A determinantal formula for the GOE Tracy-Widom distribution

Patrik L. Ferrari and Herbert Spohn

Technische Universität München

Zentrum Mathematik and Physik Department

e-mails: ferrari@ma.tum.de, spohn@ma.tum.de

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Abstract

Investigating the long time asymptotics of the totally asymmetric simple exclusion process, Sasamoto obtains rather indirectly a formula for the GOE Tracy-Widom distribution. We establish that his novel formula indeed agrees with more standard expressions.

1 Introduction

The Gaussian orthogonal ensemble (GOE) of random matrices is a probability distribution on the set of $N \times N$ real symmetric matrices defined through

$$Z^{-1} e^{-\operatorname{Tr}(H^2)/4N} \mathrm{d}H. \tag{1}$$

Z is the normalization constant and $\mathrm{d}H = \prod_{1 \leq i \leq j \leq N} \mathrm{d}H_{i,j}$. The induced statistics of eigenvalues can be studied through the method of Pfaffians. Of particular interest for us is the statistics of the largest eigenvalue, E_1. As proved by Tracy and Widom [8], the limit

$$\lim_{N \to \infty} \mathbb{P}\big(E_1 \leq 2N + sN^{1/3}\big) = F_1(s) \tag{2}$$

exists, $\mathbb{P}$ being our generic symbol for probability of the event in parenthesis. F_1 is called the GOE Tracy-Widom distribution function. Following [3] it can be expressed in terms of a Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$ as follows,

$$F_1(s)^2 = \det\big(\mathbb{1} - P_s(K + |g\rangle\langle f|)P_s\big), \tag{3}$$

where K is the Airy kernel defined through

$$\begin{aligned} K(x,y) &= \int_{\mathbb{R}_+} \mathrm{d}\lambda \, \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda), \\ g(x) &= \mathrm{Ai}(x), \\ f(y) &= 1 - \int_{\mathbb{R}_+} \mathrm{d}\lambda \, \mathrm{Ai}(y+\lambda), \end{aligned} \tag{4}$$

and P_s is the projection onto the interval $[s, \infty)$.

The GOE Tracy-Widom distribution $F_1(s)$ turns up also in the theory of one-dimensional growth process in the KPZ universality class, KPZ standing for Kardar-Parisi-Zhang [4]. Let us denote the height profile of the growth process at time t by $h(x,t)$, either $x \in \mathbb{R}$ or $x \in \mathbb{Z}$. One then starts the growth process with flat initial conditions, meaning $h(x,0) = 0$, and considers the height above the origin $x = 0$ at growth time t. For large t it is expected that

$$h(0,t) = c_1 t + c_2 t^{1/3} \xi_1. \tag{5}$$

Here c_1 and c_2 are constants depending on the details of the model and ξ_1 is a random amplitude with

$$\mathbb{P}(\xi_1 \le s) = F_1(s). \tag{6}$$

For the polynuclear growth (PNG) model the height $h(0,t)$ is related to the length of the longest increasing subsequence of symmetrized random permutations [5], for which Baik and Rains [1] indeed prove the asymptotics (5), (6), see [2] for further developments along this line. Very recently Sasamoto [6] succeeds in proving the corresponding result for the totally asymmetric simple exclusion process (TASEP). If $\eta_j(t)$ denotes the occupation variable at $j \in \mathbb{Z}$ at time t, then the TASEP height is given by

$$h(j,t) = \begin{cases} 2N_t + \sum_{i=1}^{j} (1 - 2\eta_i(t)) & \text{for } j \ge 1, \\ 2N_t & \text{for } j = 0, \\ 2N_t - \sum_{i=j+1}^{0} (1 - 2\eta_i(t)) & \text{for } j \le -1, \end{cases} \tag{7}$$

with N_t denoting the number of particles which passed through the bond $(0,1)$ up to time t. The flat initial condition for the TASEP is $\ldots 0\,1\,0\,1\,0\,1 \ldots$. For technical reasons Sasamoto takes instead $\ldots 0\,1\,0\,1\,0\,0\,0\,0\,0 \ldots$ and studies the asymptotics of $h(-3t/2, t)$ for large t with the result

$$h(-3t/2, t) = \tfrac{1}{2} t + \tfrac{1}{2} t^{1/3} \xi_{\mathrm{SA}}. \tag{8}$$

The distribution function of the random amplitude $\xi_{\rm SA}$ is

$$\mathbb{P}(\xi_{\rm SA} \leq s) = F_{\rm SA}(s) \tag{9}$$

with

$$F_{\rm SA}(s) = \det(\mathbb{1} - P_s A P_s). \tag{10}$$

Here A has the kernel $A(x,y) = \frac{1}{2}\,\mathrm{Ai}((x+y)/2)$ and, as before, the Fredholm determinant is in $L^2(\mathbb{R})$.

