

RANDOM MATRICES AND DETERMINANTAL PROCESSES

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1. INTRODUCTION

Eigenvalues of random matrices have a rich mathematical structure and are a source of interesting distributions and processes. These distributions are natural statistical models in many problems in quantum physics, [15]. They occur for example, at least conjecturally, in the statistics of spectra of quantized models whose classical dynamics is chaotic, [4]. Random matrix statistics is also seen in the statistics of zeros of L-functions in number theory, [23].

In recent years we have seen a new development where probability distributions from random matrix theory appear as limit laws in models of a statistical mechanical nature, namely in certain random growth and random tiling problems. This came as a surprise and has added a new side to random matrix theory. It clearly shows that the limit probability measures coming out of random matrix theory are natural limit probability distributions.

Only very special models, which are exactly solvable in a certain sense, can be analyzed. In these notes we will survey two models, random domino tilings of the Aztec diamond and a one-dimensional local random growth model, the corner growth model. We will also discuss relations between these two models. Underlying the exact solvability of the models is the fact that they can be mapped to families of non-intersecting paths and that these in turn lead to determinantal point processes. Point processes with determinantal correlation functions have emerged as an interesting class of point processes, with a rich structure and many interesting examples, [33].

2. POINT PROCESSES

2.1. General theory. We will need some general facts about point processes, but we will only survey those aspects that will be directly relevant for the present exposition, see [9]. Let Λ be a complete separable metric space and let $\mathcal{N}(\Lambda)$ denote the space of all counting measures ξ on Λ for which $\xi(B) < \infty$ for every bounded $B \subseteq \Lambda$. We say that ξ is *boundedly finite*. A counting measure is a measure ξ whose values on bounded Borel sets in Λ is a non-negative integer. Typically Λ will be \mathbb{R} , \mathbb{Z} , some subset of these or the disjoint union of several copies of \mathbb{R} or \mathbb{Z} . We can define a σ -algebra \mathcal{F} on $\mathcal{N}(\Lambda)$ by taking the smallest σ -algebra for which $A \rightarrow \xi(A)$ is measurable for all Borel sets A in Λ .

If B is a bounded set $\xi(B)$ is finite and we can write

$$(2.1) \quad \xi|_B = \sum_{i=1}^{\xi(B)} \delta_{x_i},$$

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for some $x_1, \dots, x_{\xi(B)} \in \Lambda$. Note that we can have $x_i = x_j$ for $i \neq j$, i.e. a multiple point. We say that ξ is *simple* if $\xi(\{x\}) \leq 1$ for all $x \in \Lambda$. The counting measure ξ can be thought of as giving a point or particle configuration in Λ . A *point process* on Λ is a probability measure \mathbb{P} on $\mathcal{N}(\Lambda)$. The point process is *simple* if $\mathbb{P}(\xi \text{ simple}) = 1$.

If the function $\phi : \Lambda \rightarrow \mathbb{C}$ has support in the bounded Borel set B we write

$$(2.2) \quad \prod_i (1 + \phi(x_i)) = \prod_{i=1}^{\xi(B)} (1 + \phi(x_i)),$$

where $x_1, \dots, x_{\xi(B)}$ are defined by (2.1). If $\xi(B) = 0$, then the right hand side of (2.2) is $= 1$ by definition. Note that if $|\phi(x)| < 1$ for all $x \in \Lambda$ we have

$$(2.3) \quad \prod_i (1 + \phi(x_i)) = \exp\left(\int_{\Lambda} \log(1 + \phi(x)) d\xi(x)\right).$$

A natural way to investigate a point process is to consider expectations of products of the form (2.2). If we take for instance $\phi = \exp(-\psi) - 1$, $\psi \geq 0$ with bounded support, we get the so called Laplace functional. We can write

$$(2.4) \quad \prod_i (1 + \phi(x_i)) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_{i_1} \neq \dots \neq x_{i_n}} \phi(x_{i_1}) \dots \phi(x_{i_n}),$$

where the sum is over all n -tuples of *distinct* points in the process and we include all permutations of the n points, which we compensate for by dividing by $n!$. We want to include all permutations since there is no ordering of the points in the process. If we have a multiple point of multiplicity k it should be counted as k distinct points occupying the same position. The $n = 0$ term in (2.4) is $= 1$ by definition. Since ϕ has bounded support, the second sum in (2.4) is actually finite almost surely. We can construct a new point process Ξ_n in Λ^n for each $n \geq 1$, by setting

$$(2.5) \quad \Xi_n = \sum_{x_{i_1} \neq \dots \neq x_{i_n}} \delta_{(x_{i_1}, \dots, x_{i_n})},$$

i.e. each n -tuple of points in the original point process ξ , including all permutations of the points, gives rise to a point in the new process.

