\mu \\
&\times \frac{1}{2\pi i} \int_{D_{\sqrt{\lambda}}} dw \exp \left(n \left[\log(1 + w) - \lambda \log \left(1 - \frac{w}{\sqrt{\lambda}} \right) - (1 + \sqrt{\lambda})w \right] \right) \\
&\times \exp \left(-n^{1/3} [(1 + \sqrt{\lambda})(\sigma + y) + \mu]w \right) \\
&\times \frac{1}{2\pi i} \int_{D_1} dz \exp \left(n \left[\lambda \log \left(1 + \frac{z}{\sqrt{\lambda}} \right) - \log(1 - z) - (1 + \sqrt{\lambda})z \right] \right) \\
&\times \exp \left(-n^{1/3} [(1 + \sqrt{\lambda})(\sigma + x) + \mu]z + \mathcal{O}(n^{1/3}w^2 + w + n^{1/3}z^2 + z) \right).
\end{aligned} \tag{83}$$

We can take the integration paths D_1 and $D_{\sqrt{\lambda}}$ to denote here circles around 1 and $\sqrt{\lambda}$ with radii 1 and $\sqrt{\lambda}$, in this way, passing through 0.

In the above formulae, integrals of form

$$\frac{1}{2\pi i} \int_\gamma dz e^{nf_0(z) + n^{2/3}f_1(z) + n^{1/3}f_2(z)} \tag{84}$$

appear. Hence, we can follow the steps of the proof of Lemma 6.1 in [5]. It can be checked that in all the cases in (75), (77), (79), (81) and (83), $w_c = z_c = 0$ is a double critical point for the functions below that appear as f_0 . We also give the Taylor expansion for later use

$$\begin{aligned}
\log(1 + w) + \frac{w^2}{2} - w &= \frac{w^3}{3} + \mathcal{O}(w^4), \\
\lambda \log \left(1 + \frac{w}{\sqrt{\lambda}} \right) + \frac{w^2}{2} - w &= \frac{1}{\sqrt{\lambda}} \frac{w^3}{3} + \mathcal{O}(w^4), \\
\log(1 + w) - \lambda \log \left(1 - \frac{w}{\sqrt{\lambda}} \right) - (1 + \sqrt{\lambda})w &= \frac{1 + \sqrt{\lambda}}{\sqrt{\lambda}} \frac{w^3}{3} + \mathcal{O}(w^4),
\end{aligned} \tag{85}$$

and their versions after taking the transformation $-f_0(-w)$ which preserves the leading terms on the right-hand side of (85).

The issue of finding step descent paths is treated in a more general setting in the proofs of Lemma 3.2 and 4.1 below. Thus the details are omitted here. As it turns out, the integration paths in (75), (77), (79), (81) and (83) are step descent for f_0 in the following sense (γ means the generic integration path):

- $\operatorname{Re}(f_0(z))$ on γ reaches its maximum at 0,
- $\operatorname{Re}(f_0(z))$ is monotone along γ except at its maximum point 0 and, if $\gamma = D_1$ or $D_{\sqrt{\lambda}}$, then except also at the antipodal points 2 or $2\sqrt{\lambda}$ respectively (where the real part reaches its minimum).

In the neighborhood of 0, we can slightly modify the paths $i\mathbb{R}$, D_1 and $D_{\sqrt{\lambda}}$ in such a way that they are still steep descent and that, close to 0, the descent is the steepest. The modification is as follows. For a small $\delta > 0$ that is given precisely later, we choose the path $\{e^{-i\pi/3}t, 0 \leq t \leq \delta\} \cup \{e^{i\pi/3}t, 0 \leq t \leq \delta\}$ close to 0. This is clearly locally steepest descent for all functions in (85). In the case of $i\mathbb{R}$, we continue with $\delta/2 + i\mathbb{R}$ to infinity. If we had a circle, we decrease the radius slightly, and we cut off a piece of it close to 0 in such a way that it matches the steepest descent path. Computations which are done in the proofs of Lemma 3.2 and 4.1 show that these modified paths are still steep descent for f_0 in all the cases in (75), (77), (79), (81) and (83).