The universality hypothesis for one-dimensional growth processes claims that in the scaling limit, up to model-dependent coefficients, the asymptotic distributions are identical. In particular, since (5) is proved for PNG, the TASEP with flat initial conditions should have the same limit distribution function, to say

$$F_{\rm SA}(s) = F_1(s). \tag{11}$$

Our contribution provides a proof for (11).

2 The identity

As written above, the s-dependence sits in the projection P_s. It will turn out to be more convenient to transfer the s-dependence into the integral kernel. From now on the determinants are understood as Fredholm determinants in $L^2(\mathbb{R}_+)$ with scalar product $\langle\cdot,\cdot\rangle$. Thus, whenever we write an integral kernel like $A(x,y)$, the arguments are understood as $x \geq 0$ and $y \geq 0$.

Let us define the operator $B(s)$ with kernel

$$B(s)(x,y) = \mathrm{Ai}(x+y+s). \tag{12}$$

By [7] $\|B(s)^2\| < 1$ and clearly $B(s)$ is symmetric. Thus also $\|B(s)\| < 1$ for all s. $B(s)$ is trace class with both positive and negative eigenvalues. Shifting the arguments in (10) by s, one notes that

$$F_{\rm SA}(s) = \det(\mathbb{1} - B(s)). \tag{13}$$

Applying the same operation to (3) yields

$$F_1(s)^2 = \det\left(\mathbb{1} - B(s)^2 - |g\rangle\langle f|\right) \tag{14}$$

with

$$\begin{aligned} g(x) &= \mathrm{Ai}(x+s) = (B(s)\delta)(x), \\ f(y) &= 1 - \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(y+\lambda+s) = ((\mathbb{1} - B(s))1)(y). \end{aligned} \tag{15}$$

Here δ is the δ-function at $x = 0$ and 1 denotes the function $1(x) = 1$ for all $x \geq 0$. δ and 1 are not in $L^2(\mathbb{R}_+)$. Since the kernel of $B(s)$ is continuous and has super-exponential decay, the action of $B(s)$ is unambiguous.

Proposition 1. *With the above definitions we have*

$$\det(\mathbb{1} - B(s)) = F_1(s). \tag{16}$$

Proof. For simplicity we suppress the explicit s-dependence of B. We rewrite

$$\begin{aligned} F_1(s)^2 &= \det\big((\mathbb{1} - B)(\mathbb{1} + B - |B\delta\rangle\langle 1|)\big) \\ &= \det(\mathbb{1} - B)\det(\mathbb{1} + B)\big(1 - \langle\delta, B(\mathbb{1} + B)^{-1}1\rangle\big) \\ &= \det(\mathbb{1} - B)\det(\mathbb{1} + B)\langle\delta, (\mathbb{1} + B)^{-1}1\rangle \end{aligned} \tag{17}$$

since $1 = \langle\delta, 1\rangle$. Thus we have to prove that

$$\det(\mathbb{1} - B) = \det(\mathbb{1} + B)\langle\delta, (\mathbb{1} + B)^{-1}1\rangle. \tag{18}$$

Taking the logarithm on both sides,

$$\ln\det(\mathbb{1} - B) = \ln\det(\mathbb{1} + B) + \ln\langle\delta, (\mathbb{1} + B)^{-1}1\rangle, \tag{19}$$

and differentiating it with respect to s results in

$$-\operatorname{Tr}\big((\mathbb{1} - B)^{-1}\frac{\partial}{\partial s}B)\big) = \operatorname{Tr}\big((\mathbb{1} + B)^{-1}\frac{\partial}{\partial s}B)\big) + \frac{\frac{\partial}{\partial s}\langle\delta, (\mathbb{1} + B)^{-1}1\rangle}{\langle\delta, (\mathbb{1} + B)^{-1}1\rangle} \tag{20}$$

where we used

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln(\det(T)) = \operatorname{Tr}\big(T^{-1}\frac{\partial}{\partial s}T\big). \tag{21}$$

Since $B(s) \to 0$ as $s \to \infty$, the integration constant for (20) vanishes and we have to establish that

$$-2\operatorname{Tr}\big((\mathbb{1} - B^2)^{-1}\frac{\partial}{\partial s}B)\big) = \frac{\frac{\partial}{\partial s}\langle\delta, (\mathbb{1} + B)^{-1}1\rangle}{\langle\delta, (\mathbb{1} + B)^{-1}1\rangle}. \tag{22}$$

Define the operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then using the cyclicity of the trace and Lemma 2,

$$\begin{aligned} -2\operatorname{Tr}\big((\mathbb{1} - B^2)^{-1}\frac{\partial}{\partial s}B)\big) &= -2\operatorname{Tr}\big((\mathbb{1} - B^2)^{-1}DB)\big) \\ &= \langle\delta, (\mathbb{1} - B^2)^{-1}B\delta\rangle. \end{aligned} \tag{23}$$

Using Lemma 3 and $D1 = 0$, one obtains

$$\langle\delta, \frac{\partial}{\partial s}(\mathbb{1} + B)^{-1}1\rangle = \langle\delta, (\mathbb{1} - B^2)^{-1}B\delta\rangle\langle\delta, (\mathbb{1} + B)^{-1}1\rangle. \tag{24}$$

Thus (22) follows from (23) and (24). ∎

Lemma 2. *Let A be a symmetric, trace class operator with smooth kernel and let $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then*

$$2\,\mathrm{Tr}(DA) = -\langle \delta, A\delta\rangle \tag{25}$$

where DA is the operator with kernel $\frac{\partial}{\partial x}A(x,y)$.