We define a measure M_n on Λ^n by setting

$$(2.6) \quad M_n(A) = \mathbb{E}[\Xi_n(A)],$$

for each bounded Borel set $A \subseteq \Lambda^n$, i.e. $M_n(A)$ is the expected number of n -tuples of distinct points that fall in A . Here we assume that the process is such that all the M_n , $n \geq 1$, are well-defined, $M_n(A) < \infty$ for bounded A . The measure M_n is an intensity measure for n -tuples of distinct points in the original process. The formula (2.4) can now be written

$$(2.7) \quad \prod_i (1 + \phi(x_i)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) \Xi_n(d^n x).$$

Assume that

$$(2.8) \quad \sum_{n=0}^{\infty} \frac{\|\phi\|_{\infty}^n}{n!} M_n(B^n) < \infty,$$

where the bounded set B contains the support of ϕ . Since,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E} \left[\left| \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) \Xi_n(d^n x) \right| \right] \leq \sum_{n=0}^{\infty} \frac{\|\phi\|_{\infty}^n}{n!} M_n(B^n)$$

it follows from Fubini's theorem that

$$(2.9) \quad \mathbb{E} \left[\prod_i (1 + \phi(x_i)) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) M_n(d^n x).$$

Consider the case when ϕ is a simple function

$$(2.10) \quad \phi(x) = \sum_{j=1}^m a_j \chi_{A_j}(x)$$

with A_1, \dots, A_m disjoint, measurable subsets of a bounded set B . Note that since the A_j 's are disjoint we have

$$1 + t\phi(x) = \prod_{j=1}^m (1 + ta_j)^{\chi_{A_j}(x)},$$

where $|t| \leq 1$, and hence

$$(2.11) \quad \prod_i (1 + t\phi(x_i)) = \prod_{j=1}^m (1 + ta_j)^{\xi(A_j)}.$$

Set $1/n! = 0$ if $n < 0$. Then, by the binomial theorem,

$$(2.12) \quad \begin{aligned} \prod_{j=1}^m (1 + ta_j)^{\xi(A_j)} &= \prod_{j=1}^m \sum_{n_j=0}^{\xi(A_j)} \frac{\xi(A_j)!}{n_j! (\xi(A_j) - n_j)!} (ta_j)^{n_j} \\ &= \sum_{n_1, \dots, n_m=0}^{\infty} \prod_{j=1}^m \frac{(ta_j)^{n_j}}{n_j!} \prod_{j=1}^m \frac{\xi(A_j)!}{(\xi(A_j) - n_j)!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n_1 + \dots + n_m = n} \binom{n}{n_1 \dots n_m} \prod_{j=1}^m a_j^{n_j} \prod_{j=1}^m \frac{\xi(A_j)!}{(\xi(A_j) - n_j)!}. \end{aligned}$$

If t, a_1, \dots, a_m are all positive, it follows from Fubini's theorem that

$$(2.13) \quad \begin{aligned} &\mathbb{E} \left[\prod_i (1 + t\phi(x_i)) \right] \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n_1 + \dots + n_m = n} \binom{n}{n_1 \dots n_m} \prod_{j=1}^m a_j^{n_j} \mathbb{E} \left[\prod_{j=1}^m \frac{\xi(A_j)!}{(\xi(A_j) - n_j)!} \right]. \end{aligned}$$