Along the integration paths, only the contribution on $\gamma \cap \{|w| \leq \delta\}$ is considered, since, by the steep descent property of the paths, the error that is made by neglecting the rest of the path is $\mathcal{O}(\exp(-c\delta^3 n))$ as $n \rightarrow \infty$ uniformly as ξ_1 and ξ_2 are in bounded intervals.

We choose now δ such that the error terms in (75), (77), (79), (81) and (83) are small enough in the δ neighborhood of 0. That is, for the difference of the integrals with and without the error in the exponent, we apply $|e^x - 1| \leq |x|e^{|x|}$. We do a change of variables $n^{1/3}w \rightarrow w$, and, by taking δ small enough, we see that the difference is $\mathcal{O}(n^{-1/3})$ uniformly.

Finally, we can extend the steepest descent path to $e^{-i\pi/3}\infty$ and $e^{i\pi/3}\infty$ on the price of a uniformly $\mathcal{O}(\exp(-c\delta^3 n))$ error again. Then, we apply the formula

$$\begin{aligned} \frac{1}{2\pi i} \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \exp\left(a\frac{z^3}{3} + bz^2 + cz\right) dz &= a^{-1/3} \exp\left(\frac{2b^3}{3a^2} - \frac{bc}{a}\right) \text{Ai}\left(\frac{b^2}{a^{4/3}} - \frac{c}{a^{1/3}}\right) \\ &= a^{-1/3} \text{Ai}^{(a^{-2/3}b)}\left(-\frac{c}{a^{1/3}}\right), \end{aligned} \quad (86)$$

and we exactly get the limits (36)–(40). From this, along with the use of symmetry given in Remark 3, (41)–(45) follow immediately. All the error terms can be bounded uniformly in ξ_1 and ξ_2 if they are in a compact interval, hence Lemma 3.1 is proved. \square

First, we give the following lemma with its full proof. In the proof of Lemma 3.2, we will use the assertion and some similar statements which will not be spelled out later, because they can be shown as Lemma 4.1.

Lemma 4.1. *There are constants $C, c > 0$ such that*

$$\left| \frac{n^{1/3}}{2\pi i} \int_{i\mathbb{R}} dw \exp\left(n \left[\log(1+w) + \frac{w^2}{2} - w \right] + \kappa_1 n^{2/3} w^2 - sn^{1/3} w \right) \right| \leq C e^{-cs} \quad (87)$$

for $s > 0$ large enough.

Proof of Lemma 4.1. We follow the lines of the proof of Proposition 5.3 in [6]. Since we are interested in large values of s , we take

$$\tilde{s} = n^{-2/3} s, \quad (88)$$

and we define

$$\tilde{f}_0(w) = \log(1+w) + \frac{w^2}{2} - w - \tilde{s}w. \quad (89)$$

For small values of \tilde{s} , this function has two critical points at $\pm\tilde{s}^{1/2}$ at first order, and we will pass through the positive one. Hence define

$$\alpha = \begin{cases} \tilde{s}^{1/2} & \text{if } \tilde{s} \leq \varepsilon, \\ \varepsilon^{1/2} & \text{if } \tilde{s} > \varepsilon, \end{cases} \quad (90)$$

for some small $\varepsilon > 0$ to be chosen later and consider the path $\Gamma = \alpha + i\mathbb{R}$. By the Cauchy theorem, the integral in (87) does not change if we modify the integration path to Γ .

