3) Interacting particle systems

3.1) Setting, examples.

- In this chapter we will consider a class of processes which is quite different from Feller Diffusion processes or the Markov Chains that we have discussed in the previous part of the lecture.
- For simplicity we will consider interacting particle systems where one can put at most one of them at a given position x and time t.
- To see the differences and introduce the new notations (typical of the domain), consider as representant of the already known processes a diffusion on $\mathbb{R}^n$.

Interacting particles moving on $\Lambda = \mathbb{Z}^d$.		Diffusion on $\mathbb{R}^n = S$
$S = \{0,1\}^{\Lambda}$, where Λ is a countable set.	State space	$S = \mathbb{R}^n$
$\eta, \xi, \zeta, \ldots$	Element in S	$x, y, z, \ldots$
$\eta_t, \xi_t, \ldots$	Configuration at time t	$X(t), Y(t), \ldots$
$\eta_t(x) \in \{0,1\}, x \in \Lambda$	Value of a configuration	$X_k(t) \in \mathbb{R}, k \in \{1, \ldots, n\}$

↳ Where $\eta_t(x) = \begin{cases} 1, & \text{if there is a particle at time } t \text{ at position } x, \\ 0, & \text{if site } x \text{ is empty at time } t. \end{cases}$

- Rem.: The evolution of the Feller processes that we will define have (of course, by def.) the Markov property, i.e., $t \mapsto \eta_t$ will be Markovian. However one is very often interested in the evolution of a projection of η_t, e.g., on the behavior of a single particle: this is not (in general) Markov anymore.

. Further on the setting:

. We consider the product topology on $S = \{0,1\}^{\Lambda}$.
Then, since $S = \prod_{i \in \Lambda} S_i$, $S_i = \{0,1\}$ are compact, by Tychonov's theorem, also <u>S is compact.</u>

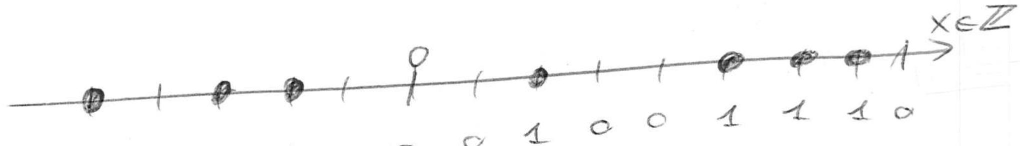
$\Rightarrow$ By Thm 2.32 it then follows that at least one stationary distribution exists.

<u>.Q.:</u> What can we say about the stationary distributions?

. This and other questions will be discussed in this chapter, but before going more in details, let us give three examples of interacting particle systems (we will study some of them in this lecture).

<u>Ex. 1) Exclusion processes.</u>

. let $\Lambda = \mathbb{Z}$. A configuration η_t is for example visualized as



. Corresponding $\eta_t(x)$: 1 0 1 1 0 0 0 1 0 0 1 1 1 0

<u>. Dynamics:</u> . A particle at site x tries to jump to another site y at a jump rate $c(x,y)$
. The jump is allowed if y is empty.
. The jumps occur independently from each other.

<u>. Rem.:</u> If $|\Lambda| < \infty$, it is just a Markov chain with

$$\mathcal{L} f(\eta) = \sum_{x,y \in \Lambda} c(x,y) \cdot \eta_x (1-\eta_y) \left[f(\eta^{x,y}) - f(\eta) \right]$$

where
$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

. We will not study this system now (but it will be treated in my summester lecture extensively enough).

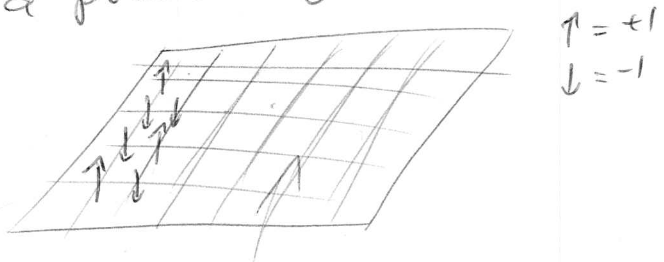
Ex. 2) Spin systems (Ising model)

- Consider $\Lambda = [-L, L]^d \cap \mathbb{Z}^d$ and to each site x (representing an atom) we assign a variable $\sigma(x) \in \{-1, 1\}$, called spin, (that is, the magnetic moment).

- To each configuration $\sigma = (\sigma(x))_{x \in \Lambda}$, we assign an energy function, aka Hamiltonian,

$$H_\Lambda(\sigma) = \underbrace{-J \sum_{\substack{x,y \in \Lambda, \\ |x-y|=1}} \sigma(x)\sigma(y)}_{\text{Interaction}} + \underbrace{h \sum_{x \in \Lambda} \sigma(x)}_{\text{External magnetic field}}$$

- At temperature $T \equiv \frac{1}{\beta}$ (at thermal equilibrium), a configuration σ has a probability to be observed given by the Gibbs distribution:



$$\mu_{\Lambda,\beta}(\sigma) = \frac{e^{-\beta H(\sigma)}}{\sum_{\tilde{\sigma} \in \{-1,1\}^\Lambda} e^{-\beta H(\tilde{\sigma})}}.$$

- Q.: What happens when $\Lambda \to \infty$?

Static picture: Consider first $J = 1$ (i.e., spins likes to have the same value as their neighbors) and $h = 0$.

$\Rightarrow$ For T large enough, the penalty $e^{-2\beta}$ payed for each time two neighboring spins have the opposite sign is not that strong and $\mu_{\Lambda,\beta} \xrightarrow[|\Lambda| \to \infty]{} \mu_\beta$ (a unique measure; disordered system)

- But in $d \geq 2$, for T small enough, $e^{-2\beta}$ is very small spins likes to have the same sign as their neighbors

$\Rightarrow$ spins likes to have the same value.

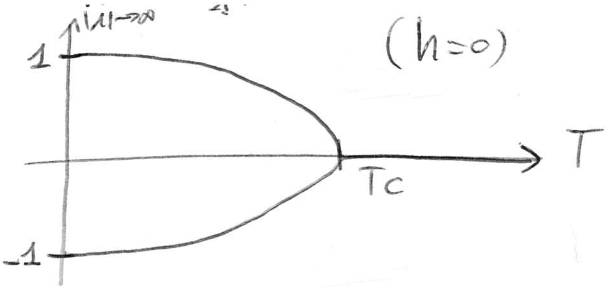
But ~~for~~ $h = 0$, $+1$ or -1 plays the same role.

Indeed, $\mu_{\Lambda,\beta} \not\xrightarrow[|\Lambda| \to \infty]{}$ unique measure.

The extremal cases are all spins $+1$ or all spins -1

- Rem.: The magnetic momentum of the modelled material is $m_\Lambda = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma(x)$; In this latter case $m_\Lambda \not\to 0$; one speaks of Spontaneous magnetisation (e.g., Magnet).

$(J>0, d=2,3)$



Dynamic picture: Assume now that we start with a given configuration σ and describe the time evolution (σ might be taken from an equilibrium configuration at some inverse temperature β and then change β). Q.: Does the system converge to a unique stationary distribution?

Example: let $\sigma \sim \mu_{\beta_0}$ and the dynamics be given as follows:

- Each spin, independently, changes sign with the following rate (for a spin at x):

$$C(x,\sigma) = e^{-\beta \sum_{y:|y-x|=1} \sigma(x)\sigma(y)}, \text{ i.e.,}$$

$$\text{"} \mathcal{L} f(\sigma) = \sum_{x\in\mathbb{Z}^d} C(x,\sigma)\left(f(\sigma^x)-f(\sigma)\right)\text{"}, \text{ with}$$

$$\sigma^x(z) = \begin{cases} \sigma(z), & z\neq x, \\ -\sigma(x), & z=x. \end{cases}$$

Rem: If spins are already all aligned (same sign) around x $\Rightarrow \sum_{y:|y-x|=1} \sigma(x)\sigma(y)$ is maximal $\Rightarrow C(x,\sigma)$ is minimal.

$\Rightarrow$ moving away from the low energy configuration is more unlikely that going towards low energy configurations.

But: # configurations where most spins are aligned is $\ll$ # disordered configuration.

$\Rightarrow$ There is a competition between energy and entropy.

Q.: What happens?

- For $T > T_c$, for any initial distribution ν, $\nu T(t) \xRightarrow{t\to\infty} \mu_\beta$.

In this case we will say that the semigroup $T(t)$ is ergodic.

Ex. 3) The voter model

- Consider first $|\Lambda| < \infty$ and $S^\Lambda = \eta \in \{0,1\}^\Lambda$. Let

$$\mathcal{L} f(\eta) = \sum_{x \in \Lambda} c(x,\eta) \left(f(\eta^x) - f(\eta) \right)$$

changes the value of $\eta(x)$ into $1-\eta(x)$

with $c(x,\eta) = \sum_{y: \eta(y) \neq \eta(x)} q(x,y)$, where $q(x,y) \geq 0$ for $x \neq y$.

- Interpretations: • At rate $q(x,y)$, the voter at x adopt the opinion of the voter at y.
  - If 0 and 1 represent two species, then at rate $q(x,y)$ the species at y invades x.
- Rem.: In this case there are always two stationary distributions, $\eta \equiv 0$ and $\eta \equiv 1$.

  Q.: Are these the only ones?

3.2) Spin systems

- One of the question that we have to answer is also under which conditions on the rates $c(x,\eta)$ is the process really a Feller process.
- Like for the issue about explosive Markov chains, also here we have to put some (sufficient) conditions that ensures that the proposed operator has a closure that is a probability generator.
- For the example of the exclusion process, we need to avoid that in finite time our system in a bounded region is influenced by what happens at "infinity".

• [illegible]

Setting: • $c(x,\eta)$ denotes a non-negative, uniformly bounded function on $\Lambda \times S$ that is continuous in η for each $x \in \Lambda$.

• For $\eta \in S$ and $x \in \Lambda$, we define

$$\eta^x \in S : \quad \eta^x(z) = \begin{cases} \eta(z), & \text{for } z \neq x, \\ 1-\eta(x), & \text{for } z = x. \end{cases}$$

$\Rightarrow$ $c(x,\eta)$ is interpreted as the rate at which η changes to η^x.

• If $|\Lambda| < \infty$, the process η_t is a finite state, continuous time Markov chain with Q-matrix

$$q(\eta,\eta^x) = c(x,\eta),$$

i.e., with generator

$$\mathcal{L}f(\eta) = \sum_{x\in\Lambda} c(x,\eta)\left[f(\eta^x) - f(\eta)\right]. \quad (*)$$

• For general countable Λ, we need to ensure that $(*)$ is summable. For that reason, consider:

$$D = \left\{ f \in C(S) \;\middle|\; |||f||| := \sum_{x\in\Lambda} \sup_{\eta\in S} |f(\eta^x) - f(\eta)| < \infty \right\}.$$

• As $c(x,\eta)$ is uniformly bounded, we can define $\mathcal{L}$ on D by $(*)$.

Rem.: The voter model is a spin system.

3.2.1) Construction of the probability generator.

• Goal: Find natural conditions on $c(x,\eta)$ s.t. $\bar{\mathcal{L}}$ is a probability generator.

• The properties ⓐ, ⓑ, ⓓ of Def. 2.6 holds already with the above mentioned assumptions, while for property ⓒ we need to work more.

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Lemma 3.1) The operator $\mathcal{L}$ defined on D satisfies:

(a) D is dense in $C(\mathcal{S}')$,

(b) If $f \in D$, $\lambda \geq 0$, $f - \lambda \mathcal{L} f = g$, then
$$\min_{\eta \in \mathcal{S}'} f(\eta) \geq \min_{\eta \in \mathcal{S}'} g(\eta),$$

(d) $1 \in D$ and $\mathcal{L} 1 = 0$.

Proof.: (d): Clearly $1 \in D$, since $1(\eta^x) - 1(\eta) = 0$. For the same reason, $\mathcal{L} 1 = 0$.

(a): Notice that D is an algebra of continuous functions on a compact set $\mathcal{S}'$.
Further, D separates points in $\mathcal{S}'$ [i.e., $\eta \neq \eta'$ implies that $\exists x \in \Lambda$ s.t. $\eta(x) \neq \eta'(x)$, and the function $f(\xi) = \xi(x)$ separates η and η' since $f(\eta) = \eta(x) \neq \eta'(x) = f(\eta')$].
Finally, D contains the constant functions.
Then, by Stone-Weierstrass theorem, any $f \in C(\mathcal{S}')$ can be given as a limit of $f_n \in D$.

(b): Let $f \in D$, $\lambda \geq 0$, $f - \lambda \mathcal{L} f = g$.
Since f is continuous and $\mathcal{S}'$ is compact,
$\exists \tilde{\eta}$ s.t. $f(\tilde{\eta})$ is minimal.
But
$$\mathcal{L} f(\tilde{\eta}) = \sum_{x \in \Lambda} \underbrace{c(x, \tilde{\eta})}_{\geq 0} \underbrace{\left(f(\tilde{\eta}^x) - f(\tilde{\eta})\right)}_{\geq 0} \geq 0$$
$$\Rightarrow \min_{\eta \in \mathcal{S}'} f(\eta) \geq f(\tilde{\eta}) \geq g(\tilde{\eta}) \geq \min_{\eta \in \mathcal{S}'} g(\eta).$$

#

To verify (c) of Def. 2.6 we need to determine an a priori bound on the solutions of $f - \lambda \mathcal{L} f = g$.

Notations:

- $\varepsilon := \inf_{\substack{x \in \Lambda \\ \eta \in \mathcal{S}'}} \left[c(x, \eta) + c(x, \eta^x) \right]$.
- $\gamma(x, y) := \sup_{\eta \in \mathcal{S}'} \left| c(x, \eta^y) - c(x, \eta) \right|$: It indicates how strongly the flip rate at site x depends on the configuration at site y.

. let $\underline{\ell^1(\Lambda)}$: Banach space of functions $\alpha : \Lambda \to \mathbb{R}$ s.t.

$$\|\alpha\| := \sum_{x \in \Lambda} |\alpha(x)| < \infty.$$

Then, the matrix γ defines an operator Γ on $\ell^1(\Lambda)$ by

$$\Gamma\alpha(y) := \sum_{\substack{x \in \Lambda, \\ x \neq y}} \alpha(x)\, \gamma(x,y),$$

which is well-defined and bounded if

$$. \; M := \sup_{x \in \Lambda} \sum_{y : y \neq x} \gamma(x,y) < \infty.$$

. In that case, $\|\Gamma\| = M$ $(*)$

. Finally, for $f \in C(S)$, $x \in \Lambda$:

$$\Delta_f(x) := \sup_{\eta \in S} |f(\eta^x) - f(\eta)|$$

← Maximal variation of $f(\eta)$ due to a "spin flip" at site x.

Then, $|||f||| = \|\Delta_f\|_{\ell^1(\Lambda)}$.