On the other hand, by (2.9),

$$(2.14) \quad \begin{aligned} \mathbb{E} \left[\prod_i (1 + \phi(x_i)) \right] &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\Lambda^n} \prod_{k=1}^n \left(\sum_{j=1}^m a_j \chi_{A_j}(x_k) \right) M_n(d^n x) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n_1 + \dots + n_m = n} \binom{n}{n_1 \dots n_m} \prod_{j=1}^m a_j^{n_j} M_n(A_1^{n_1} \times \dots \times A_m^{n_m}) \end{aligned}$$

Hence, for any bounded, disjoint Borel sets A_1, \dots, A_m in Λ , and n_i , $1 \leq i \leq m$, such that $1 \leq n_i \leq n$ and $n_1 + \dots + n_m = n$,

$$(2.15) \quad M_n(A_1^{n_1} \times \dots \times A_m^{n_m}) = \mathbb{E} \left[\prod_{i=1}^m \frac{(\xi(A_i))!}{(\xi(A_i) - n_i)!} \right].$$

This can be used as an alternative definition of the measure M_n . If X is a random variable, $\mathbb{E}(X^k)$ is the k :th moment of X , and $\mathbb{E}(X(X-1)\dots(X-k+1))$ is the k 'th *factorial moment* of X . For this reason M_n is called the factorial moment measure since, by (3.8), can be defined using joint factorial moments.

In many cases there is a natural reference measure λ on Λ like Lebesgue measure on \mathbb{R} or the standard counting measure on \mathbb{Z} . We can then ask if the factorial moment measure M_n has a density with respect to λ^n on Λ^n .

Definition 2.1. *If M_n is absolutely continuous with respect to λ^n on Λ^n , i.e.*

$$(2.16) \quad M_n(A_1, \dots, A_n) = \int_{A_1 \times \dots \times A_n} \rho_n(x_1, \dots, x_n) d\lambda(x_1) \dots d\lambda(x_n)$$

for all Borel sets A_i in Λ , we call $\rho_n(x_1, \dots, x_n)$ the n 'th correlation function or correlation density. They are also called product densities.

We will be dealing with point processes for which all correlation functions exist. In many cases if we are given the correlation functions $(\rho_n)_{n \geq 1}$ the process is uniquely determined. As can be guessed from above, the uniqueness problem is closely related to the classical moment problem.

In the case of a simple point process on \mathbb{R} we can get some intuition for the correlation functions as follows. Let $A_i = [y_i, y_i + \Delta y_i]$, $1 \leq i \leq n$, be disjoint intervals. If the Δy_i are small we expect there to be either one or no particle in each A_i . Hence, typically, the product $\xi(A_1) \dots \xi(A_n)$ is 1 if there is exactly one particle in each A_i and 0 otherwise. From (2.16) we then expect

$$(2.17) \quad \rho_n(y_1, \dots, y_n) = \lim_{\Delta y_i \rightarrow 0} \frac{\mathbb{P}[\text{one particle in each } [y_i, y_i + \Delta y_i], 1 \leq i \leq n]}{\Delta y_1 \dots \Delta y_n}.$$

Note that $\rho_n(y_1, \dots, y_1)$ is not a probability density. The function $\rho_1(y)$ is the density of particles at y , but since we have many particles the event of finding a particle at y_1 and the event of finding a particle at y_2 are not disjoint even if $y_1 \neq y_2$. We should think of $\rho_n(y_1, \dots, y_1)$ as particle densities not probability densities. It follows from the argument above that if we have a simple point process on \mathbb{Z} (or some other countable or finite set), then $\rho_n(y_1, \dots, y_n)$ is exactly the probability of finding particles at y_1, \dots, y_n .

The next proposition follows from (2.8) and (2.9). The condition (2.18) below implies (2.8) and we get (2.9), which is exactly (3.12) by the definition of the correlation functions.

Proposition 2.2. *Consider a point process all of whose correlation functions exist. Let ϕ be a complex-valued, bounded, measurable function with bounded support. Assume that the support of ϕ is contained in the bounded, measurable set B and that*

$$(2.18) \quad \sum_{n=0}^{\infty} \frac{\|\phi\|_{\infty}^n}{n!} \int_{B^n} \rho_n(x_1, \dots, x_n) d^n \lambda(x) < \infty.$$

Then,

$$(2.19) \quad \mathbb{E} \left[\prod_j (1 + \phi(x_j)) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) \rho_n(x_1, \dots, x_n) d^n \lambda(x).$$

Here the product in the expectation in the left hand side is defined by (2.2).