The path Γ is steep descent for the function $\text{Re}(\tilde{f}_0(w))$, since

$$\frac{d}{dt} \text{Re}(\tilde{f}_0(\alpha + it)) = -t \underbrace{\left(1 - \frac{1}{(1+\alpha)^2 + t^2}\right)}_{\geq 0 \text{ for } \alpha, t > 0}. \quad (91)$$

Define

$$Q(\alpha) = \exp\left(\text{Re}\left(n\tilde{f}_0(\alpha) + n^{2/3}\kappa_1\alpha^2\right)\right). \quad (92)$$

Let $\Gamma_\delta = \{\alpha + it, |t| \leq \delta\}$. By the steep descent property of Γ , the contribution of the integral over $\Gamma \setminus \Gamma_\delta$ in (87) is bounded by $Q(\alpha)\mathcal{O}(e^{-cn})$ where $c > 0$ does not depend on n . The integral on Γ_δ can be bounded by

$$Q(\alpha) \left| \frac{n^{1/3}}{2\pi i} \int_{\Gamma_\delta} dw \exp\left(n(\tilde{f}_0(w) - \tilde{f}_0(\alpha)) + n^{2/3}\kappa_1(w^2 - \alpha^2)\right) \right|. \quad (93)$$

By series expansion,

$$\text{Re}(\tilde{f}_0(\alpha + it) - \tilde{f}_0(\alpha)) = -\gamma t^2(1 + \mathcal{O}(t)) \quad (94)$$

with

$$\gamma = \frac{1}{2} \left(1 - \frac{1}{(1+\alpha)^2}\right). \quad (95)$$

After a change of variable $w = \alpha + it$, (93) is written as

$$\begin{aligned} & Q(\alpha) \frac{n^{1/3}}{2\pi} \int_{-\delta}^{\delta} dt \exp\left(-\gamma t^2 n(1 + \mathcal{O}(t)) (1 + \mathcal{O}(n^{-1/3}))\right) \\ & \leq Q(\alpha) \frac{n^{1/3}}{2\pi} \int_{-\delta}^{\delta} dt \exp\left(-\frac{\gamma t^2 n}{2}\right) \leq Q(\alpha) \frac{1}{\sqrt{2\pi\gamma n^{1/3}}} \end{aligned} \quad (96)$$

for δ small enough and n large enough. The estimate above is the largest if γ is small. Note that, by (95), (90) and (88),

$$\gamma n^{1/3} \sim \alpha n^{1/3} \sim \tilde{s}^{1/2} n^{1/3} \sim s^{1/2} \quad (97)$$

which is large if s is large enough, so the integral in (93) is at most constant times $Q(\alpha)$.

Hence, it remains to bound $Q(\alpha)$ exponentially in s . For this end, we use the Taylor expansion

$$\tilde{f}_0(w) = \left(\frac{w^3}{3} - \tilde{s}w \right) (1 + \mathcal{O}(w)). \quad (98)$$

If $\tilde{s} \leq \varepsilon$, then

$$\begin{aligned} Q(\alpha) &= \exp \left(\left(-\frac{2}{3} n \tilde{s}^{3/2} + \kappa_1 n^{2/3} \tilde{s} \right) (1 + \mathcal{O}(\sqrt{\varepsilon})) \right) \\ &= \exp \left(\left(-\frac{2}{3} s^{3/2} + \kappa_1 s \right) (1 + \mathcal{O}(\sqrt{\varepsilon})) \right) \end{aligned} \quad (99)$$

where the first term in the exponent dominates as s is large, so this is even stronger than what we had to prove.

If $\tilde{s} > \varepsilon$, then

$$Q(\alpha) = \exp \left(\left(n \sqrt{\varepsilon} \left(\frac{\varepsilon}{3} - \tilde{s} \right) + \kappa_1 n^{2/3} \varepsilon \right) (1 + \mathcal{O}(\sqrt{\varepsilon})) \right) \quad (100)$$

where $\varepsilon/3 - \tilde{s} \leq -\frac{2}{3}\tilde{s}$. Hence, the first term in the exponent is about $-\frac{2}{3}\sqrt{\varepsilon}n^{1/3}s$ which dominates the second term that is of order $\varepsilon n^{2/3} \sim s$. Therefore, for a given $\varepsilon > 0$, n can be chosen so large that

$$Q(\alpha) \leq \exp \left(-\frac{1}{3} \sqrt{\varepsilon} n^{1/3} s \right). \quad (101)$$