. $\underline{\text{Prop. 3.2)}}$ Assume that either

ⓐ $f \in D$

or

ⓑ f is continuous and $c(x,\cdot) \equiv 0$ for all but finitely many $x \in \Lambda$.

. If $f - \lambda \Omega f = g \in D$, $\lambda > 0$, and

$$\lambda M < 1 + \lambda \varepsilon,$$

then $\Delta_f \leq \left[(1+\lambda\varepsilon)\mathbb{1} - \lambda\Gamma\right]^{-1} \Delta_g$ (coordinatewise),

where $\left[(1+\lambda\varepsilon)\mathbb{1} - \lambda\Gamma\right]^{-1} \alpha \equiv \dfrac{1}{1+\lambda\varepsilon} \displaystyle\sum_{k=0}^{\infty} \left(\frac{\lambda}{1+\lambda\varepsilon}\right)^k \Gamma^k \alpha.$

$(*)$: Indeed, $\|\Gamma\| \overset{\text{def.}}{=} \sup_{\alpha : \|\alpha\|=1} \|\Gamma\alpha\| = \sup_{\alpha : \|\alpha\|=1} \sum_{x \in \Lambda} \Big| \sum_{y \neq x} \alpha(x) \underbrace{\gamma(x,y)}_{\geq 0} \Big|$

$= \sup_{\alpha : \|\alpha\|=1} \sum_{x} \sum_{y \neq x} |\alpha(x)|\, \gamma(x,y) \leq M \sup_{\alpha : \|\alpha\|=1} \sum_{x} |\alpha(x)| = M$

as the sup will occurs when all α has the same sign.

$\geq \sum_{y \neq x} \gamma(z,y)$, $\forall z \in \Lambda$ $\Rightarrow \|\Gamma\| = M$.
$\alpha(x) = \delta_z(x)$

. Where will this a-priori bound be useful?

. We will want to bound $\Delta_{T(t)g}$. By Thm 2.10,

$$\Delta_{T(t)g} = \lim_{n\to\infty} \Delta_{(\mathbb{1}-\frac{t}{n}\mathcal{L})^{-n}g} \underset{\substack{\text{Prop 1.2}\\ +\text{ iterations}}}{\leq} \lim_{n\to\infty} \left((1+\tfrac{t\varepsilon}{n})\mathbb{1} - \tfrac{t}{n}\Gamma\right)^{-n}\Delta_g$$

$$= e^{-\varepsilon t} e^{t\Gamma}\Delta_g, \ \forall g \in C(S).$$

$\Rightarrow$ We have a bound on the influence of spin flips on g at time t in terms of the one at time $t=0$.

. Proof of Prop 3.2:

• The series is well-defined. Indeed,

$$\left\| \sum_{k=0}^{\infty} \left(\frac{\lambda}{1+\lambda\varepsilon}\right)^k \Gamma^k \alpha \right\| \leq \sum_{k=0}^{\infty} \frac{\lambda^k \|\Gamma\|^k \|\alpha\|}{(1+\lambda\varepsilon)^k} \underset{\|\Gamma\|=M}{\overset{\lambda M < 1+\lambda\varepsilon}{\leq}} \frac{\|\alpha\|}{1-\frac{\lambda M}{1+\lambda\varepsilon}} < \infty,$$

. Let $f - \lambda\mathcal{L}f = g$.

$$\Rightarrow \quad g(\eta^u) - g(\eta) = f(\eta^u) - \lambda\mathcal{L}f(\eta^u) - f(\eta) + \lambda\mathcal{L}f(\eta)$$

with $\mathcal{L}f(\eta) = \sum_x c(x,\eta)\left[f(\eta^x) - f(\eta)\right]$

$$= f(\eta^u) - f(\eta) - \lambda\sum_x c(x,\eta^u)\left[f((\eta^u)^x) - f(\eta^u)\right]$$
$$+ \lambda\sum_x c(x,\eta)\left[f(\eta^x) - f(\eta)\right]$$
$$= f(\eta^u) - f(\eta) - \lambda\sum_{x\neq u} c(x,\eta^u)\left[f((\eta^u)^x) - f(\eta^u)\right]$$
$$+ \lambda\sum_{x\neq u} c(x,\eta)\left[f(\eta^x) - f(\eta)\right]$$
$$- \lambda c(u,\eta^u)\Big[f(\underbrace{(\eta^u)^u}_{=\eta}) - f(\eta^u)\Big] + \lambda c(u,\eta)\left[f(\eta^u) - f(\eta)\right]$$

$$= \left[f(\eta^u) - f(\eta)\right]\cdot\overbrace{\left[1+\lambda c(u,\eta^u) + \lambda c(u,\eta)\right]}^{\geq 1+\lambda\varepsilon \ :-)}$$
$$- \sum_{x\neq u}\left[c(x,\eta^u)\left[f((\eta^u)^x) - f(\eta^u)\right] - c(x,\eta)\left[f(\eta^u) - f(\eta)\right]\right] \quad (*)$$

Now, as $f(\eta^u)-f(\eta)$ is a continuous function of η for each $u\in\Lambda$, $\exists\,\tilde\eta$ s.t. $f(\tilde\eta^u)-f(\tilde\eta)=\sup_\eta\left(f(\eta^u)-f(\eta)\right)$

Further, as $f(\xi^u)-f(\xi)=-\{f(\eta^u)-f(\eta)\}$ for $\xi=\eta^u$, the set $\{f(\eta^u)-f(\eta),\ \eta\in S\}$ is symmetric.

$$\Rightarrow\quad f(\tilde\eta^u)-f(\tilde\eta)=\sup_\eta\left|f(\eta^u)-f(\eta)\right|=\Delta_f(u).$$

$$\Rightarrow\ \forall\,\eta\in S\quad f(\eta^u)-f(\eta)\le f(\tilde\eta^u)-f(\tilde\eta).$$

In particular, for $\eta=\tilde\eta^x$ we get:

$$f((\tilde\eta^u)^x)-f(\tilde\eta^u)=f((\tilde\eta^x)^u)-f(\tilde\eta^u)$$
$$\le f(\tilde\eta^x)-f(\tilde\eta).$$

Therefore, using (*), i.e.,

$$\left(f(\eta^u)-f(\eta)\right)\left(1+\lambda c(u,\eta^u)+\lambda c(u,\eta)\right)=g(\eta^u)-g(\eta)$$
$$+\lambda\sum_{x\ne u}\left[c(x,\eta^u)\left(f((\eta^u)^x)-f(\eta^u)\right)-c(x,\eta)\left(f(\eta^x)-f(\eta)\right)\right]$$

taking $\sup_\eta$ and using the inequality $c(u,\eta^u)+c(u,\eta)\ge\varepsilon$, we get

$$(1+\lambda\varepsilon)\cdot\Delta_f(u)\le\Delta_g(u)+\lambda\sum_{x\ne u}\left(c(x,\tilde\eta^u)-c(x,\tilde\eta)\right)\left[f(\tilde\eta^x)-f(\tilde\eta)\right]$$
$$\le\Delta_g(u)+\lambda\sum_{x\ne u}\gamma(x,u)\,\Delta_f(x).\qquad(**)$$

$\gamma(x,u)\ge|c(x,\eta^u)-c(x,\eta)|$ $\forall\eta$

Rem.: If assumption (b) holds $\Rightarrow$ $\sum_{x\ne u}$ is actually a finite sum so that it actually implies that $f\in D$ too (here use $g\in D$ and $\sum_u\sum_{x\ne u}=\sum_x\sum_{u\ne x}\dots$).

(**) becomes: $(1+\lambda\varepsilon)\Delta_f\le\Delta_g+\lambda\Gamma\Delta_f$ and iterating it we obtain

$$\Delta f\le\frac{\Delta g}{1+\lambda\varepsilon}+\frac{\lambda}{(1+\lambda\varepsilon)^2}\Gamma\Delta g+\dots=\underbrace{\frac{1}{1+\lambda\varepsilon}}\sum_{\ell\ge0}\left(\frac{\lambda}{1+\lambda\varepsilon}\right)^\ell\Gamma^\ell\Delta g.\ \#$$

Now we can prove the existence of a probability generator for spin systems.

<u>Thm. 3.3)</u> Assume $M < \infty$. Then, $\bar{\mathcal{L}}$ is a probability generator associated with the semigroup $T(t)$.
Further,
$$\Delta_{T(t)f} \leq e^{-\varepsilon t} e^{t\Gamma} \Delta_f .$$

In particular, if $f \in D$, then $T(t)f \in D$ too and
$$|||T(t)f||| < e^{(M-\varepsilon)t} \cdot |||f|||.$$

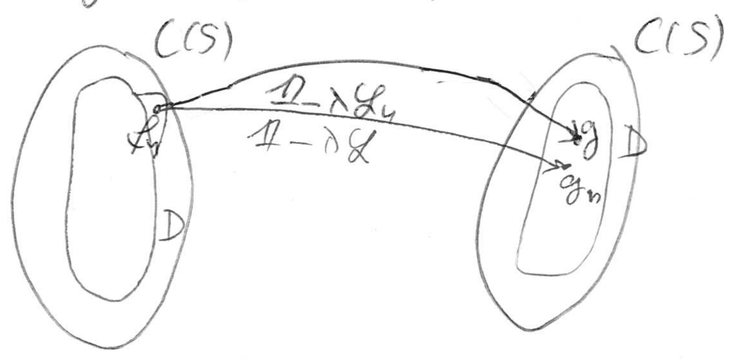
<u>Proof.</u>: By Lemma 4.1, the fact that $\mathcal{L}1 = \bar{\mathcal{L}}1 = 0$ and using Prop 2.24 ⓐ, it follows that Properties ⓐ, ⓑ, ⓓ of Def. 2.6 holds.
It remains to verify property ⓒ, i.e., for $\lambda > 0$ small enough, $\mathcal{R}(\mathbb{1} - \lambda\bar{\mathcal{L}}) = C(S)$.

- We take Λ_n finite with $(\Lambda_n)_{n \geq 1}$ an increasing sequence with $\Lambda_n \nearrow \Lambda$ as $n \to \infty$.

Define $\mathcal{L}_n f(\eta) = \sum_{x \in \Lambda_n} c(x,\eta)\left(f(\eta^x) - f(\eta)\right)$ for $f \in C(S)$.

- $\mathcal{L}_n$ is the generator of the spin system in which we modify only coordinates in Λ_n and keep frozen $\{\eta(x), x \notin \Lambda_n\}$.

Then, since $\mathcal{L}_n$ is a bounded operator, by Lemma 2.12, $\mathcal{R}(\mathbb{1} - \lambda\mathcal{L}_n) = C(S)$ for λ small enough, and by Prop. 2.24 ⓒ, $\mathcal{R}(\mathbb{1} - \lambda\mathcal{L}_n) = C(S)$ for all $\lambda > 0$.



. For given $\underline{\underline{g \in D}}$, $\exists f_n \in C(S)$ s.t. $f_n - \lambda \mathcal{L}_n f_n = g$.

. We can now apply Prop. 4.2 since $\mathcal{L}_n$ satisfies condition ⓑ of Prop 3.2.

$\Rightarrow$ For $\lambda \ll 1$ s.t. $\lambda M < 1 + \lambda \varepsilon$, then by Prop. 4.2. we have that actually $f_n \in D$ too (and not in $C(S) \setminus D$).

$\Rightarrow$ $\mathcal{L} f_n$ is well defined and so is

$$g_n = f_n - \lambda \mathcal{L} f_n \quad \in \mathcal{R}(\mathbb{1} - \lambda \mathcal{L}).$$

. Thus, $\| g_n - g \| = \lambda \| (\mathcal{L} - \mathcal{L}_n) f_n \| \leq \lambda \cdot \underbrace{\left(\sup_{x, \eta} c(x, \eta) \right)}_{\equiv K} \cdot \sum_{x \notin \Lambda_n} \Delta_{f_n}(x)$

$$\underset{\text{Prop 4.2.}}{\leq} \lambda \cdot K \cdot \sum_{x \notin \Lambda_n} ((1 + \lambda \varepsilon) \mathbb{1} - \lambda \Gamma)^{-1} \Delta_g(x) \cdot \circledast$$

. Since $\Delta_g \in \ell^1(\Lambda)$ (by $|||g||| = \|\Delta_g\|_{\ell^1(\Lambda)}$ and $g \in D \Leftrightarrow |||g||| < \infty$)

$\Rightarrow \lim_{n \to \infty} \circledast = 0$. Thus, $g_n \to g$ as $n \to \infty$.

Therefore, $\quad g \in \overline{\mathcal{R}(\mathbb{1} - \lambda \mathcal{L})} \Rightarrow D \subset \overline{\mathcal{R}(\mathbb{1} - \lambda \mathcal{L})}$.

. But D is dense in $C(S) \Rightarrow \mathcal{R}(\mathbb{1} - \lambda \mathcal{L})$ is also dense in $C(S)$.

Therefore, by Prop. 2.24 ⓓ, $\mathcal{R}(\mathbb{1} - \lambda \overline{\mathcal{L}}) = C(S)$ too, which is Property ⓒ of Def. 2.6 for $\overline{\mathcal{L}}$.

. Finally, the inequality in Prop. 3.2. rewrites,

$$\Delta_{(\mathbb{1} - \lambda \mathcal{L})^{-1} g} \leq ((1 + \lambda \varepsilon) \mathbb{1} - \lambda \Gamma)^{-1} \Delta g$$

$$\Rightarrow \Delta_{(\mathbb{1} - \frac{t}{n} \mathcal{L})^{-n} g} \leq \left((1 + \tfrac{t}{n} \varepsilon) \mathbb{1} - \tfrac{t}{n} \Gamma \right)^{-n} \Delta g$$

Then by Thm 2.10 and taking $n \to \infty$ we get

$$\Delta_{T(t) g} \leq e^{-\varepsilon t} e^{t \Gamma} \Delta g, \quad \forall g \in C(S). \quad \#$$

3.2.2) Ergodicity of spin systems

. One important question for Stoch. Processes is whether it converges to a stationary state regardless of the initial state. This property is known as ergodicity.

Def. 3.4) The spin system η_t with semigroup $T(t)$ is said to be <u>ergodic</u>, if its stationary distribution μ is unique, and

$$\lim_{t\to\infty} T(t)f(\eta) = \int_S f\,d\mu ,$$

$\forall\, \eta$ and $f \in C(S)$, i.e., the spin system is ergodic if

$$\nu T(t) \Rightarrow \mu$$

$\forall$ probability measure ν.

. <u>Recall</u>:
- . $\varepsilon = \inf_{\substack{x\in\Lambda \\ \eta\in S}} [c(x,\eta) + c(x,\eta^x)]$; $\gamma(x,u) = \sup_{\eta\in S} |c(x,\eta^u) - c(x,\eta)|$
- . $M := \sup_{x\in\Lambda} \sum_{u:x\neq u} \gamma(x,u) < \infty$.

. The next is a generic statement over ergodicity. It is not so "powerful" as Thm 3.3 as the issue of ergodicity is in special cases subtle and it is not expected to be resolved by a general thm.

Thm. 3.5) If $M < \varepsilon$, then η_t is ergodic.