We can think of the left hand side of (2.19) as a generating function for the correlation functions. Below we will see that (2.19) is useful for computing interesting probabilities. The condition (2.18) is not intended to be optimal but it will suffice for our purposes.

We also have a kind of converse of proposition 2.2.

Proposition 2.3. *Let $(u_n)_{n \geq 1}$ be a sequence of measurable functions $u_n : \Lambda^n \rightarrow \mathbb{R}$. Assume that for any simple, measurable function ϕ with bounded support, our point process satisfies*

$$(2.20) \quad \mathbb{E} \left[\prod_i (1 + \phi(x_i)) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) u_n(x_1, \dots, x_n) d^n \lambda(x),$$

with a convergent right hand side. Then all the correlation functions ρ_n , $n \geq 1$, exist and $\rho_n = u_n$.

Proof. Arguing as above in (2.13) and (2.14) we see that

$$(2.21) \quad \begin{aligned} M_n(A_1^{n_1} \times \dots \times A_m^{n_m}) &= \mathbb{E} \left[\prod_{j=1}^m \frac{\xi(A_j)!}{(\xi(A_j) - n_j)!} \right] \\ &= \int_{A_1^{n_1} \times \dots \times A_m^{n_m}} u_N(x_1, \dots, x_n) d^n \lambda(x). \end{aligned}$$

This proves the proposition by the definition of the correlation functions and (2.15). \square

Proposition 2.2 is useful when we want to compute *gap probabilities*, i.e. the probability that there is no particle in a certain set. If B is a bounded, measurable set and (2.18) holds with $\phi = -\chi_B$, then

$$(2.22) \quad \mathbb{P}[\text{no particle in } B] = \mathbb{E} \left[\prod_j (1 - \chi_B(x_j)) \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \rho_n(x_1, \dots, x_n) d^n \lambda(x)$$

Below we will be interested in processes on \mathbb{R} , or a subset of \mathbb{R} , which have a last or rightmost particle. Consider a point process ξ on \mathbb{R} . If there is a t such that $\xi(t, \infty) < \infty$, we say that ξ has a *last particle*. This will then be true for all t , since $\xi(A) < \infty$ for any bounded set. If $x_1 \leq \dots \leq x_{n(\xi)}$ are the finitely many particles in (t, ∞) , we define $x_{\max}(\xi) = x_{n(\xi)}$, the *position of the last particle*. The distribution function $\mathbb{P}[x_{\max}(\xi) \leq t]$ is called the *last particle distribution*. If $\mathbb{E}[\xi(t, \infty)] < \infty$ for some $t \in \mathbb{R}$, then ξ has a last particle almost surely.

Proposition 2.4. *Consider a point process ξ on \mathbb{R} or a subset of \mathbb{R} , all whose correlation functions exist, and assume that*

$$(2.23) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(t, \infty)^n} \rho_n(x_1, \dots, x_n) d^n \lambda(x) < \infty$$

for each $t \in \mathbb{R}$. Then the process ξ has a last particle and

$$(2.24) \quad \mathbb{P}[x_{\max}(\xi) \leq t] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \rho_n(x_1, \dots, x_n) d^n \lambda(x).$$

Proof. It follows from the $n = 1$ term in (2.23) that $\mathbb{E}[\xi(t, \infty)] < \infty$ and hence the process has a last particle almost surely. Take $t < s$. Proposition 2.2 implies that

$$\mathbb{P}[\text{no particle in } (t, s)] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t, s)^n} \rho_n(x_1, \dots, x_n) d^n \lambda(x).$$

We see from (2.23) and the dominated convergence theorem that we can let $s \rightarrow \infty$ and obtain (2.24). \square

Let us consider some examples of point processes.