This finishes the proof. \square

Proof of Lemma 3.2. By Remark 3, it is enough to prove all the bounds for $i = 1$. The assertion (46) can be shown as follows. Lemma 4.1 with $s = \sigma + \xi_2 + \mu$ applies for the integral with respect to w in (75) and provides an exponentially decaying bound in μ . If we prove a similar statement for the integral with respect to z , then (46) follows for $i = 1$ along with the uniformity assertion for ξ_1 and ξ_2 . We omit the details of the proof of the bound on the z -integral here, because they are very similar to that of Lemma 4.1, but we give that for the second integral in (75). We consider the function

$$g_0^A(z) = -\log(1-z) - \frac{z^2}{2} - z - \tilde{s}z \quad (102)$$

and the path $\{1 - \rho e^{i\phi}\}$ with $0 < \rho \leq 1$. It is steep descent, because

$$\frac{d}{d\phi} \operatorname{Re}(g_0^A(1 - \rho e^{i\phi})) = -\rho \sin \phi \underbrace{(2(1 - \rho \cos \phi) + \tilde{s})}_{\geq 0}. \quad (103)$$

For (47), we use Lemma 4.1 and the fact that, for the function

$$g_0^B(z) = \lambda \log \left(1 + \frac{z}{\sqrt{\lambda}} \right) - \log(1 - z) - (1 + \sqrt{\lambda})z - \tilde{s}z, \quad (104)$$

the path $\{1 - \rho e^{i\phi}\}$ with $0 < \rho \leq 1$ is steep descent. Indeed

$$\frac{d}{d\phi} \operatorname{Re}(g_0^B(1 - \rho e^{i\phi})) = -\rho \sin \phi \left((1 + \sqrt{\lambda}) \left(1 - \frac{\lambda}{|\sqrt{\lambda} + 1 - \rho e^{i\phi}|^2} \right) + \tilde{s} \right) \quad (105)$$

where the last factor between the outermost parenthesis is certainly positive.

For proving (48), we take the function

$$f_0^\beta(w) = \lambda \log \left(1 + \frac{w}{\sqrt{\lambda}} \right) + \frac{w^2}{2} - w - \tilde{s}w. \quad (106)$$

The steep descent path is $\alpha + i\mathbb{R}$ for $\alpha \geq 0$ by

$$\frac{d}{dt} \operatorname{Re}(f_0^\beta(\alpha + it)) = -t \underbrace{\left(1 - \frac{\lambda}{(\sqrt{\lambda} + \alpha)^2 + t^2} \right)}_{\geq 0}. \quad (107)$$

Finally, for the function

$$f_0^C(w) = \log(1 + w) - \lambda \log \left(1 - \frac{w}{\sqrt{\lambda}} \right) - (1 + \sqrt{\lambda})w - \tilde{s}w, \quad (108)$$

we choose the steep descent path $\{\sqrt{\lambda} - \rho e^{i\phi}\}$ with $0 < \rho \leq \sqrt{\lambda}$. The steep descent property is shown by

$$\frac{d}{d\phi} \operatorname{Re}(f_0^C(\sqrt{\lambda} - \rho e^{i\phi})) = -\rho \sin \phi \left((1 + \sqrt{\lambda}) \left(1 - \frac{1}{|\sqrt{\lambda} + 1 - \rho e^{i\phi}|^2} \right) + \tilde{s} \right) \quad (109)$$

where the factor between the outermost parenthesis is positive. This fact, together with the steep descent property of $g_0^A(z)$ in (102), yields (49).

Using the steep descent paths for $f_0^C(w)$ in (108) and for $g_0^B(z)$ in (104), we can mimic the proof of Lemma 4.1 to get exponential bounds on the integrals in (83). It completes the proof of Lemma 3.2. \square