Proof. The claim follows from spectral representation of A and the identity

$$\int_{\mathbb{R}_+} \mathrm{d}x f'(x)f(x) = -f(0)f(0) - \int_{\mathbb{R}_+} \mathrm{d}x f(x)f'(x). \tag{26}$$

□

Lemma 3. *It holds*

$$\frac{\partial}{\partial s}(\mathbb{1}+B)^{-1} = (\mathbb{1}-B^2)^{-1}BD + (\mathbb{1}-B^2)^{-1}|B\delta\rangle\langle\delta(\mathbb{1}+B)^{-1}|. \tag{27}$$

Proof. First notice that $\frac{\partial}{\partial s}B \equiv \dot{B} = DB$. For any test function f,

$$\begin{aligned}(\dot{B}f)(x) &= \int_{\mathbb{R}_+} \mathrm{d}y \partial_y \mathrm{Ai}(x+y+s)f(y) \\ &= -\mathrm{Ai}(x+s)f(0) - \int_{\mathbb{R}_+} \mathrm{d}y\, \mathrm{Ai}(x+y+s)f'(y).\end{aligned} \tag{28}$$

Thus, using the notation $P = |B\delta\rangle\langle\delta|$, one has

$$DB = -BD - P. \tag{29}$$

Since $\|B\| < 1$, we can expand $\frac{\partial}{\partial s}(\mathbb{1}+B)^{-1}$ in a power series and get

$$\frac{\partial}{\partial s}(\mathbb{1}+B)^{-1} = \sum_{n\geq 1}(-1)^n \frac{\partial}{\partial s}B^n = \sum_{n\geq 1}(-1)^n \sum_{k=0}^{n-1} B^k D B^{n-k}. \tag{30}$$

Using recursively (29) we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} B^k D B^{n-k} &= -\frac{1-(-1)^n}{2}B^n D + \sum_{j=0}^{n-1}\sum_{k=j}^{n-1}(-1)^{j+1}B^k P B^{n-k-1} \\ &= -\frac{1-(-1)^n}{2}B^n D + \sum_{k=0}^{n-1}\frac{1+(-1)^k}{2}B^k P B^{n-k-1}.\end{aligned} \tag{31}$$

Inserting (31) into (30) and exchanging the sums results in

$$\begin{aligned}\frac{\partial}{\partial s}(\mathbb{1}+B)^{-1} &= \sum_{n\geq 1} B^{2n+1}D + \sum_{k\geq 0}\sum_{n\geq k+1}\frac{1+(-1)^k}{2}B^k P(-B)^{n-(k+1)} \\ &= (\mathbb{1}-B^2)^{-1}BD + (\mathbb{1}-B^2)^{-1}P(\mathbb{1}+B)^{-1}.\end{aligned} \tag{32}$$

□

3 Outlook

The asymptotic distribution of the largest eigenvalue is also known for Gaussian unitary ensemble of Hermitian matrices ($\beta = 2$) and Gaussian symplectic ensemble of quaternionic symmetric matrices ($\beta = 4$). As just established, for $\beta = 1$,

$$F_1(s) = \det(\mathbb{1} - B(s)), \tag{33}$$

and, for $\beta = 2$,

$$F_2(s) = \det(\mathbb{1} - B(s)^2), \tag{34}$$

which might indicate that $F_4(s)$ equals $\det(\mathbb{1} - B(s)^4)$. This is however incorrect, since the decay of $\det(\mathbb{1} - B(s)^4)$ for large s is too rapid. Rather one has

$$F_4(s/\sqrt{2}) = \frac{1}{2}\big(\det(\mathbb{1} - B(s)) + \det(\mathbb{1} + B(s))\big). \tag{35}$$

This last identity is obtained as follows. Let $U(s) = \frac{1}{2}\int_s^\infty q(x)\mathrm{d}s$ with q the unique solution of the Painlevé II equation $q'' = sq + 2q^3$ with $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$. Then the Tracy-Widom distributions for $\beta = 1$ and $\beta = 4$ are given by

$$F_1(s) = \exp(-U(s))F_2(s)^{1/2}, \quad F_4(s/\sqrt{2}) = \cosh(U(s))F_2(s)^{1/2}, \tag{36}$$

see [8]. Thus $F_4(s/\sqrt{2}) = \frac{1}{2}(F_1(s) + F_2(s)/F_1(s))$, from which (35) is deduced.

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