Example 2.5. A classical and basic example of a point process is the Poisson process on \mathbb{R} with density $\rho(x)$, where ρ is locally L^1 . Let A_1, \dots, A_m be disjoint, bounded sets in \mathbb{R} . Then $\xi(A_i)$ are independent Poisson random variables with parameter $\int_{A_i} \rho(x) dx$, $1 \leq i \leq m$. Hence with ϕ as in (2.10),

$$\begin{aligned} \mathbb{E} \left[\prod_i (1 + \phi(x_i)) \right] &= \prod_{j=1}^m \mathbb{E} \left[\prod_{j=1}^m (1 + a_j)^{\xi(A_j)} \right] \\ &= \prod_{j=1}^m \left(\sum_{k=0}^{\infty} \frac{(1 + a_j)^k}{k!} \left(\int_{A_j} \rho(x) dx \right)^k e^{-\int_{A_j} \rho(x) dx} \right) = e^{\int_{\mathbb{R}} \phi(x) \rho(x) dx} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{j=1}^n \phi(x_j) \prod_{j=1}^n \rho(x_j) d^n x. \end{aligned}$$

It follows from proposition 2.3 that the correlation functions are given by

$$\rho_n(x_1, \dots, x_n) = \rho(x_1) \dots \rho(x_n),$$

which reflects the independence of particles at different locations. If $\rho(x)$ is integrable in $[t, \infty)$ the process has a last particle almost surely and

$$(2.25) \quad \mathbb{P}[x_{\max}(\xi) \leq t] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \rho(x_1) \dots \rho(x_n) d^n \lambda(x) = \exp\left(-\int_t^{\infty} \rho(x) dx\right).$$

Example 2.6. If $u_N(x_1, \dots, x_N)$ is a symmetric probability density on \mathbb{R}^N , then $(x_1, \dots, x_N) \rightarrow \sum_{i=1}^N \delta_{x_i}$ maps the probability measure with density u_N to a finite point process on \mathbb{R} . The correlation functions are given by

$$(2.26) \quad \rho_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} u_N(x_1, \dots, x_N) dx_{n+1} \dots dx_N.$$

i.e. they are multiples of the marginal densities. This is not difficult to see using proposition 2.3. When point processes defined in this way are studied (2.26) is often taken as the definition of the correlation functions.

Example 2.7. Let \mathcal{H}_N be the space of all $N \times N$ Hermitian matrices. This space can be identified with \mathbb{R}^{N^2} , since we have N^2 independent real numbers. If μ_N is a

probability measure on \mathcal{H}_N and $\{\lambda_1(M), \dots, \lambda_N(M)\}$ denotes the set of eigenvalues of $M \in \mathcal{H}_N$, then

$$(2.27) \quad \mathcal{H}_N \ni M \rightarrow \sum_{j=1}^N \delta_{\lambda_j(M)}$$

maps μ_N to a finite point process on \mathbb{R} .

If dM is Lebesgue measure on \mathcal{H}_N , then

$$(2.28) \quad d\mu_N(M) = \frac{1}{Z_N} e^{-\text{Tr} M^2} dM$$

is a Gaussian probability measure on \mathcal{H}_N called the GUE (*Gaussian Unitary Ensemble*). It can be shown, [26], that for any symmetric, continuous function on \mathbb{R}^N with compact support

$$(2.29) \quad \int_{\mathcal{H}_N} f(\lambda_1(M), \dots, \lambda_N(M)) d\mu_N(M) = \int_{\mathbb{R}^N} f(x_1, \dots, x_N) u_N(x_1, \dots, x_N) d^N x,$$

where

$$(2.30) \quad u_N(x_1, \dots, x_N) = \frac{1}{Z_N N!} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{j=1}^N e^{-x_j^2}$$

is the induced eigenvalue measure on \mathbb{R}^N . Hence the point process on \mathbb{R} defined by (2.30) has correlation functions given by (2.26).

We will show below that the correlation functions for the GUE eigenvalue process have a particularly nice determinantal form. This leads us to introduce so-called determinantal processes.

2.2. Determinantal processes. Determinantal processes are characterized by the fact that their correlation functions have a certain determinantal form.