References

- [1] M. Adler, P.L. Ferrari, and P. van Moerbeke, *Non-intersecting random walks in the neighborhood of a symmetric tacnode*, arXiv:1007.1163; To appear in Ann. Probab. (2010).
- [2] M. Adler, K. Johansson, and P. van Moerbeke, *Double Aztec diamonds and the tacnode process*, arXiv:1112.5532 (2011).
- [3] J. Ben Hough, M. Krishnapur, Y. Peres, and B. Virág, *Determinantal processes and independence*, Probability Surveys **3** (2006), 206–229.
- [4] A. Borodin and M. Duits, *Limits of determinantal processes near a tacnode*, Ann. Inst. H. Poincaré (B) **47** (2011), no. 1, 243–258.
- [5] A. Borodin and P.L. Ferrari, *Anisotropic growth of random surfaces in $2 + 1$ dimensions*, arXiv:0804.3035 (2008).
- [6] A. Borodin and P.L. Ferrari, *Large time asymptotics of growth models on space-like paths I: PushASEP*, Electron. J. Probab. **13** (2008), 1380–1418.
- [7] A. Borodin and J. Kuan, *Random surface growth with a wall and Plancherel measures for $O(\infty)$* , Comm. Pure Appl. Math. **63** (2010), 831–894.
- [8] S. Delvaux, A. Kuijlaars, and L. Zhang, *Critical behavior of non-intersecting Brownian motions at a tacnode*, Comm. Pure Appl. Math. **64** (2011), 1305–1383.
- [9] P.L. Ferrari, *Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues*, Comm. Math. Phys. **252** (2004), 77–109.
- [10] P.L. Ferrari and H. Spohn, *Step fluctuations for a faceted crystal*, J. Stat. Phys. **113** (2003), 1–46.
- [11] K. Johansson, *Non-intersecting paths, random tilings and random matrices*, Probab. Theory Related Fields **123** (2002), 225–280.
- [12] K. Johansson, *Discrete polynuclear growth and determinantal processes*, Comm. Math. Phys. **242** (2003), 277–329.
- [13] K. Johansson, *Non-intersecting, simple, symmetric random walks and the extended Hahn kernel*, Ann. Inst. Fourier **55** (2005), 2129–2145.
- [14] K. Johansson, *Random matrices and determinantal processes*, Mathematical Statistical Physics, Session LXXXIII: Lecture Notes of the Les Houches Summer School 2005 (A. Bovier, F. Dunlop, A. van Enter, F. den Hollander, and J. Dalibard, eds.), Elsevier Science, 2006, pp. 1–56.

- [15] K. Johansson, *Non-colliding Brownian Motions and the extended tacnode process*, arXiv:1105.4027 (2011).
- [16] S. Karlin and L. McGregor, *Coincidence probabilities*, Pacific J. **9** (1959), 1141–1164.
- [17] M. Katori and H. Tanemura, *Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems*, J. Math. Phys. **45** (2004), 3058–3085.
- [18] M. Katori and H. Tanemura, *Infinite systems of noncolliding generalized meanders and Riemann–Liouville differintegrals*, Probab. Theory Relat. Fields **138** (2007), 113–156.
- [19] R. Lyons, *Determinantal probability measures*, Publ. Math. Inst. Hautes Etudes Sci. **98** (2003), 167–212.
- [20] T. Nagao and P.J. Forrester, *Multilevel dynamical correlation functions for Dyson’s Brownian motion model of random matrices*, Phys. Lett. A **247** (1998), 42–46.
- [21] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), 581–603.
- [22] A. Okounkov and N. Reshetikhin, *Random skew plane partitions and the Pearcey process*, Comm. Math. Phys. **269** (2007), 571–609.
- [23] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and the Airy process*, J. Stat. Phys. **108** (2002), 1071–1106.
- [24] T. Sasamoto and T. Imamura, *Fluctuations of a one-dimensional polynuclear growth model in a half space*, J. Stat. Phys. **115** (2004), 749–803.
- [25] A.B. Soshnikov, *Determinantal random fields*, Encyclopedia of Mathematical Physics (J.-P. Francoise, G. Naber, and T. S. Tsun, eds.), Elsevier, Oxford, 2006, pp. 47–53.
- [26] H. Spohn, *Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals*, Physica A **369** (2006), 71–99.
- [27] C. Tracy and H. Widom, *The Pearcey Process*, Comm. Math. Phys. **263** (2006), 381–400.
- [28] C.A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), 151–174.