. <u>Proof</u>: Consider $\mu \in I$ (set of invariant measures) and let ν be any probability measure on $S=\{0,1\}^\Lambda$. Then, $\forall f \in D$,

$$(*)\quad \left|\int_S f\,d\mu - \int_S f\,d(\nu T(t))\right| = \left|\int [T(t)f(\eta) - T(t)f(\zeta)]\,d(\mu\otimes\nu)\right.$$

$\Rightarrow$ We need to bound $|T(t)f(\eta) - T(t)f(\zeta)|$ uniformly in η, ζ.

- For that purpose, let $\eta, \zeta \in S'$ ~~be~~ given. Then there are (infinite) sequences $\eta^{(n)} \in S'$ and $x_n \in \Lambda$ distincts s.t. $\eta^{(0)} = \eta$, $\eta^{(n+1)} = \eta^{(n), x_n}$ and $\zeta = \lim_{n\to\infty} \eta^{(n)}$.

Thus, for $f \in C(S')$,

$$
\begin{aligned}
|f(\eta) - f(\zeta)| &= \Big|\lim_{n\to\infty} \big(f(\eta^{(0)}) - f(\eta^{(n)})\big)\Big| \\
&= \Big|\lim_{n\to\infty} \sum_{k=1}^{n} \big[f(\eta^{(k-1)}) - f(\eta^{(k)})\big]\Big| \\
&\leq \sum_{k=1}^{\infty} |f(\eta^{(k-1)}) - f(\eta^{(k)})| \\
&\leq \sum_{x\in\Lambda} \Delta_f(x).
\end{aligned}
$$

$$\Rightarrow \sup_{\eta,\zeta\in S'} |f(\eta) - f(\zeta)| \leq \underbrace{\sum_{x\in\Lambda} \Delta_f(x)}_{= \|\Delta_f\|_{\ell^1(\Lambda)}} \overset{\text{def}}{=} |||f|||.$$

Thus, $\sup_{\eta,\zeta\in S'} |T(t)f(\eta) - T(t)f(\zeta)| \leq |||T(t)f||| \overset{\text{Thm 3.3}}{\leq} e^{(M-\varepsilon)t} |||f|||$, $f \in D$.

Plugging this in (*) we obtain

$$\Big|\int_{S'} f\,d\mu - \int_{S'} f\,d(\nu T(t))\Big| \leq e^{(M-\varepsilon)t} |||f||| \xrightarrow[t\to\infty]{} 0$$

as $M - \varepsilon < 0$ (and for $f \in D$).

As D is dense in $C(S')$, $\nu T(t) \Rightarrow \mu$.

$\Rightarrow$ Convergence and uniqueness holds at once. #

Example for the contact process.

- Let Λ be a graph of bounded degree.

Notation: "$x \sim y$" $\Leftrightarrow$ x and y are neighboring vertices of Λ.

- The contact process with parameter $\varrho > 0$ has flip rates

$$c(x,\eta) = \begin{cases} 1 & , \text{ if } \eta(x) = 1, \\ \varrho \, \#\{y \sim x : \eta(y) = 1\} & , \text{ if } \eta(x) = 0. \end{cases}$$

This means that the spin at x flips from 1 to 0 at rate 1, and from 0 to 1 at rate proportional to the number of neighbors with spin value 1.

- Clearly, δ_0, the pointmass on $\eta \equiv 0$, is stationary.

Q.: Are there any other?

- In this case:
  - $\varepsilon = 1$ (attained e.g. if $\eta(x) = 1$ and $\eta(y) = 0$ for all $y \sim x$)
  - $M = \varrho \cdot \max_{x \in \Lambda} \deg(x)$ (indeed, $\gamma(x,u) \equiv 0$ if $u \nsim x$ and also if $\eta(x) = 1$.

$$\Rightarrow \sup_{x \in \Lambda} \sum_{u : x \neq u} \gamma(x,u) = \sup_x \sum_{u \sim x} \underbrace{\sup_\eta |c(x,\eta^u) - c(x,\eta)|}_{= \varrho}$$

$\Rightarrow$ By Theorem 3.3 the process is well-defined (since $M < \infty$) and by Theorem 3.5, $I = \{\delta_0\}$ if $\varrho < \frac{1}{\max_{x \in \Lambda} \deg(x)} \equiv \varrho_c$.

Rem.: What happens for $\varrho > \varrho_c$ is not answered by Thm 3.5.

Example for the noisy voter model.

- Let $p(x,y)$ be the transition probabilities for a discrete time Markov chain on Λ with $p(x,x) = 0$, $\forall x \in \Lambda$.
- Let $\beta, \delta \geq 0$. The noisy voter model has spin flip rates

$$c(x,\eta) = \sum_{y : \eta(y) \neq \eta(x)} p(x,y) + \begin{cases} \beta & , \text{ if } \eta(x) = 0, \\ \delta & , \text{ if } \eta(x) = 1. \end{cases}$$

Q.: Is this model well-defined? Is it ergodic?

. let us compute ε and M.

. If $\eta(x)=0$; $c(x,\eta)=\beta+\sum_{y:\eta(y)=1} p(x,y)$ and

$$c(x,\eta^x)=\delta+\sum_{y:\eta(y)=0} p(x,y)$$

$$\Rightarrow c(x,\eta)+c(x,\eta^x)=\beta+\delta+\underbrace{\sum_y p(x,y)}_{(\text{Since } p(x,x)=0)}=\beta+\delta+1.$$

(Similarly, if $\eta(x)=1$)

$$\Rightarrow \boxed{\varepsilon=\beta+\delta+1}.$$

. $\gamma(x,u)=\sup_\eta |c(x,\eta^u)-c(x,\eta)|$:

If $\eta(x)=0$: ($u\neq x$) $c(x,\eta)=\beta+\sum_{y:\eta(y)=1} p(x,y)$; $c(x,\eta^u)=\beta+\sum_{\substack{y:\eta(y)=1\\ y\neq u}} p(x,y)$

$$+\, p(x,u)\,\delta_{\eta(u),0}$$

$$\Rightarrow |c(x,\eta^u)-c(x,\eta)|=p(x,u)\underbrace{|\delta_{\eta(u),1}-\delta_{\eta(u),0}|}_{=1}$$

(Similarly for $\eta(x)=1$)

$$\Rightarrow \gamma(x,u)=p(x,u) \Rightarrow \boxed{M=\sup_{x\in\Lambda}\sum_{u:u\neq x}\gamma(x,u)=1.}$$

. Therefore, by Thm 3.3 the process is well defined.

. By Thm 3.5, if $\delta+\beta>0$ ($\Leftrightarrow \varepsilon>M$), the spin system is ergodic.

Q.: Is $\beta+\delta>0$ necessary for ergodicity?

A: Yes; Indeed, if $\beta=\delta=0\Rightarrow$ $\nu=\delta_0$ and $\nu=\delta_1$ are both invariant measures !

3.2.3) Coupling of spin systems and attractiveness

- A coupling of two stochastic processes is a construction of the two processes on a common probability space (having marginals the two original processes).

- Clearly one can consider just the product space, but this does not give any useful insights. To be useful one has to choose the coupling in a clever way!
- First some definitions.

Def. 3.6)
- For $\eta, \zeta \in \{0,1\}^{\Lambda} \equiv S$, we say that $\eta \leq \zeta$ if $\eta(x) \leq \zeta(x)$ for all $x \in \Lambda$.
- A function $f \in C(S)$ is <u>increasing</u> if $f(\eta) \leq f(\zeta)$ whenever $\eta \leq \zeta$. The set of increasing functions is denoted by $\mathcal{M}$.
- Two probability measures are <u>stochastically ordered</u>, $\mu \leq \nu$, if $\int_S f \, d\mu \leq \int_S f \, d\nu, \quad \forall f \in \mathcal{M}$.

Thm. 3.7) Consider η_t and ζ_t spin systems with flip rates $c_1(x,\eta)$ and $c_2(x,\zeta)$ resp..

If $\eta \leq \zeta$ implies that

$$c_1(x,\eta) \leq c_2(x,\zeta) \quad \text{for } \eta(x) = \zeta(x) = 0,$$
$$c_1(x,\eta) \geq c_2(x,\zeta) \quad \text{for } \eta(x) = \zeta(x) = 1,$$

then $\exists$ coupling (η_t, ζ_t) for initial configurations satisfying $\eta \leq \zeta$ s.t.

$$\mathbb{P}^{(\eta,\zeta)}(\eta_t \leq \zeta_t \text{ for all } t \geq 0) = 1.$$

Proof: We need to specify the transition rates for $(\eta(x), \zeta(x))$ for each x if $\eta \le \zeta$.

They are the following:

$$(0,0) \longrightarrow \begin{cases} (1,1) & \text{at rate } c_1(x,\eta), \quad (a) \\ (0,1) & \text{at rate } c_2(x,\zeta) - c_1(x,\eta), \quad (b) \end{cases}$$

$$(0,1) \longrightarrow \begin{cases} (0,0) & \text{at rate } c_2(x,\zeta), \quad (c) \\ (1,1) & \text{at rate } c_1(x,\eta), \quad (d) \end{cases}$$

$$(1,1) \longrightarrow \begin{cases} (0,0) & \text{at rate } c_2(x,\zeta), \quad (e) \\ (0,1) & \text{at rate } c_1(x,\eta) - c_2(x,\zeta). \quad (f) \end{cases}$$

- By assumptions $\Rightarrow$ all these rates are positive.
- **Rem.:** The marginals are the given by:

For η:
- $0 \to 1$: rate $c_1(x,\eta)$ ((a) and (d)).
- $1 \to 0$: rate $c_1(x,\eta)$ ((e) + (f)).

For ζ:
- $0 \to 1$: rate $c_2(x,\zeta)$ ((a) + (b)).
- $1 \to 0$: rate $c_2(x,\zeta)$ ((c) and (e)).

$\Rightarrow$ The process (η_t, ζ_t) is a Feller process on $\{(0,0), (0,1), (1,1)\}^{\Lambda}$. A slight modification of the proof of Thm 3.3 gives the existence of the process. #

7.1.'14

- If the two marginal processes, i.e., if $c_1 \equiv c_2$, then the assumptions of Thm 3.7 are the ones for an attractive spin system.

Def. 3.8) A spin system with rates $c(x,\eta)$ is attractive if

$$\begin{cases} c(x,\eta) \le c(x,\zeta) & \text{for } \eta(x)=\zeta(x)=0, \\ c(x,\eta) \ge c(x,\zeta) & \text{for } \eta(x)=\zeta(x)=1, \end{cases}$$

whenever $\eta \le \zeta$.

As a corollary of Thm 3.7 we have:

Cor. 3.9) For an attractive spin system, $\exists$ coupling (η_t, ζ_t) of two copies of the process for initial configurations satisfying $\eta \le \zeta$ s.t.

$$\mathbb{P}^{(\eta,\zeta)}(\eta_t \le \zeta_t \text{ for all } t \ge 0) = 1.$$

. Q.: Is an attractive spin system ergodic?

: First a Lemma.

Lemma 3.10) Let $T(t)$ be the semigroup of an attractive spin system. Then:

ⓐ $f \in \mathcal{M}$ implies $T(t)f \in \mathcal{M}$,

ⓑ $\mu \le \nu$ implies $\mu T(t) \le \nu T(t)$.

. Proof (exercises).

. To (try to) answer to the above question, consider as initial states $(\eta \equiv 0, \zeta \equiv 1)$. The rates in the proof of Thm 3.7 which introduce/remove discrepancies in (η,ζ) are:

$(0,0) \to (0,1)$	: $c(x,\zeta) - c(x,\eta)$	$(0,1) \to (0,0)$	: $c(x,\zeta)$
$(1,1) \to (0,1)$	: $c(x,\eta) - c(x,\zeta)$	$(0,1) \to (1,1)$	: $c(x,\eta)$

. One sees that the coupling tends to remove discrepancies, since the rate $(0,0) \to (0,1)$ is smaller than the rate $(0,1) \to (0,0)$ (similarly for $(1,1)$). This is the reason of the name "attractive": two ordered initial conditions tends to become the same by local spin flips.

. Thm 3.11) Let $T(t)$ be the semigroup of an attractive spin system. Then,

ⓐ $\delta_0 T(s) \le \delta_0 T(t)$ and $\delta_1 T(s) \ge \delta_1 T(t)$ for $s \le t$,

ⓑ $\underline{\nu} \equiv \lim_{t\to\infty} \delta_0 T(t)$ and $\bar{\nu} \equiv \lim_{t\to\infty} \delta_1 T(t)$ exists and are stationary,

ⓒ $\delta_0 T(t) \le \mu T(t) \le \delta_1 T(t)$, $\forall \mu$.

ⓓ Any weak limit ν of $\mu T(t)$ along a sequence of times tending to ∞ satisfies $\underline{\nu} \le \nu \le \bar{\nu}$.

Proof: (Exercises).

Corollary 3.12) An attractive spin system is ergodic $\iff \underline{\nu} = \bar{\nu}$.

. Remark: Stochastic ordering is a quite strong relation. In particular, it implies equality of joint distribution from equality of one-point distributions!

Lemma 3.13) If $\mu_1 \le \mu_2$ are two probability measures on $S = \{0,1\}^{\Lambda}$ with $\mu_1\{\eta : \eta(x)=1\} = \mu_2\{\eta : \eta(x)=1\}$ for all $x \in \Lambda$, then $\mu_1 = \mu_2$.

Proof: We are going to show that $\forall A \subset \Lambda$,

$$\mu_1\{\eta : \eta(x)=1 \text{ for all } x\in A\} = \mu_2\{\eta : \eta(x)=1 \text{ for all } x\in A\}. \quad (***)$$

Assume this to be true for all A with $|A|=n$ (true for $n=1$).
Let $\tilde{A}$ with $|\tilde{A}|=n+1$.
Then, $\mu_1\{\eta : \eta(x)=1, x\in\tilde{A}\} \equiv \int \mathbb{1}_{\tilde{A}}(\eta)\, d\mu_1(\eta)$

$$\overset{(*)}{\leq} \int \mathbb{1}_{\tilde{A}}(\eta)\, d\mu_2(\eta) = \mu_2\{\eta : \eta(x)=1, x\in\tilde{A}\}$$

since $\mu_1 \leq \mu_2$ and $f=\mathbb{1}_{\tilde{A}}$ is an increasing function

Further, $\mathbb{1}_{\tilde{A}}(\eta) = \prod_{x\in\tilde{A}} \mathbb{1}_x(\eta) = \prod_{x\in\tilde{A}} [\underbrace{(\mathbb{1}_x(\eta)-1)}_{:=\mathbb{1}_x^c(\eta)}+1]$

$$= \sum_{\substack{B\subset\tilde{A}\\ |B|\leq n}} \mathbb{1}_B^c(\eta) + \mathbb{1}_{\tilde{A}}^c(\eta). \quad (**)$$

Assume that for some $\tilde{A}$ with $|\tilde{A}|=n+1$, the inequality $(*)$ is strict. Then, by $(**)$ and the fact that for $|B|\leq n$ we have equalities,

$$\int \mathbb{1}_{\tilde{A}}(\eta)\,d\mu_1(\eta) < \int \mathbb{1}_{\tilde{A}}(\eta)\,d\mu_2(\eta)$$

$$\overset{(**)}{\|} \qquad\qquad \overset{(**)}{\|}$$

$$\underbrace{\sum_{\substack{B\subset\tilde{A}\\ |B|\leq n}} \int \mathbb{1}_B^c(\eta)\,d\mu_1(\eta) + \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_1(\eta) \;\Big|\; \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_2(\eta) + \sum_{\substack{B\subset\tilde{A}\\ |B|\leq n}} \int \mathbb{1}_B^c(\eta)\,d\mu_2(\eta)}_{= \text{ by induction}}$$

$$\Rightarrow \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_1(\eta) < \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_2(\eta)$$

But $g=-\mathbb{1}_{\tilde{A}}^c$ is increasing (?) (Not complete proof).

$$\Rightarrow \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_1(\eta) \geq \int \mathbb{1}_{\tilde{A}}^c(\eta)\,d\mu_2(\eta),$$ which is a contradiction.