Definition 2.8. Consider a point process ξ on a complete separable metric space Λ , with reference measure λ , all of whose correlation functions ρ_n exist. If there is a function $K : \Lambda \times \Lambda \rightarrow \mathbb{C}$ such that

$$(2.31) \quad \rho_n(x_1, \dots, x_N) = \det(K(x_i, x_j))_{i,j=1}^n$$

for all $x_1, \dots, x_n \in \Lambda$, $n \geq 1$, then we say that ξ is a determinantal point process. We call K the correlation kernel of the process.

We can view the correlation kernel K as an integral kernel of an operator K on $L^2(\Lambda, \lambda)$,

$$(2.32) \quad Kf(x) = \int_{\Lambda} K(x, y) f(y) d\lambda(y)$$

provided the right hand side is well-defined.

Consider a determinantal process on Λ . Let $\phi \in L^\infty(\Lambda, \lambda)$ have bounded support in B . Then by proposition 2.2

$$\begin{aligned}
\mathbb{E} \left[\prod_j (1 + \phi(x_j)) \right] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n \phi(x_j) \det(K(x_i, x_j))_{i,j=1}^n d^n \lambda(x) \\
(2.33) \quad &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B^n} \prod_{j=1}^n \phi(x_j) \det(K(x_i, x_j))_{i,j=1}^n d^n \lambda(x)
\end{aligned}$$

provided

$$(2.34) \quad \sum_{n=0}^{\infty} \frac{\|\phi\|_{\infty}^n}{n!} \int_{B^n} \det(K(x_i, x_j))_{i,j=1}^n d^n \lambda(x) < \infty.$$

The estimate (2.34) can usually be proved using Hadamard's inequality. The expansion (2.33) can be taken as the definition of the Fredholm determinant $\det(I + \chi_B K \chi_B \phi)_{L^2(\Lambda)} = \det(I + K\phi)_{L^2(B)}$. Here $K\phi$ is the operator on $L^2(B)$ with kernel $K(x, y)\phi(y)$. There are other ways of defining the Fredholm determinant for so called trace class operators, namely $\det(I + K\phi)_{L^2(B)} = \prod_i (1 + \lambda_i)$, where $\{\lambda_i\}$ are all the eigenvalues of the operator $K\phi$ on $L^2(B)$. If $K(x, y)\phi(y)$ defines a trace class operator on $L^2(B)$ and $\text{Tr } K\phi = \int_B K(x, x)d\lambda(x)$, then these two definitions agree. See [14] for more on Fredholm determinants.

Proposition 2.9. *Consider a determinantal point process ξ on a subset Λ of \mathbb{R} with a hermitian correlation kernel $K(x, y)$, i.e. $K(y, x) = \overline{K(x, y)}$. Assume that $K(x, y)$ defines a trace class operator K on $L^2(t, \infty)$ for each $t \in \mathbb{R}$, and that*

$$(2.35) \quad \text{Tr } K = \int_t^{\infty} K(x, x)d\lambda(x) < \infty.$$

Then ξ has a last particle almost surely and

$$(2.36) \quad \mathbb{P}[x_{\max}(\xi) \leq t] = \det(I - K)_{L^2(t, \infty)}.$$

Proof. This follows from proposition 2.4 and the above discussion provided we can prove (2.34). Since correlation functions are non-negative and $K(y, x) = \overline{K(x, y)}$, the matrix $(K(x_i, x_j))_{1 \leq i, j \leq n}$ is positive definite. In that case Hadamard's inequality says that $\det(K(x_i, x_j))_{1 \leq i, j \leq n} \leq \prod_{j=1}^n K(x_j, x_j)$. Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{C^n}{n!} \int_{(t, \infty)^n} \det(K(x_i, x_j))_{1 \leq i, j \leq n} d^n \lambda(x) &\leq \sum_{n=0}^{\infty} \frac{C^n}{n!} \left(\int_t^{\infty} K(x, x)d\lambda(x) \right)^n \\
&= \exp \left(C \int_t^{\infty} K(x, x)d\lambda(x) \right),
\end{aligned}$$

by (2.35). □

Consider for example the *Airy kernel*,

$$(2.37) \quad A(x, y) = \int_0^{\infty} \text{Ai}(x+t)\text{Ai}(y+t)dt.$$

This is a Hermitian kernel and $\int_t^{\infty} A(x, x)dx < \infty$ for any real t . It can be shown that $A(x, y)\chi_{(t, \infty)}$ is the kernel of a trace class operator, and that there is a point process ξ on \mathbb{R} , the *Airy kernel point process*, which is determinantal with kernel $A(x, y)$. This follows from general theory, see [33]. We have

$$(2.38) \quad F_{\text{TW}}(t) \doteq \mathbb{P}[x_{\max}(\xi) \leq t] = \det(I - A)_{L^2(t, \infty)}.$$