$\Rightarrow (***)$ holds for $n+1$ too. By induction it holds for all A.

By inclusion-exclusion we have equality on all cylinder sets and by Charathéodory extension theorem we finish the proof. #

3.2.4) Application of coupling and monotonicity

- In the exercises you will see how to prove ergodicity for the noisy voter model using the monotonicity statements of Thm 3.11.

- Here we show that the one-dimensional stochastic Ising model is also ergodic for all $\beta \geq 0$.

Thm 3.14) The one-dimensional stochastic Ising model, in which $\Lambda = \mathbb{Z}$, $\beta \geq 0$ is a parameter, and

$$c(x,\eta) = \exp\Big(-\beta \sum_{y: y\sim x} (2\eta(x)-1)(2\eta(y)-1)\Big),$$

is ergodic for all $\beta \geq 0$.

Proof: The idea is to compare the original dynamics with the one where outside a region of radius m everything is frozen. We will have an inequality of the stationary measure and $\bar{\nu}$ (resp. $\underline{\nu}$) and then we take $m \to \infty$.

- Denote $K = \sup_{x,\eta} c(x,\eta) \ (= e^{2\beta})$. For $m \in \mathbb{N}$, define

$$c_m(x,\eta) = \begin{cases} c(x,\eta), & \text{if } |x| < m, \\ 0, & \text{if } |x| \geq m \text{ and } \eta(x) = 1, \\ K, & \text{if } |x| \geq m \text{ and } \eta(x) = 0. \end{cases}$$

$$\Rightarrow \left.\begin{cases} \text{for } \eta(x) = \zeta(x) = 0, & c(x,\eta) \leq c_m(x,\eta) \\ \text{for } \eta(x) = \zeta(x) = 1, & c(x,\eta) \geq c_m(x,\eta) \end{cases}\right\} \text{which are the requirements of Theorem 3.7.}$$

- let $\eta_0 = \zeta_0 \equiv 1$ and η_t (resp. ζ_t) be spin systems with flip rates $c(x,\eta)$ (resp. $c(x,\zeta)$).

By Thm 3.7 there exists a coupling s.t. $\eta_t \le \zeta_t$ for all $t \ge 0$.

. Now, the process ζ_t restricted to $\Lambda_m = \{-m+1, \dots, m-1\}$ is an irreducible Markov Chain on $\{0,1\}^{\Lambda_m}$.

$\Rightarrow$ It converges weakly to its stationary distribution, μ_m.

$\Rightarrow$ $\boxed{\bar{\nu} \le \mu_m.}$ $\leftrightarrow$ $\left\{\begin{array}{l}\text{In the sense that } \bar{\nu} \text{ is projected onto } \{0,1\}^{\Lambda_m} \\ \text{or } \mu_m \text{ defined on } \{0,1\}^{\mathbb{Z}} \text{ by setting} \\ \mu_m\{\eta : \eta(x) = 1\} = 1 \text{ for } |x| \ge m.\end{array}\right.$

. The next task is to identify μ_m (it will not be product measure).

. Let μ the measure on $\{0,1\}^{\mathbb{Z}}$ given by a two-sided discrete time Markov chain with transition matrix

$$\mathbb{P} = \frac{1}{e^{\beta} + e^{-\beta}} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}, \text{ i.e.,}$$

$$\mu\{\eta : \eta(x) = \zeta(x), k \le x \le l\} = \frac{1}{2} \prod_{n=k}^{l-1} p(\zeta(n), \zeta(n+1)), \text{ for } k \le l, \zeta \in S'.$$

. One can verify that the conditional measure

$$\mu(\cdot \mid \eta(x) = 1 \text{ for all } |x| \ge m)$$

is reversible for the chain ζ_t (easy) $\Rightarrow$ it is stationary

which means that $\mu_m = \mu(\cdot \mid \eta(x) = 1 \text{ for all } |x| \ge m)$.

. The next point is to verify that $\mu = \lim_{m \to \infty} \mu_m$.

The finite-dim. distribution of μ_m are given by:

$$\mu_m\{\eta : \eta(x) = \zeta(x) \text{ for } k \le x \le l\} = \frac{(\mathbb{P}^{k+m})(1, \zeta(k)) \cdot \prod_{n=k}^{l-1} P(\zeta(n), \zeta(n+1)) (\mathbb{P}^{m-l})(\zeta(l), 1)}{(\mathbb{P}^{2m})(1,1)}$$

for $-m < k < l < m$.

• Now, $\lim_{n\to\infty} (P^n)(u,v) = \mu(\eta : \eta(x) = v) = \frac{1}{2}$, $\forall\, u,v \in \{0,1\}$,
by the convergence for finite state Markov chains.

Thus $$\frac{(P^{k+m})(1,\zeta(k))\,(P^{m-\ell})(\zeta(\ell),1)}{(P^{2m})(1,1)} \xrightarrow[m\to\infty]{} \frac{1}{2}, \text{ i.e., } \mu = \mu_m .$$

. We have proven that $\bar{\nu} \le \mu$.

. By exchanging the roles of 0's and 1's one similarly obtains $\underline{\nu} \ge \mu$.

But $\underline{\nu} \le \bar{\nu} \Rightarrow \mu \le \underline{\nu} \le \bar{\nu} \le \mu$, i.e. $\underline{\nu} = \bar{\nu} = \mu$
and by Corollary 3.12 we conclude that the stochastic Ising model in one dimension is ergodic for any $\beta \ge 0$. #

Corollary 3.15) The unique stationary distribution of the 1-d stochastic Ising model with $\beta \ge 0$ is the measure μ with finite-dim. distributions given by $(\star)$.

13.1.14

3.2.5) Correlation inequalities.

. Attractiveness of spin systems is closely connected with positivity of correlations.

. **Def. 3.16)** A probability measure μ on S' is said to have positive correlations if

$$\int_{S'} f g \, d\mu \geq \int_{S'} f \, d\mu \int_{S'} g \, d\mu \quad , \forall f, g \in \mathcal{M}.$$

. Rem.: The above formula can be rewritten as

$$Cov_{\mu}(f, g) \equiv \mathbb{E}_{\mu}(f \cdot g) - \mathbb{E}_{\mu}(f) \mathbb{E}_{\mu}(g) \geq 0 \quad , \forall f, g \in \mathcal{M}.$$

. Here is a result that can be used to verify that $\mu_1 \leq \mu_2$.

. **Thm. 3.17)** . Let $S' = \{0,1\}^{\Lambda}$, $|\Lambda| < \infty$, For $\eta, \zeta \in S'$, define $\eta \vee \zeta$ and $\eta \wedge \zeta$ by

$$(\eta \vee \zeta)(x) = \max\{\eta(x), \zeta(x)\},$$
$$(\eta \wedge \zeta)(x) = \min\{\eta(x), \zeta(x)\}.$$

. Assume that μ_1 and μ_2 are probability measures on S' assigning a strictly positive probability to each point of S'.

. If

$$\mu_1(\eta \wedge \zeta)\, \mu_2(\eta \vee \zeta) \geq \mu_1(\eta)\, \mu_2(\zeta),$$

$\forall \eta, \zeta \in S'$, then $\mu_1 \leq \mu_2$.

. Proof.: Idea: Define Markov chain (η_t, ζ_t) on $\{(\eta, \zeta) \in S' \times S' \mid \eta \leq \zeta\}$ with the properties

ⓐ η_t is irreducible with stationary measure μ_1,

ⓑ ζ_t is irreducible " " μ_2.

. Assume this to be made. Then, for $f \in \mathcal{M}$,

$$\mathbb{E}^{\eta}(f(\eta_t)) \le \mathbb{E}^{\zeta}(f(\zeta_t)) \text{ whenever } \eta \le \zeta.$$

Taking then $t \to \infty$ one obtains

$$\int_S f\, d\mu_1 \le \int_S f\, d\mu_2 \;, \forall f \in \mathcal{M}, \text{ i.e., } \mu_1 \le \mu_2.$$

. Now we need to construct the desired coupling.

. <u>Marginal processes:</u>

. <u>For η_t:</u>
$$\begin{cases} \cdot\ \eta \to \eta^x \text{ at rate } 1 \quad \text{if } \eta(x)=0, \\ \cdot\ \eta \to \eta^x \text{ at rate } \dfrac{\mu_1(\eta^x)}{\mu_1(\eta)} \quad \text{if } \eta(x)=1. \end{cases}$$

. <u>For ζ_t:</u>
$$\begin{cases} \cdot\ \zeta \to \zeta^x \text{ at rate } 1 \quad \text{if } \zeta(x)=0, \\ \cdot\ \zeta \to \zeta^x \text{ at rate } \dfrac{\mu_2(\zeta^x)}{\mu_2(\zeta)} \quad \text{if } \zeta(x)=1. \end{cases}$$

. The two chains satisfy detailed-balance (i.e., they are reversible with respect to μ_1 and μ_2 respectively)
$\Rightarrow \mu_1$ and μ_2 are the stationary measures for the two chains.

[Need to verify: $\mu_k(\eta) \cdot c(x,\eta) = \mu_k(\eta^x)\, c(x,\eta^x)$].

. <u>Coupling:</u> For $\eta \le \zeta$:

$$\begin{cases} \cdot\ (\eta,\zeta) \to (\eta^x,\zeta) \text{ at rate } 1, \text{ if } \eta(x)=0, \zeta(x)=1, \\ \cdot\ (\eta,\zeta) \to (\eta,\zeta^x) \text{ at rate } \dfrac{\mu_2(\zeta^x)}{\mu_2(\zeta)}, \text{ if } \eta(x)=0, \zeta(x)=1, \\ \cdot\ (\eta,\zeta) \to (\eta^x,\zeta^x) \text{ at rate } 1, \text{ if } \eta(x)=\zeta(x)=0, \\ \cdot\ (\eta,\zeta) \to (\eta^x,\zeta^x) \text{ at rate } \dfrac{\mu_2(\zeta^x)}{\mu_2(\zeta)}, \text{ if } \eta(x)=\zeta(x)=1, \\ \cdot\ (\eta,\zeta) \to (\eta^x,\zeta) \text{ at rate } \dfrac{\mu_1(\eta^x)}{\mu_1(\eta)} - \dfrac{\mu_2(\zeta^x)}{\mu_2(\zeta)}, \text{ if } \eta(x)=\zeta(x)=1. \end{cases}$$

. With other notations, the flip rate at x are:

$$\begin{cases} (0,1) \to (1,1) & \text{at rate } 1, \\ (0,1) \to (0,0) & \text{at rate } \frac{\mu_2(\zeta^x)}{\mu_2(\zeta)}, \\ (0,0) \to (1,1) & \text{at rate } 1, \\ (1,1) \to (0,0) & \text{at rate } \frac{\mu_2(\zeta^x)}{\mu_2(\zeta)}, \\ (1,1) \to (0,1) & \text{at rate } \frac{\mu_1(\eta^x)}{\mu_1(\eta)} - \frac{\mu_2(\zeta^x)}{\mu_2(\zeta)}. \end{cases}$$

. This has the right marginals.

. Further, it is well-defined if

$$\mu_1(\eta^x)\mu_2(\zeta) \geq \mu_1(\eta)\mu_2(\zeta^x) \quad (*)$$

whenever $\eta \leq \zeta$ and $\eta(x) = \zeta(x) = 1$.

. The assumption of the theorem for η, ζ^x writes:

$$\mu_1(\eta \wedge \zeta^x)\mu_2(\eta \vee \zeta^x) \geq \mu_1(\eta)\mu_2(\zeta^x).$$

But for $\eta(x) = \zeta(x) = 1$, $\zeta^x(x) = 0$ and using $\eta \leq \zeta$,

$$\Rightarrow \eta \wedge \zeta^x = \eta^x, \quad \eta \vee \zeta^x = \zeta, \text{ i.e., } (*) \text{ holds.}$$

#

. <u>Remark</u>: The assumption in Thm 3.17 is sufficient but not necessary: It is indeed a quite strong condition.

Cor. 3.18 (FKG inequalities) Assume that μ is a probability measure on S' which assigns a strict positive probability to each point of S'.

If μ satisfies

$$\mu(\eta \wedge \zeta)\mu(\eta \vee \zeta) \geq \mu(\eta)\mu(\zeta), \quad \forall \eta, \zeta \in S',$$

then μ has positive correlations.

Proof: let $f, g \in \mathcal{M}$. W.l.o.g. take $g > 0$ (as Def. 3.16 is not affected by a shift by a constant).

Define
$$\begin{cases} \mu_1 = \mu, \\ \mu_2(\eta) = \dfrac{g(\eta)\,\mu(\eta)}{\int_S g(\tilde{\eta})\, d\mu(\tilde{\eta})}. \end{cases}$$

Then, for $\eta, \zeta \in S$,

$$\mu_1(\eta \wedge \zeta)\,\mu_2(\eta \vee \zeta) = \frac{\mu(\eta\wedge\zeta)\, g(\eta\vee\zeta)\,\mu(\eta\vee\zeta)}{\int g\, d\mu}$$
$$\overset{\text{hyp.}}{\geq} \frac{\mu(\eta)\,\mu(\zeta)}{\int g\, d\mu} \cdot \underbrace{g(\eta\vee\zeta)}_{\underset{g\in\mathcal{M}}{\geq} g(\zeta)}$$
$$\geq \mu_1(\eta)\,\mu_2(\zeta).$$

$\Rightarrow$ By Thm 3.17, $\mu_1 \leq \mu_2$, i.e., $\forall f \in \mathcal{M}$, $\int f\, d\mu_1 \leq \int f\, d\mu_2$.

Explicitly, $\displaystyle \int f\, d\mu \leq \frac{\int f \cdot g\, d\mu}{\int g\, d\mu}$, which is qed #

Remark: In Thm 3.17 and Cor 3.18, Λ is finite. However, we can use them also for Λ countable. For that, it is enough to note that from the def. of the two properties ($\mu_1 \leq \mu_2$, μ positive correlations) it is enough to verify the properties for the projections on $\{0,1\}^T$ of the measures for all finite $T \subset \Lambda$.