The distribution function $F_{\text{TW}}(t)$ for the last particle in the Airy kernel point process is a natural scaling limit of certain finite determinantal point processes. We call it the *Tracy-Widom distribution*, [36].

We will now look at some general ways of getting finite determinantal point processes. It is possible to get interesting infinite point processes by looking at scaling limits of these. These limiting point processes are typically of a few standard types, e.g. the Airy kernel point process is obtained when we scale around the last particle in some finite point processes on \mathbb{R} . These infinite point processes are natural scaling limits and it is an interesting problem to understand how universal they are. In section 4 we will see the Airy kernel point process arising as a scaling limit of a finite point process associated with a random domino tiling of the so-called Aztec diamond. It also occurs as the scaling limit of GUE around the largest eigenvalue, see example 2.12.

The following determinantal identity, we will call the *generalized Cauchy-Binet identity*. If we take $\Lambda = \{1, \dots, M\}$, λ as counting measure on Λ , $\phi_i(k) = a_{ik}$ and $\psi_i(k) = b_{ik}$, $M \geq N$, we get the classical Cauchy-Binet identity.

Proposition 2.10. *Let $(\Lambda, \mathcal{B}, \lambda)$ be a measure space, and let ϕ_j, ψ_j , $1 \leq i, j \leq N$, be measurable functions such that $\phi_i \psi_j$ is integrable for any i, j . Then,*

$$(2.39) \quad \det \left(\int_{\Lambda} \phi_i(x) \psi_j(x) d\lambda(x) \right)_{1 \leq i, j \leq N} = \frac{1}{N!} \int_{\Lambda^N} \det(\phi_i(x_j))_{1 \leq i, j \leq N} \det(\psi_i(x_j))_{1 \leq i, j \leq N} d^N \lambda(x).$$

Proof. This is a computation,

$$\begin{aligned} & \det \left(\int_{\Lambda} \phi_i(x) \psi_j(x) d\lambda(x) \right)_{1 \leq i, j \leq N} = \int_{\Lambda^N} \det(\phi_i(x_i) \psi_j(x_i))_{1 \leq i, j \leq N} d^N \lambda(x) \\ &= \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_i) \det(\psi_j(x_i))_{1 \leq i, j \leq N} d^N \lambda(x) \\ &= \int_{\Lambda^N} \prod_{i=1}^N \phi_i(x_{\sigma(i)}) \det(\psi_j(x_{\sigma(i)}))_{1 \leq i, j \leq N} d^N \lambda(x) \\ &= \int_{\Lambda^N} \text{sgn}(\sigma) \prod_{i=1}^N \phi_i(x_{\sigma(i)}) \det(\psi_j(x_i))_{1 \leq i, j \leq N} d^N \lambda(x) \\ &= \frac{1}{N!} \int_{\Lambda^N} \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{i=1}^N \phi_i(x_{\sigma(i)}) \det(\psi_j(x_i))_{1 \leq i, j \leq N} d^N \lambda(x). \end{aligned}$$

The first equality follows immediately using the definition of the determinant. In the third we have permuted the variables using an arbitrary permutation $\sigma \in S_N$, and in the fourth equality we used the antisymmetry of the determinant. The last equality follows since the integral is independent of σ . The final expression is exactly what we want by the definition of the determinant. \square