Rem.: To verify the conditions of Cor. 3.18 one needs to have an expression for μ explicit enough. This is in general not available.

The following result is useful if that information is not available.

Thm 3.19) For a spin system η_t the following are equivalent:

ⓐ η_t is attractive,

ⓑ If μ has positive correlations, then so has $\mu T(t)$, $\forall t$;

Proof: ⓐ ⇒ ⓑ: For $f, g \in \mathcal{M} \cap D$,

$$\mathcal{L} fg - f \mathcal{L} g - g \mathcal{L} f \geq 0. \quad (*)$$

Indeed,
$$\mathcal{L} = \sum_{x \in \Lambda} c(x,\eta) \Big\{ (f(\eta^x) g(\eta^x) - f(\eta) g(\eta)) - f(\eta)(g(\eta^x) - g(\eta)) - g(\eta)(f(\eta^x) - g(\eta)) \Big\}$$

$$= \sum_{x \in \Lambda} c(x,\eta) \underbrace{(f(\eta^x) - f(\eta))(g(\eta^x) - g(\eta))}_{\text{Either both } \geq 0 \text{ or both } \leq 0.} \geq 0$$

Set $F(t) := T(t)(fg) - (T(t)f)(T(t)g)$.

Then,
$$\frac{d}{dt} F(t) \underset{\substack{\uparrow \\ \text{Thm 2.10 ⓑ} \\ \text{Thm 3.3}}}{=} \mathcal{L}[T(t)(fg)] - (T(t)f)(\mathcal{L} T(t) g) - (\mathcal{L} T(t) f)(T(t) g)$$

Now, by Lemma 3.10 ⓐ and Thm 3.3 $\Big\}$: $T(t)f,\ T(t)g \in \mathcal{M} \cap D$

$\Rightarrow$ By $(*)$: $\mathcal{L}(T(t)f \cdot T(t)g) \geq T(t)f \cdot \mathcal{L} T(t) g + T(t) g \cdot \mathcal{L} T(t) f$

$$\Rightarrow \frac{d}{dt} F(t) \geq \mathcal{L}\Big(T(t)(fg) - (T(t)f)(T(t)g)\Big)$$
$$= \mathcal{L} F(t).$$

. Thus, $\frac{d}{ds}\big(T(t-s)F(s)\big) = T(t-s)F'(s) - \overbrace{\mathcal{L}\, T(t-s)}^{\text{commutes by Thm 2.10 (b)}} F(s)$
$$= T(t-s)\left[F'(s) - \mathcal{L}F(s)\right] \geq 0.$$

$$\Rightarrow F(t) = T(t-s)F(s)\big|_{s=t} \geq T(t-s)F(s)\big|_{s=0} = T(t)F(0) = 0$$

$$\Rightarrow \boxed{T(t)(fg) \geq (T(t)f)(T(t)g). \quad (**)}$$

. As $\mathcal{M} \cap D$ is dense in $\mathcal{M}$, (**) holds also for all $f, g \in \mathcal{M}$.

. Now, assume that μ has positive correlations.
Then, for $f, g \in \mathcal{M}$,

$$\int fg \, d(\mu T(t)) = \int T(t)(fg)\, d\mu$$
$$\overset{(**)}{\geq} \int (T(t)f)(T(t)g)\, d\mu$$
$$\underset{\text{positive corr.}}{\geq} \left(\int T(t)f\, d\mu\right)\left(\int T(t)g\, d\mu\right)$$
$$= \left(\int f\, d(\mu T(t))\right)\left(\int g\, d(\mu T(t))\right).$$

$\Rightarrow$ Also $\mu T(t)$ has positive correlations.

(b) $\Rightarrow$ (a): See exercises.

#

. Rem: As a consequence, $f = \mathbb{1}_A$, $g = \mathbb{1}_B$, $A \cap B = \emptyset$
$\Rightarrow$ for η_t attractive with μ having positive correlations

$\Rightarrow \mathbb{P}$(at time t, $A \cup B$ is filled) $\geq \mathbb{P}$(at time t, A is filled) $\cdot \mathbb{P}$(at time t, B is filled)

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Remark: Every pointmass has positive correlations.

$\Rightarrow$ By Thm 3.19, for an attractive spin system, $\delta_\eta T(t)$ has positive correlations $\forall t \geq 0$, $\eta \in S'$.

Corollary 3.20) For an attractive spin system, $\bar{\nu}$ and $\underline{\nu}$ have positive correlations.

Proof: As $\bar{\nu} = \lim_{t\to\infty} \delta_1 T(t)$ and $\underline{\nu} = \lim_{t\to\infty} \delta_0 T(t)$ and $\delta_\eta T(t)$ has positive correlations $\forall t \geq 0 \Rightarrow$ also $\bar{\nu}$ and $\underline{\nu}$ have the same property. #

Prop. 3.21) Assume $\mu_1 \leq \mu_2$ and μ_1, μ_2 having positive correlations. Then, also
$$\mu_\lambda := \lambda \mu_1 + (1-\lambda)\mu_2, \quad \lambda \in [0,1]$$
has positive correlations.

Proof: let $f, g \in \mathcal{M}$. Denote $F_k = \int f \, d\mu_k$, $G_k = \int g \, d\mu_k$.

Then, $\int fg \, d\mu_\lambda - \left(\int f \, d\mu_\lambda\right)\left(\int g \, d\mu_\lambda\right)$

$$= \lambda \int fg \, d\mu_1 + (1-\lambda)\int fg \, d\mu_2 - \left(\int f d\mu_\lambda\right)\left(\int g d\mu_\lambda\right)$$

$$\geq \lambda F_1 \cdot G_1 + (1-\lambda) F_2 \cdot G_2 - \lambda^2 F_1 G_1 - (1-\lambda)^2 F_2 G_2 - \lambda(1-\lambda)\left(F_1 G_2 + F_2 G_1\right)$$

$$= \lambda(1-\lambda)(F_2 - F_1)(G_2 - G_1) \geq 0. \qquad \#$$

. Finally, one more result.

. Corollary 3.22) All product measures on $\{0,1\}^{\Lambda}$ have positive correlations.

. Proof.: . Any product measure on $\{0,1\}^{\Lambda}$ can be described by setting the one-point density, say $\rho(x) \in [0,1]$, $x \in \Lambda$.

. Consider the spin system with flip rates

$$c(x,\eta) = \begin{cases} \rho(x) & \text{if } \eta(x)=0, \\ 1-\rho(x) & \text{if } \eta(x)=1. \end{cases}$$

Then, by exercise 1, Sheet 8, η_t is ergodic.

. The unique stationary distribution is the product measure with density $\rho(x)$.

. Further, if $\eta(x)=\zeta(x)=0$, $c(x,\eta) = \rho(x) \overset{(\leq)}{=} c(x,\zeta)$
and if $\eta(x)=\zeta(x)=1$, $c(x,\eta) = 1-\rho(x) \overset{(\geq)}{=} c(x,\zeta)$, whenever $\eta \leq \zeta$.

$\Rightarrow$ The spin system is attractive.

. By Theorem 3.19 and ergodicity,

$$\delta_1 T(t)$$

has positive correlations for any $t \geq 0$.

. But since $\lim_{t\to\infty} \delta_1 T(t) = \mu$ (weakly) $\Rightarrow \mu$ has positive correlations.

#

3.3) The voter model

- Finally, we consider a precise model, the (linear) voter model and we will derive properties like the stationary measures.
- First we need to define the model and see that it is well-defined.

Def. 3.23)
- Let $q(x,y) \geq 0$ for all $x \neq y \in \Lambda$. The (linear) voter model η_t is the spin system with flip rates given by

$$c(x,\eta) = \sum_{y:\eta(y)\neq\eta(x)} q(x,y).$$

- Interpretations: ⓐ At each site $x \in \Lambda$, there is a voter that can have two opinions, 0 or 1. At rate $q(x,y)$ the voter at x adopts the opinion of the voter at y.

ⓑ Two species compete for territory. 0 and 1 represent the two species and, at rate $q(x,y)$, the species at y invades x.

Q: What shall we assume on $q(x,y)$ to be sure that the model is well-defined?

- Let us compute the coefficient M.

Remind that $M = \sup_{x\in\Lambda} \sum_{u:u\neq x} \gamma(x,u)$ where $\gamma(x,u) = \sup_{\eta\in S} |c(x,\eta^u) - c(x,\eta)|$.

For $x \neq u$: $\eta^u(x) = \eta(x)$

$$\Rightarrow \left\{ \begin{aligned} c(x,\eta) &= \sum_{\substack{y\neq u \\ \eta(y)\neq\eta(x)}} q(x,y) + q(x,u)\,\mathbb{1}_{\eta(u)\neq\eta(x)} \\ c(x,\eta^u) &= \sum_{\substack{y\neq u \\ \eta(y)\neq\eta(x)}} q(x,y) + q(x,u)\,\mathbb{1}_{\eta(u)=\eta(x)} \end{aligned} \right.$$

$$\Rightarrow \gamma(x,u) = q(x,u). \Rightarrow \boxed{M = \sup_{x\in\Lambda} \sum_{u:u\neq x} q(x,u).}$$

⇒ Assumption: $\boxed{M = \sup_{x \in \Lambda} \sum_{u: u \neq x} q(x,u) < \infty}$

. Remark: . This condition is exactly the sufficient condition in Corollary 1.18 ⓐ for the Markov chain with Q-matrix $q(x,y)$ to be nonexplosive !

. Q: Why?

A: . In this model, the "information" moves around as the Markov chain with Q-matrix $q(x,y)$.

. Now, the model will be well-defined if any finite region do not get information from ∞ in a finite time, i.e., the Markov chain has to be nonexplosive.

3.3.1) Duality in the (linear) voter model.

. First we introduce what will turn out to be the duality function.

. Let us define the function

$$H(\eta, A) := \prod_{x \in A} \eta(x) = \mathbb{1}_{\{\eta \equiv 1 \text{ on } A\}},$$

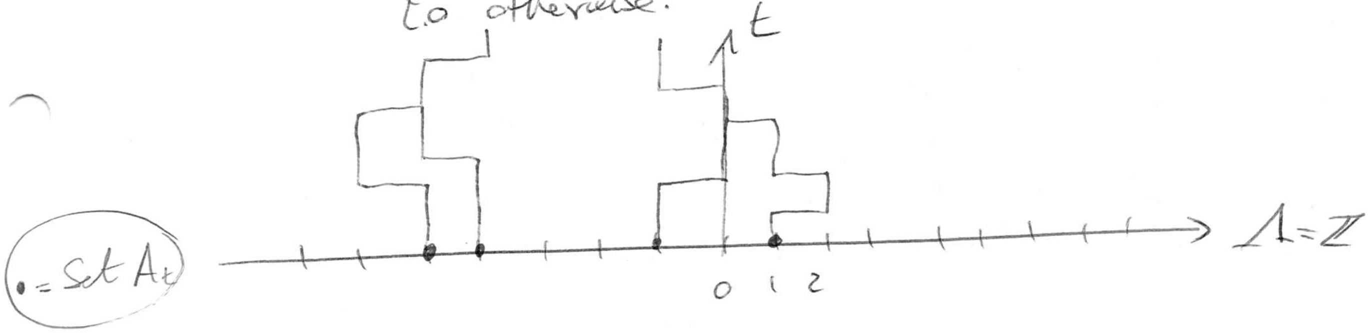
for any configuration $\eta \in \{0,1\}^{\Lambda}$ and finite subsets A of Λ.

. The dual process A_t is a collection of coalescing Markov chains with Q-matrix $q(x,y)$.
More precisely, the points in A_t move independently according to Markov chains with Q-matrices $q(x,y)$ until two chains meet. From that time, these chains coalesce and continues as the same Markov chain.

Formally: A_t is a Markov chain on the collection of finite subsets of Λ with Q-matrix:

$$\begin{cases} \tilde{q}(A, (A \setminus \{x\}) \cup \{y\}) = q(x,y) & \text{for } x \in A, y \notin A, \\ \tilde{q}(A, A \setminus \{x\}) = \sum\limits_{y \in A : y \neq x} q(x,y) & \text{for } x \in A. \end{cases}$$

Example: $q(x,y) = \begin{cases} \frac{1}{2} & \text{if } y = x \pm 1, \\ -1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$



Remark: • Clearly, $|A_t|$ is decreasing and if the Markov chain in background is recurrent, eventually one will have $|A_t| = 1$.

Lemma 3.24) The dual chain A_t is nonexplosive.

Proof:
• It holds: $c(A) = \sum\limits_{B \neq A} \tilde{q}(A,B) = \sum\limits_{x \in A} \sum\limits_{y : y \neq x} q(x,y) \leq M \cdot |A|$.

• Corollary 1.18 ⓐ can not be applied directly to the full chain as $\sup\limits_{A} c(A) = \infty$. However, it can be applied to $\{A : |A| \leq n\}$ for any $n \in \mathbb{N}$. Since $|A_t|$ is decreasing, $|A_t| \leq |A_0|$. So if we start from $|A_0| < \infty$, Corollary 1.18 ⓐ implies that the dual chain is nonexplosive.

#

. Now that the two processes are well-defined, we can state the duality result.

Theorem 3.25) The processes η_t and A_t are dual with respect to $H(\eta, A)$, i.e.,

$$\mathbb{E}^{\eta}\left(H(\eta_t, A)\right) = \mathbb{E}^{A}\left(H(\eta, A_t)\right), \ \forall \eta \in \{0,1\}^{\Lambda}, A \text{ finite subset of } \Lambda.$$

Proof: For the proof we can use the generator approach and apply Thm 2.36.

. Let $\mathcal{L}_1$ denote the generator of the voter model. Note that $H(\eta, A)$ depends on η only through its values in A, $\{\eta(x), x \in A\}$.

Thus,

$$\mathcal{L}_1 H(\eta, A) = \sum_{\substack{x \in A, y \in \Lambda \\ \eta(y) \neq \eta(x)}} q(x,y)\Big(\underbrace{H(\eta^x, A)}_{= \mathbb{1}_{\{\eta^x \equiv 1 \text{ on } A\}}} - \underbrace{H(\eta, A)}_{= \mathbb{1}_{\{\eta \equiv 1 \text{ on } A\}}}\Big)$$

$$= \sum_{\substack{x \in A, y \in \Lambda \\ \eta(y) \neq \eta(x)}} q(x,y) \underbrace{(1 - 2\eta(x))}_{\left(\equiv \begin{cases} 1 & \text{if } \eta(x) = 0, \\ -1 & \text{if } \eta(x) = 1. \end{cases}\right)} H(\eta, A \setminus \{x\})$$

$$= \sum_{x \in A, y \in \Lambda} q(x,y)\, H(\eta, A \setminus \{x\}) \cdot (\eta(y) - \eta(x))$$

$$= \sum_{x \in A, y \in \Lambda} q(x,y) \left(H(\eta, (A \setminus \{x\}) \cup \{y\}) - H(\eta, A) \right)$$

$$= \sum_{B \subseteq \Lambda} \tilde{q}(A,B) \left(H(\eta, B) - H(\eta, A) \right).$$

$$= \mathcal{L}_2 H(\eta, A)$$, where $\mathcal{L}_2$ is the generator of the Markov chain of the dual process.

The result then follows by Thm 2.36. #

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Rem.: The usefulness of duality in this case is also related with the fact that the dual process does not grow.

$\Rightarrow$ Indeed, the joint distribution of n coordinates of η_t depend only on the distributions of n coordinates at a time in the initial distribution.

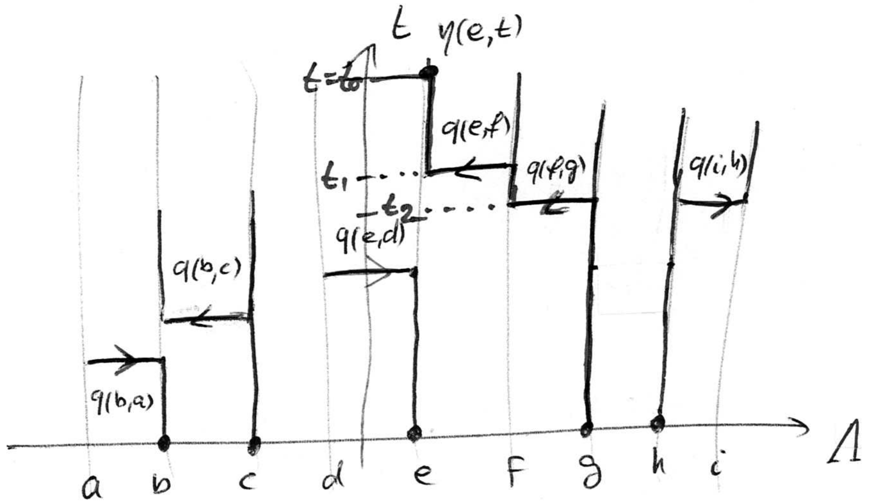
$$\mu T(t)(\{\eta \equiv 1 \text{ on } A\}) = \int_S \mathbb{E}^{\eta}(H(\eta_t, A))\,d\mu$$

$$\overset{\text{duality}}{=} \int_S \mathbb{E}^{A}(H(\eta, A_t))\,d\mu$$

$$= \int_S \sum_{B, |B| \leq |A|} \mathbb{P}^{A}(A_t = B)\, H(\eta, B)\,d\mu$$

$$= \sum_{B, |B| \leq |A|} \mathbb{P}^{A}(A_t = B) \cdot \mu(\{\eta \equiv 1 \text{ on } B\}).$$

3.3.2) Probabilistic interpretation of the duality.



$\bullet = \{\eta(x) \equiv 1\}$

- Look backwards in time $\eta(e,t) \Rightarrow \eta(e, t_0) = \eta(f, t_1) = \eta(g, t_2) = \eta(g, 0)$.
- In general, looking backwards in time, there is a random sequence (x_i, t_i) with $0 < t_n < \dots < t_1 < t$ s.t.

$\eta(x, t) = \eta(x_1, t_1) = \dots = \eta(x_n, t_n) = \eta(x_n, 0)$.

- $\boxed{\text{The path } (x, t), (x_1, t_1), \dots, (x_n, t_n) \text{ is a Markov chain } Y_x(t) \text{ starting at } x \text{ with Q-matrix } q'}$

. Further, two of these chains starting at different points are not independent; whenever they meet, they coalesce.

. By this construction, $\boxed{\eta_t(x) = \eta_0(Y_x(t))}$.

. The next goal is to use this to obtain results on the invariant measures.

Lemma 3.23) Let the initial distribution be the product measure ν_ρ with density ρ, i.e., having finite-dim. distributions

$$\nu_\rho(\{\eta \equiv 1 \text{ on } A\}) = \rho^{|A|}.$$

Then $\lim_{t\to\infty} \mathbb{P}(\eta_t(x) = \eta_t(y)) = 1 \Leftrightarrow \lim_{t\to\infty} \mathbb{P}(Y_x(t) \neq Y_y(t)) = 0$

Proof: . We have $\mathbb{P}(\eta_t(x) = 1) = \mathbb{P}(\eta_0(Y_x(t)) = 1) = \rho$.

Also, $\mathbb{P}(\eta_t(x) = \eta_t(y) = 1) = \mathbb{P}(\eta_0(Y_x(t)) = \eta_0(Y_y(t)) = 1)$

$Y_x(t), Y_y(t)$ indep. of ν_ρ

$$= \rho \cdot \mathbb{P}(Y_x(t) = Y_y(t)) + \rho^2 \mathbb{P}(Y_x(t) \neq Y_y(t)).$$

$\Rightarrow$ $\eta_t(x)$ and $\eta_t(y)$ becomes perfectly correlated in the $t\to\infty$ limit $\Leftrightarrow$ two indep. copies of the chain eventually meet with proba one. #

. Goal: Find all stationary measures.

. Setting: For simplicity, consider $\Lambda = \mathbb{Z}^d$, $d \in \mathbb{N}$, and q to be translation invariant: $q(x,y) = q(0, y-x)$ + irreducible.

. Rem.: One might guess from Lemma 3.23 that whenever the Markov chains will be recurrent, the only stationary measures will be δ_0 and δ_1, while in the recurrent case something else can happen.

Notations:

- In what follows we will denote by $X_1(t), X_2(t), \dots$ independent random walks with transition rates $q(x,y)$ starting at points $x_1, x_2, \dots$.
- Also, denote $Z(t) = X_1(t) - X_2(t)$.

3.3.3) Stationary measures — the recurrent case.

Thm 3.24) Assume that $Z(t)$ is recurrent.

ⓐ $\forall \eta \in \{0,1\}^\Lambda$ and $x_1, x_2 \in \Lambda$,
$$\lim_{t\to\infty} \mathbb{P}^\eta\left(\eta_t(x_1) \neq \eta_t(x_2)\right) = 0.$$

ⓑ $\mathcal{I}_e = \{\delta_0, \delta_1\}$.

ⓒ A necessary and sufficient condition on μ for
$$\lim_{t\to\infty} \mu T(t) = \lambda \delta_1 + (1-\lambda)\delta_0$$
is
$$\lim_{t\to\infty} \sum_{y\in\Lambda} p_t(x,y)\, \mu(\{\eta(y)=1\}) = \lambda, \quad \forall x \in \Lambda,$$
where $p_t(x,y)$ are the transition probabilities for $X_i(t)$

Proof: ⓐ let
$$\tau = \inf\{t>0 \mid X_1(t) = X_2(t)\} = \inf\{t>0 \mid Z(t) = 0\}.$$

Now, by duality,
$$\mathbb{P}^\eta\left(\eta_t(x_1) \neq \eta_t(x_2)\right) = \mathbb{P}\left(\eta(X_1(t)) \neq \eta(X_2(t)),\ t<\tau\right) \leq \mathbb{P}(\tau > t).$$

But since Z is recurrent, $\lim_{t\to\infty} \mathbb{P}(\tau > t) = 0$. ✓

(b) Let us consider a $\underline{\mu \in \mathcal{I}}$.

Then, $\mu(\{\eta(x_1) \neq \eta(x_2)\}) = \mu T(t)(\{\eta(x_1) \neq \eta(x_2)\})$

$$= \int_S \mathbb{P}^\eta(\eta_t(x_1) \neq \eta_t(x_2))\, d\mu$$

$$\xrightarrow[t\to\infty]{} 0 \text{ by (a).}$$

Therefore, η is constant a.s. with respect to μ.

(c) Take any initial distribution μ. Then,

$$\mu T(t)(\{\eta(x_1)=1\}) = \int_S \mathbb{P}^\eta(\eta_t(x_1)=1)\, d\mu \overset{\text{duality}}{=} \int_S \mathbb{P}(\eta(X_1(t))=1)\, d\mu$$

$$= \sum_{y\in\Lambda} p_t(x_1,y)\, \mu(\{\eta(y)=1\}). \quad (*)$$

Furthermore, take any finite $A \subset \Lambda$ with $x_1 \in A$,

$$\mu T(t)(\{\eta(x_1)=1\}) - \mu T(t)(\{\eta(x)=1 \text{ for all } x\in A\})$$

$$= \int_S \left[\mathbb{P}^\eta(\eta_t(x_1)=1) - \mathbb{P}^\eta(\eta_t(x)=1 \text{ for all } x\in A)\right] d\mu \quad (**)$$

$$\xrightarrow[t\to\infty]{} 0 \text{ by (a).}$$

. So, if $\lim_{t\to\infty} \mu T(t) = \lambda \cdot \delta_1 + (1-\lambda)\delta_0$, then by $(*)$

$$\lim_{t\to\infty} \sum_{y\in\Lambda} p_t(x_1,y)\,\mu(\{\eta(y)=1\}) = \lim_{t\to\infty} \mu T(t)(\{\eta(x_1)=1\}) = \lambda.$$

. On the other hand, if $\lim_{t\to\infty} \sum_{y\in\Lambda} p_t(x,y)\,\mu(\{\eta(y)=1\}) = \lambda,\ \forall x\in\Lambda$,

then by $(**)$, $\lim_{t\to\infty} \mu T(t)(\{\eta(x)=1 \text{ for all } x\in A\}) = \lim_{t\to\infty} \mu T(t)(\{\eta(x_1)=1\})$

$$\overset{(*)}{=} \lim_{t\to\infty} \sum_{y\in\Lambda} p_t(x_1,y)\,\mu(\{\eta(y)=1\}) = \lambda, \quad \forall A\subset\Lambda,$$

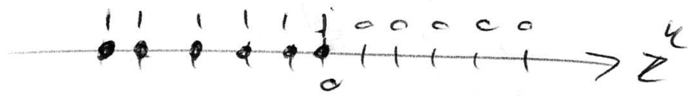
$\Rightarrow \lim_{t\to\infty} \mu T(t) = \lambda\,\delta_1 + (1-\lambda)\,\delta_0.$

#

Remark: Naively, from Thm 3.24 ⓐ, one might think that $\eta_t(x)$ changes only finitely many times $\forall x$. This is not always the case.

Example: $\Lambda = \mathbb{Z}$, $q(x,y) = 1$ for $|x-y| = 1$, $q(x,y) = 0$ if $|x-y| > 1$.

Consider the initial configuration:



$\Rightarrow$ At time t, η_t has the same form as η_0 except that the right-most "1" is at a random location given by a simple symmetric random walk.

This being recurrent $\Rightarrow \eta_t(x)$ changes ∞-often.

3.3.4) Stationary measures – the transient case.

- In this section we assume that the random walk $Z(t)$ is transient.

- Here is one non-trivial one-parameter family of stationary distributions.

  – Let us start with product measure ν_ρ, $\rho \in (0,1)$ fixed.

  $\Rightarrow$ For any finite $A \subset \Lambda$,

$$\nu_\rho T(t)(\eta \equiv 1 \text{ on } A) = \int_S \mathbb{P}^\eta(\eta_t \equiv 1 \text{ on } A)\, d\nu_\rho$$

$$\overset{\text{duality}}{=} \int_S \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t)\, d\nu_\rho \qquad (*)$$

$$= \sum_B \mathbb{P}^A(A_t = B) \underbrace{\int_S \mathbb{1}_{(\eta \equiv 1 \text{ on } B)}\, d\nu_\rho}_{= \rho^{|B|}}$$

$$= \mathbb{E}^A(\rho^{|A_t|}).$$

- Since $|A_t|$ is nonincreasing in t, its limit exists (an unlike in the recurrent case it does not have to be 1).

- We denote by $|A_\infty|$ the limit of $|A_t|$ (although A_∞ as a set does not exists).

- Therefore, taking $t\to\infty$ in ⊛ we get a measure μ_s,

$$\mu_s = \lim_{t\to\infty} \nu_s T(t),$$

which is stationary by Lemma 2.34. It satisfies,

$$\mu_s(\eta \equiv 1 \text{ on } A) = \mathbb{E}^A\left(s^{|A_\infty|}\right).$$

- Next we would like to derive:

  - → properties of μ_s
  - → see that all stationary measures can be expressed in terms of μ_s.

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- One important ingredient is the Green function for $Z(t)$:

$$G(x,y) := \int_0^\infty \mathbb{P}^x(Z(t)=y)\,dt \equiv \mathbb{E}\left(\text{Time spent at } y \text{ by } Z(t) \text{ starting at } x\right).$$

- $G(x,y) < \infty$ since $Z(t)$ is transient.

- For any $A \subset \Lambda$, $g(A) := \mathbb{P}^A(|A_t| < |A| \text{ for some } t>0)$

Here are two preliminary results.

Lemma 3.25) ⓐ If $A \subseteq B$, then $g(A) \le g(B)$.

ⓑ If $|A| \geq 2$, then $g(A) \le \sum_{B\subset A,\ |B|=2} g(B)$.

Proof: Take $x_1, x_2, \ldots$ to be distinct.

ⓐ Then, $g(\{x_1, \ldots, x_n\}) = \mathbb{P}\big(X_i(t) = X_j(t)$ for some $1 \leq i < j \leq n$ and $t > 0\big)$.

from which ⓐ follows straightforwardly.

ⓑ $g(\{x_1, \ldots, x_n\}) \leq \sum_{1 \leq i < j \leq n} \mathbb{P}(X_i(t) = X_j(t)$ for some $t > 0)$

$$= \sum_{1 \leq i < j \leq n} g(\{x_i, x_j\}). \;\#$$

Since Z is transient, if we start two walks very far apart, then they will typically not meet so that the function g can be zero. More precisely;

Lemma 3.26) ⓐ $\lim_{|x| \to \infty} g(\{0, x\}) = 0$.

ⓑ $\forall\, n \in \mathbb{N}$, $\lim_{t \to \infty} g(\{X_1(t), \ldots, X_n(t)\}) = 0$ a.s.

ⓒ $\forall$ initial state A_0, $\lim_{t \to \infty} g(A_t) = 0$ a.s.

Proof: ⓐ $\mathbb{P}^x(Z(2t) = y) = \sum_{z \in \Lambda} \mathbb{P}^x(Z(t) = z)\, \mathbb{P}^z(Z(t) = y)$

$$\overset{C.S.}{\leq} \Big(\sum_{z \in \Lambda} \big(\mathbb{P}^x(Z(t) = z)\big)^2\Big)^{1/2} \Big(\sum_{z \in \Lambda} \big(\mathbb{P}^z(Z(t) = y)\big)^2\Big)^{1/2}.$$

Now, the random walk $Z(t)$ is <u>symmetric</u>, since the transition rates are given by:

$$\begin{cases} Z(t) \longrightarrow Z(t) + n \text{ at rate } \underset{X_1 \to X_1 + n \;\; X_2 \to X_2 - n}{q(0,n)} + q(0,-n) \\ Z(t) \longrightarrow Z(t) - n \text{ at rate } \underset{X_1 \to X_1 - n \;\; X_2 \to X_2 + n}{q(0,-n)} + q(0,n) \end{cases}$$

$$\Rightarrow \sum_{z\in\Lambda} \left(\mathbb{P}^y(Z(t)=z)\right)^2 = \sum_{z\in\Lambda} \mathbb{P}^y(Z(t)=z)\cdot \mathbb{P}^z(Z(t)=x)$$
$$= \mathbb{P}^x(Z(2t)=x).$$

$$\Rightarrow \underline{\mathbb{P}^x(Z(2t)=y)} \le \left(\mathbb{P}^x(Z(2t)=x)\right)^{1/2}\cdot\left(\mathbb{P}^y(Z(2t)=y)\right)^{1/2}$$
$$\underset{\text{tr. inv.}}{=} \underline{\mathbb{P}^0(Z(2t)=0)}. \quad (*)$$

. Further, $\int_t^\infty \mathbb{P}^x(Z(s)=y)\,ds = \mathbb{E}^x\left(\int_t^\infty \mathbb{1}_{(Z(s)=y)}\,ds\right)$ $(**)$

$$\underset{\text{(strong MP)}}{=} \mathbb{P}^x(Z(s)=y \text{ for some } s>t)\cdot G(y,y).$$

By $(*)$, the l.h.s. of $(**)$ is maximised for $x=y$.

$$\Rightarrow \int_t^\infty \mathbb{P}^y(Z(s)=y)\,ds \le \int_t^\infty \mathbb{P}^y(Z(s)=x)\,ds$$

$$\mathbb{P}^y(Z(s)=y \text{ for some } s>t).\underbrace{G(y,y)}_{\underset{\text{tr. inv.}}{=} G(0,0)} \qquad \underset{\text{tr. inv.}}{=} \mathbb{P}^0(Z(s)=0 \text{ for some } s>t).G(0,0)$$

$$\Rightarrow \mathbb{P}^y(Z(s)=y \text{ for some } s>t) \le \mathbb{P}^0(Z(s)=0 \text{ for some } s>t). \quad (***)$$

Now consider $g(\{0,x\})$:

$$g(\{0,x\}) = \mathbb{P}^x(Z(s)=0 \text{ for some } s>0)$$
$$\le \mathbb{P}^y(Z(s)=0 \text{ for some } 0\le s\le t)$$
$$+ \mathbb{P}^x(Z(s)=0 \text{ for some } s>t)$$
$$\underset{\substack{\text{symmetry}\\ + (***)}}{\le} \mathbb{P}^0(Z(s)=x \text{ for some } 0\le s\le t)$$
$$+ \mathbb{P}^0(Z(s)=0 \text{ for some } s>t).$$

Note that $\mathbb{P}^0(Z(s)=0 \text{ for some } s>t)$ is indep of x and goes to 0 as $t\to\infty$ since $Z(t)$ is transient.

. Denote by $\tau_x = \inf\{t>0 \mid Z(t)=x\}$.

Then, $\mathbb{P}^0(Z(\tau_x+s)=x \mid \mathcal{F}_{\tau_x}) \underset{\text{strong MP}}{=} \mathbb{P}^x(Z(s)=x) \geq \exp(-2Ms)$ ($\star$)

with $M = \sup_{x\in\Lambda} \sum_{u:u\neq x} q(x,u)$, because the time spent at x is exponentially distributed with parameter $\leq 2M$.

$$\Rightarrow \int_0^1 \mathbb{E}^0\left(\mathbb{1}_{\{\tau_x<t\}} (\star)\right) ds \Rightarrow \int_0^1 ds\, \mathbb{P}^0(Z(\tau_x+s)=x; \mathbb{1}_{\{\tau_x<t\}}) \quad (\star\star)$$

$$\geq \mathbb{P}^0(\tau_x \leq t) \underbrace{\int_0^1 ds\, e^{-2Ms}}_{= \frac{1-e^{-2M}}{2M}}$$

l.h.s. of ($\star\star$) $\leq \mathbb{E}^0\left(\int_0^{t+1} \mathbb{1}_{\{Z(s)=x\}} ds\right)$

$$\Rightarrow \sum_{x\in\Lambda} \mathbb{P}^0(\tau_x \leq t) \leq \frac{2M}{1-e^{-2M}} \cdot \underbrace{\sum_{x\in\Lambda} \mathbb{E}^0\left(\int_0^{t+1} \mathbb{1}_{\{Z(s)=x\}} ds\right)}_{= t+1}.$$

$$\Rightarrow \mathbb{P}^0(Z(s)=x \text{ for some } 0\leq s\leq t) = \mathbb{P}^0(\tau_x \leq t) \xrightarrow[x\to\infty]{} 0, \ \forall t$$

(since the above series converges).

$$\Rightarrow \lim_{x\to\infty} g(\{0,x\}) \leq \lim_{t\to\infty} \lim_{x\to\infty} \{\mathbb{P}^0(Z(s)=x \text{ for some } 0\leq s\leq t) + \mathbb{P}^0(Z(s)=0 \text{ for some } s>t\} = 0. \checkmark$$

(b) We have:

$$g(\{X_1(t),\dots,X_n(t)\}) \underset{\text{Lemma 3.25 (b)}}{\leq} \sum_{1\leq i<j\leq n} g(\{X_i(t),X_j(t)\})$$

$$\underset{\text{tr. inv.}}{=} \sum_{1\leq i<j\leq n} g(\{0, X_j(t)-X_i(t)\})$$

$$\xrightarrow[t\to\infty]{} 0 \quad \text{a.s.}$$

by (a) and the transience of $Z(t)$ (that implies $Z(t)\to\infty$ a.s. as $t\to\infty$).

ⓒ If $|A_0|=n$, couple A_t with $\{X_1(t),\dots,X_n(t)\}$ s.t.

$$A_t \subseteq \{X_1(t),\dots,X_n(t)\} \text{ for all } t.$$

(One does it by retaining only one of the coalescing particles in A_t).

Then, by Lemma 3.25ⓐ, $g(A_t) \le g(\{X_1(t),\dots,X_n(t)\})$ and by ⓑ this $\to 0$ as $t\to\infty$.

#

Now we are ready to discuss some properties of μ_ρ.

Def. 3.27) A translation invariant probability measure μ on $\{0,1\}^\Lambda$ is said to be mixing if it satisfies:

$$\lim_{x\to\infty} \mu(\{\eta : \eta\equiv 1 \text{ on } A\cup(B+x)\}) = \mu(\{\eta:\eta\equiv 1 \text{ on } A\})\cdot \mu(\{\eta:\eta\equiv 1 \text{ on } B\})$$

for any finite sets $A,B\subset\Lambda$.

Thm 3.28) ⓐ For all finite sets $A\subset\Lambda$,

$$0\le \mu_\rho(\eta\equiv 1 \text{ on } A) - \rho^{|A|} \le g(A).$$

ⓑ The measure μ_ρ is translation invariant and mixing.

ⓒ $\mu_\rho(\eta(x)=1)=\rho$.

ⓓ The covariances under μ_ρ are given by

$$\mathrm{Cov}_{\mu_\rho}(\eta(x),\eta(y)) = \rho(1-\rho)\,\frac{G(x,y)}{G(0,0)}.$$

Proof: ⓐ We have

$$\mu_\rho(\eta\equiv 1 \text{ on } A) - \rho^{|A|} = \mathbb{E}^A\left(\rho^{|A_\infty|} - \rho^{|A|}\right) \geq 0$$

since $|A_\infty| \leq |A|$ and $0 \leq \rho \leq 1$

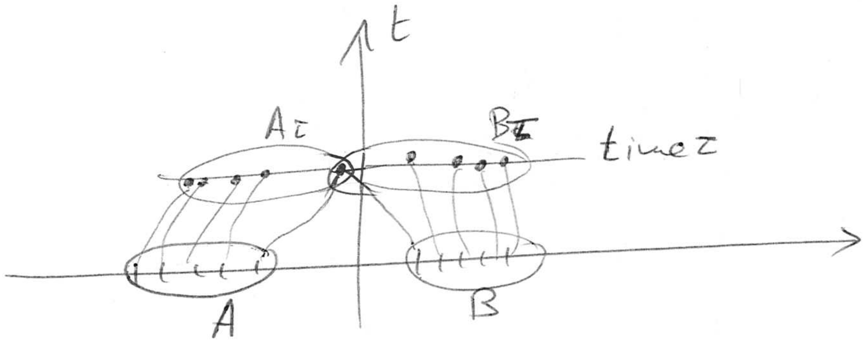
$$= \mathbb{E}^A\Big(\underbrace{(\rho^{|A_\infty|} - \rho^{|A|})}_{|\,|\leq 1}\cdot \mathbb{1}_{|A_\infty|<|A|}\Big)$$

$$\leq g(A). \quad ✓$$

ⓑ . Translation invariance follows from the fact that $\nu_\rho T(t)$ is translation invariant $\forall t \geq 0$.

. Mixing: Consider three copies A_t, B_t, C_t of the dual process and couple them as follows:

$$\begin{cases} \cdot\ A_0 = A \\ \cdot\ B_0 = B \\ \cdot\ A_t \text{ and } B_t \text{ are independent} \\ \cdot\ C_t = A_t \cup B_t \quad \text{for } t \leq \tau = \inf\{s>0 \mid A_s \cap B_s \neq \emptyset\} \\ \cdot\ \text{For } t>\tau, \text{ let } C_t \text{ evolves independently of } A_t \text{ and } B_t. \end{cases}$$



Then, $\left|\mu_\rho(\eta\equiv 1 \text{ on } A\cup B) - \mu_\rho(\eta\equiv 1 \text{ on } A)\mu_\rho(\eta \equiv 1 \text{ on } B)\right|$

$$= \left|\mathbb{E}\left(\rho^{|C_\infty|}\right) - \mathbb{E}\left(\rho^{|A_\infty|+|B_\infty|}\right)\right| \qquad (\equiv 0 \text{ on configurations with } \tau=\infty)$$

$$= \Big|\mathbb{E}\Big(\underbrace{(\rho^{|C_\infty|} - \rho^{|A_\infty|+|B_\infty|})}_{|\,|\leq 1}\mathbb{1}(\tau<\infty)\Big)\Big|$$

$$\leq \mathbb{P}(\tau<\infty) \leq \sum_{u\in A, v\in B} g(\{u,v\}).$$

By taking $B \to B+x$ and using Lemma 3.26 ⓐ, mixing follows. ✓

(c) $\mu_\rho(\eta(x)=1) = \mu_\rho(\eta \equiv 1 \text{ on } \{x\}) = \mathbb{E}^{\{x\}}(\rho^{|A_\infty|}) = \rho.$

(d) $\mathrm{Cov}_{\mu_\rho}(\eta(x),\eta(y)) = \mu_\rho(\eta(x)=\eta(y)=1) - (\mu_\rho(\eta(x)=1))(\mu_\rho(\eta(y)=1))$

$$= \mathbb{E}^{\{x,y\}}(\rho^{|A_\infty|} - \rho^2)$$

$$= \rho(1-\rho)\,\mathbb{P}^{\{x,y\}}(|A_\infty|=1)$$

$$= \rho(1-\rho)\,\mathbb{P}^{x-y}(Z(t)=0 \text{ for some } t\geq 0)$$

$$= \rho(1-\rho)\frac{G(x,y)}{G(0,0)}$$ since from (**) of the proof of Lemma 3.26 with $t=0$ gives:

$$\underbrace{\int_0^\infty \mathbb{P}^x(Z(s)=y)\,ds}_{=G(x,y)} = \mathbb{P}^x(Z(s)=y \text{ for some } s\geq 0)\cdot G(0,0)$$

#

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. The question that has still to be answered is whether the measures μ_ρ give us all the invariant measures.

Thm 3.29) $\boxed{I \text{ is the closed convex hull of } \{\mu_\rho : 0\leq\rho\leq 1\}.}$

[Consider any $\mu\in I$. To show: μ can be written as a mixture of $\{\mu_\rho, 0\leq\rho\leq 1\}$, i.e., there exists a measure γ on $[0,1]$ s.t. $\mu = \int_0^1 \mu_\rho\,\gamma(d\rho)$.

Proof: The idea is to show that μ is in a sense "exchangable".

We have: $\mu(\eta\equiv 1 \text{ on } A) = \mu T(t)(\eta\equiv 1 \text{ on } A)$

$$= \int_{S'} \mathbb{P}^\eta(\eta_t\equiv 1 \text{ on } A)\,d\mu(\eta)$$

$$\overset{\text{duality}}{=} \int_{S'} \mathbb{P}^A(\eta\equiv 1 \text{ on } A_t)\,d\mu(\eta) \qquad (*)$$

$$= \sum_B \mathbb{P}^A(A_t=B)\,\mu(\eta\equiv 1 \text{ on } B).$$

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Some notations: $\cdot\ h(A) := \mu(\eta \equiv 1 \text{ on } A)$

$\cdot\ V_t f(A) := \mathbb{E}^A(f(A_t))$

$\cdot\ U_t f(x_1, \dots, x_n) = \mathbb{E}^{x_1, \dots, x_n}(f(X_1(t), \dots, X_n(t)))$.

Then, ⊛ writes: $h(A) = V_t h(A)$. (⁎⁎)

. Further, if $A = \{x_1, \dots, x_n\}$, $U_t f(A) \equiv \mathbb{E}^{x_1, \dots, x_n}(f(\{X_1(t), \dots, X_n(t)\}))$.

. let us couple A_t and $\{X_1(t), \dots, X_n(t)\}$ s.t.:

$$\begin{cases} \to |A_0| = n \\ \to A_t \subseteq \{X_1(t), \dots, X_n(t)\} \text{ for all } t \geq 0. \end{cases}$$

Then, for any f s.t. $0 \leq f(A) \leq 1$ for all A, it holds

$$|V_t f(A) - U_t f(A)| \overset{A = \{x_1, \dots, x_n\}}{=}$$

$$= |\mathbb{E}^A(f(A_t)) - \mathbb{E}^{x_1, \dots, x_n}(f(\{X_1(t), \dots, X_n(t)\}))|$$

$$\leq \mathbb{P}^A(|A_t| < n) = g(A).$$

$$\Rightarrow |h(A) - U_t h(A)| \overset{(**)}{=} |V_t h(A) - U_t h(A)|$$

$$\leq g(A). \quad (***)$$

Moreover, $|U_s h(A) - U_{t+s} h(A)| \leq U_s g(A) = g(A_s)$ (semigroup property)

. By Lemma 3.26(c), $\lim_{s \to \infty} U_s g(A) = 0$ a.s.

$\Rightarrow \lim_{s \to \infty} U_s h(A)$ exists.

. Also, with $H(A) := \lim_{s \to \infty} U_s h(A)$, it holds

$$\mathbb{E}^A(H(A_t)) = \mathbb{E}^A(U_t H(A)) = H(A), \text{ i.e.,}$$

H is harmonic for the random walk $(X_1(t), \dots, X_n(t))$ on $(\mathbb{Z}^d)^n$.

. This random walk is irreducible by assumption.

Then, since H is harmonic and bounded $\Longrightarrow$ it is constant.
(see e.g. Thm A.58 of Liggett Appendix

Thus, $\exists$ constants $c(n)$, $n \in \mathbb{N}$ s.t.

$$H(A) = \lim_{s\to\infty} U_s h(A) = c(|A|), \ \forall A.$$

. The constants c depends on h, which on its turn depends on μ.

. Take $t \to \infty$ into (***) leads to

$$|h(A) - c(|A|)| \le g(A). \quad (****)$$

Rem.: $g(A)$ is small if the points of A are far away (see Lemma 3.25 ⓑ and 3.26 ⓑ).
($\Rightarrow$ In this case $h(A) \approx c(n)$ with $n = |A|$).

. To show that μ is a mixture of μ_ρ, recall by Thm 3.28 ⓐ that if the points of A are far apart,

$$\mu_\rho(\eta \equiv 1 \text{ on } A) \approx \nu_\rho(\eta \equiv 1 \text{ on } A).$$

$\Rightarrow$ in that case we expect that $C(n)$ is a mixture of $\nu_\rho(\eta \equiv 1 \text{ on } A) = \rho^n$, for $|A| = n$.

. Strategy: ⓐ Find that $c(n)$ is a mixture of ρ^n
ⓑ Show that μ_ρ is the same mixture of ν_ρ.

. ⓐ: Show that $\exists$ probability measure γ on $[0,1]$ s.t.

$$c(n) = \int_0^1 \rho^n \gamma(d\rho).$$

We use the following Lemma.

Lemma 3.30: (Thm A.32 of Liggett Appendix)

$$\left\{c(n) = \int_0^1 \rho^n \gamma(d\rho), n \ge 0\right\} \iff c(0) = 1 \text{ and } \forall m, n \ge 0, \ \sum_{k=0}^{n} \binom{n}{k} (-1)^k c(k+m) \ge 0. \quad (\star)$$

. Let us check $(\star)$: Fix $m, n \ge 0$. Let A_i be a sequence of sets of size $m+n$ s.t. $g(A_i) \to 0$.
By Lemma 3.26 ⓑ, such a sequence exists (take $A_i \equiv A_{t_i}$ for any choice of increasing sequences $t_i \uparrow \infty$).

. Decompose $A_i = B_i \cup C_i$ with $|B_i| = m$, $|C_i| = n$.

By Lemma 3.25 ⓐ, $\begin{cases} g(B_i) \le g(A_i) \xrightarrow[i\to\infty]{} 0, \\ g(C_i) \le g(A_i) \longrightarrow 0. \end{cases}$

. Further, inclusion-exclusion tells us:

$$\mu(\eta \equiv 1 \text{ on } B_i,\ \eta \equiv 0 \text{ on } C_i) = \sum_{F \subset C_i} (-1)^{|F|}\, h(B_i \cup F).$$

Taking $i \to \infty$ and applying (***), i.e., $\lim_{i\to\infty} h(B_i \cup F) = c(|B_i \cup F|)$,

we get: $0 \le \lim_{i\to\infty} \mu(\eta \equiv 1 \text{ on } B_i, \eta \equiv 0 \text{ on } C_i) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, c(k+m)$,

i.e., (✡) holds.

$\Rightarrow$ We have shown that $c(n) = \int_0^1 s^n \gamma(ds)$.

ⓑ let us define $\mu^* := \int_0^1 \mu_s\, \gamma(ds)$ and

accordingly $h^*(A) = \mu^*(\eta \equiv 1 \text{ on } A)$.

. Obviously $\mu^* \in I$. Thus (*) can be applied to μ^* giving

$h^*(A) = V_t\, h^*(A)$.

. By Thm 3.28 ⓐ, $|h^*(A) - c(|A|)| =$

$$= \left| \int_0^1 \gamma(ds) \left(\mu_s(\eta \equiv 1 \text{ on } A) - s^n \right) \right| \le g(A).$$

Therefore,

$$|h^*(A) - h(A)| \le \underbrace{|h^*(A) - c(|A|)|}_{\le g(A)} + \underbrace{|h(A) - c(|A|)|}_{\le g(A)\ (***)}$$

$$\le 2 \cdot g(A).$$

. Further, $|h^*(A) - h(A)| = |V_t(h^*(A) - h(A))|$

$$\le 2\, V_t\, g(A) = 2 \cdot g(A_t).$$

By Lemma 3.26 ⓒ, $\lim_{t\to\infty} V_t\, g(A) = 0 \Rightarrow h(A) = h^*(A)\ \forall A \subset \Lambda$

$\Rightarrow \mu = \mu^*$. #

3.4) The threshold voter model

. There are also non-linear voter model.
Here we shortly consider a threshold voter model for which there is a graphical construction that allows to define the process. Graphical constructions turns out to exist for several other particle systems, for instance in the exclusion process (see summer semester lecture).

. Def. 3.31) . Let $\mathcal{N}$ be a neighborhood of $0 \in \mathbb{Z}^d$ that is obtained by intersecting $\mathbb{Z}^d$ with any compact, convex, symmetric set in $\mathbb{R}^d$.

. For a positive integer T, the <u>threshold voter model</u> with neighborhood $\mathcal{N}$ and threshold T is the one with vote function

$$c(x,\eta) = \begin{cases} 1 & , \text{ if } \#\{y \in x+\mathcal{N} : \eta(x) \neq \eta(y)\} \geq T, \\ 0 & , \text{ otherwise} \end{cases}$$

Rem.: If T is large, the process will <u>coexist</u>, i.e., there exists an invariant measure that is not a mixture of δ_0 and δ_1.
<u>For example</u>, take $d=1$, $\mathcal{N} = \{-1,0,1\}$, $T=2$ and consider the following configuration:

$$\ldots 11\,00\,11\,00\,11\,00\,11\,00 \ldots$$

Then, this is a <u>trap</u> for the process.

. 3.4.1) The graphical construction

. Let $\{N_x, x \in \mathbb{Z}^d\}$ be independent Poisson processes with intensity (rate) 1.

$\Rightarrow \eta_t$ is defined by saying that

$$\eta_t(x) \neq \eta_{t^-}(x) \Longleftrightarrow \left\{ \begin{array}{l} t \text{ is an event time of } N_x, \text{ and} \\ \#\{y \in x+\mathcal{W} : \eta_t(y) \neq \eta_t(x)\} \geq T \end{array} \right\}.$$

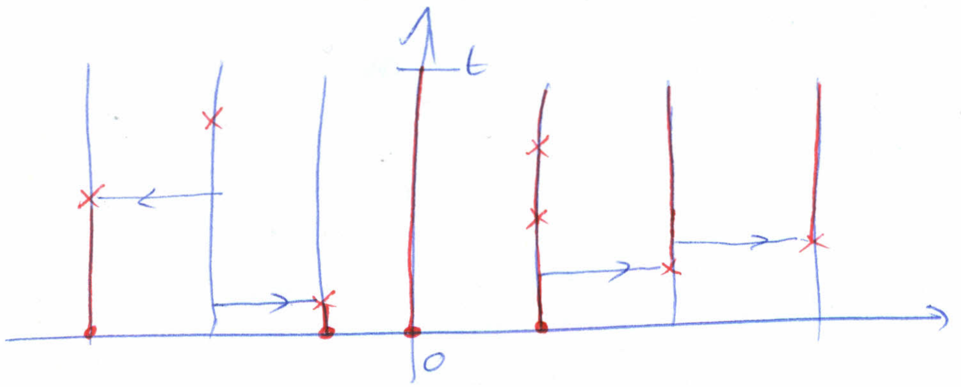

Illustration for $d=1$, $\mathcal{W} = \{-1,0,1\}$, $T=1$.

$\times$ = Poisson events.

— = value 1 of spin.

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. The small technical issue of this construction is that it works only if finitely many previous decisions are relevant in deciding whether to flip the configuration at site x and time t.

. $d=1$: In one-dimension this is easy.

Indeed, $\forall T$ and t, with probability one, $\exists$ ∞-many y's s.t. N_x has no event times in $[0,t]$ for any $x \in [y,..,y+T]$.

$\Rightarrow \mathbb{Z}$ is divided into finite "islands" that have no influence on each other up to time t.

. $d \geq 2$: In this case is slightly more complicated:

→ first one construct the process by a percolation argument

→ once the process is constructed for $t \in [0,\varepsilon]$, recursively it can be extended to any time.

Explicitely: ⓐ Fix a $t>0$

ⓑ Construct a random oriented graph with verted set $\mathbb{Z}^d$ placing an edge from x to y if N_x has an event time by time t and $y \in x + \mathcal{N}$.

ⓒ By a "potential" path of length n, we mean a sequence $x_0, x_1, \ldots, x_n$ of distinct vertices in $\mathbb{Z}^d$ s.t. $x_{i+1} \in x_i + \mathcal{N}$ for each i.
There are at most $(|\mathcal{N}|-1)^n$ potential paths of length n starting at x.
Further, the probability that any one of them is a path (in the oriented graph) is $(1-e^{-t})^n$.

$\Rightarrow$ If t is so small that $(|\mathcal{N}|-1)(1-e^{-t})<1$, then the expected number of vertices connected to x in the random graph is finite. Thus, only finitely many sites can influence the evolution of $\eta_s(x)$, for $s \in [0,t]$.

3.4.2) Properties of threshold voter model

. We will discuss fixation, clustering, and coexistence. For this, let us shortly define these concepts:

Def 3.32) . The process η_t coexists if $\exists$ invariant measure that is not a mixture of δ_0 and δ_1.

. The process η_t clusters if

$$\lim_{t\to\infty} \mathbb{P}^\eta(\eta_t(x) \neq \eta_t(y)) = 0.$$

. The process η_t fixates if each $\eta_t(x)$ flips only finitely often for every initial configuration.

. First we are going to see that for large values of the threshold the process fixates.

Thm 3.33) If $T > \frac{|\mathcal{N}|-1}{2}$, then the process fixates.

Proof: For any $\varepsilon>0$, let us define the weight of $\eta \in S'$ by

$$w_\varepsilon(\eta) := \sum_{\substack{x,y:\\ x-y\in\mathcal{N},\\ \eta(x)\neq\eta(y)}} e^{-\varepsilon|x+y|}.$$

. For all $\eta \in \{0,1\}^{\mathbb{Z}^d}$, $w_\varepsilon(\eta) < \infty$.

. We have:

. $$w_\varepsilon(\eta^u) - w_\varepsilon(\eta) = \sum_{\substack{x,y:\, x-y\in\mathcal{N},\\ \eta^u(x)\neq\eta^u(y),\\ \eta(x)=\eta(y)}} e^{-\varepsilon|x+y|} - \sum_{\substack{x,y:\, x-y\in\mathcal{N},\\ \eta^u(x)=\eta^u(y),\\ \eta(x)\neq\eta(y)}} e^{-\varepsilon|x+y|}$$

$$= 2\cdot \sum_{\substack{y\in u+\mathcal{N},\, y\neq u\\ \eta(y)=\eta(u)}} e^{-\varepsilon|u+y|} - 2\sum_{\substack{y\in u+\mathcal{N},\\ \eta(y)\neq\eta(u)}} e^{-\varepsilon|u+y|} \quad (*)$$

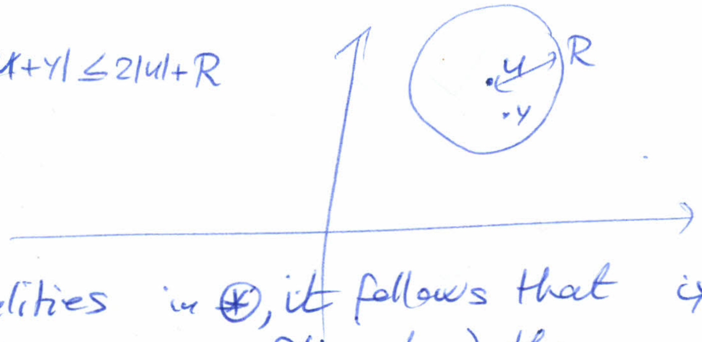
. If a spin flip can occur

$$\Rightarrow \#\{y\in u+\mathcal{N} : \eta(y)\neq\eta(u)\} \geq T.$$

. Thus, $\#\{y\in u+\mathcal{N} : \eta(y)=\eta(u),\, y\neq u\} \leq |\mathcal{N}|-T-1$.

. let $R = \sup\{|x| : x\in\mathcal{N}\}$, then

$$2|u|-R \leq |u+y| \leq 2|u|+R$$



. Plugging these inequalities in $(*)$, it follows that if $c(u,\eta)=1$ (i.e., if there was a flip at u), then

$$w_\varepsilon(\eta^u) - w_\varepsilon(\eta) \leq 2\cdot e^{-2\varepsilon|u|}\left((|\mathcal{N}|-T-1)e^{\varepsilon R} - T e^{-\varepsilon R}\right).$$

- We assumed that $|\mathcal{N}|-T-1 < T$

$\Rightarrow$ for ε small enough,

$$(|\mathcal{N}|-T-1)e^{\varepsilon R} - Te^{-\varepsilon R} < 0.$$

- Since every flip at u decreases w_ε by at least a certain amount and $w_\varepsilon \geq 0$, there can only be finitely many flips at u. #

- The second property we consider is <u>clustering in one dimension.</u>

<u>Thm. 3.34)</u> The threshold voter model in $d=1$ with $\mathcal{N} = \{-T, \dots, T\}$, $T \geq 1$, clusters.

- One can also prove that under suitable conditions the threshold voter model <u>coexists</u>. Basically, one needs that the neighborhood is large enough and the threshold value not too high.

<u>Thm 3.35)</u> $\exists$ $c>0$ s.t.:

- if a sequence of threshold voter models η_t^n with thresholds T_n and neighborhoods $\mathcal{N}_n = \{x \in \mathbb{Z}^d : |x| \leq n\}$ satisfies

$$\limsup_{n\to\infty} \frac{T_n}{|\mathcal{N}_n|} < c,$$

then η_t^n coexists for all sufficiently large n.

. Outlook: The voter model is only one of the interacting particle systems that have been (and are) studied.

. Some more examples:

(a) Population dynamics: Interacting particle systems are also used is some population dynamics model, where the state space is for example $(0,1)^{\mathbb{Z}^d}$ or $\mathbb{R}_+^{\mathbb{Z}^d}$ or $\mathbb{Z}_+^{\mathbb{Z}^d}$. The $\eta_t(x)$ is then the population size at position x or the fraction of a given gene.

. The generator is then a sum of more elementary generators, describing for example mutations, competition, natural death and birth processes.

(b) Exclusion process: Another well-studied model is the exclusion process. The state space is as in the spin system we considered, but one key difference is that $\eta_t(x)=1$ represents a particle and the dynamics preserves the number of particles: instead of spin flips we have particle exchanges between sites.

. For this model there is a graphical construction and there is in some cases duality.

. When the dynamics is "asymmetric", it belongs to a universality class of stochastic growth models displaying non-gaussian types of CLT and with nice links to random matrix theory (see summer semester lecture for details!).