Consider now the measure

$$(2.40) \quad u_N(x) d^N \lambda(x) = \frac{1}{N! Z_N} \det(\phi_i(x_j))_{1 \leq i, j \leq N} \det(\psi_i(x_j))_{1 \leq i, j \leq N} d^N \lambda(x)$$

on Λ^N , where

$$(2.41) \quad Z_N = \frac{1}{N!} \int_{\Lambda^N} \det(\phi_i(x_j))_{1 \leq i, j \leq N} \det(\psi_i(x_j))_{1 \leq i, j \leq N} d^N \lambda(x),$$

and we assume that $Z_N \neq 0$. If $u_N(x) \geq 0$, then (2.40) is a probability measure on Λ^N . It follows from the generalized Cauchy-Binet identity (2.39) that $Z_N = \det A$, where $A = (a_{ij})_{1 \leq i, j \leq N}$, and

$$(2.42) \quad a_{ij} = \int_{\Lambda} \phi_i(x) \psi_j(x) d\lambda(x).$$

Proposition 2.11. *Let $(\Lambda, \mathcal{B}, \lambda)$ be a measure space and let ϕ_i, ψ_i be as in proposition 2.10. Assume that Z_N given by (2.41) is $\neq 0$. Then the matrix A defined by (2.42) is invertible and we can define*

$$(2.43) \quad K_N(x, y) = \sum_{i, j=1}^N \psi_i(x) (A^{-1})_{ij} \phi_j(y).$$

If $g \in L^\infty(X)$, we have the following identity

$$(2.44) \quad \int_{\Lambda^N} \prod_{j=1}^N (1+g(x_j)) u_N(x) d^N \lambda(x) = \sum_{n=0}^N \frac{1}{n!} \int_{\Lambda^n} \prod_{j=1}^n g(x_j) \det(K_N(x_i, x_j))_{1 \leq i, j \leq n} d^n \lambda(x),$$

with $u_N(x)$ given by (2.40).

Proof. That A is invertible follows from the fact that $\det A = Z_N \neq 0$ by (2.39). The proof of (2.44) is based on the determinant expansion

$$(2.45) \quad \det(I + C) = \sum_{n=0}^N \frac{1}{n!} \sum_{i_1, \dots, i_n=1}^N \det(C_{i_r, i_s})_{1 \leq r, s \leq n},$$

where C is an arbitrary $N \times N$ -matrix, and the formula (2.39). The identity (2.45) is a consequence of multilinearity of the determinant and expansion along columns. It follows from (2.39) and (2.42) that

$$(2.46) \quad \begin{aligned} & \int_{\Lambda^N} \prod_{j=1}^N (1 + g(x_j)) u_N(x) d^N \lambda(x) \\ &= \frac{\det(a_{jk} + \int_{\Lambda} \phi_j(x) \psi_k(x) g(x) d\lambda(x))}{\det(a_{jk})} \\ &= \det \left(\delta_{jk} + \sum_{i=1}^N (A^{-1})_{ji} \int_{\Lambda} \phi_i(x) \psi_k(x) g(x) d\lambda(x) \right) \\ &= \det \left(\delta_{jk} \int_{\Lambda} f_j(x) h_k(x) d\lambda(x) \right), \end{aligned}$$

where $f_j(x) = \sum_{i=1}^N (A^{-1})_{ji} \phi_i(x)$ and $h_k(x) = g(x) \psi_k(x)$. Using (2.45) we see that the last expression in (2.46) can be written

$$\begin{aligned} & \sum_{n=0}^N \frac{1}{n!} \sum_{i_1, \dots, i_n=1}^N \det \left(\int_{\Lambda} f_{i_j}(x) h_{i_k}(x) d\lambda(x) \right)_{1 \leq j, k \leq n} \\ &= \sum_{n=0}^N \frac{1}{n!} \sum_{i_1, \dots, i_n=1}^N \frac{1}{n!} \int_{\Lambda^n} \det(f_{i_j}(x_k))_{1 \leq j, k \leq n} \det(h_{i_j}(x_k))_{1 \leq j, k \leq n} d^n \lambda(x) \\ &= \sum_{n=0}^N \frac{1}{n!} \int_{\Lambda^n} \det \left(\sum_{i=1}^N f_i(x_j) h_i(x_k) \right)_{1 \leq j, k \leq n} d^n \lambda(x), \end{aligned}$$

where we have used the identity (2.39) in the two last equalities. Since

$$\sum_{j=1}^N f_j(x) h_j(x) = g(x) \sum_{i,j=1}^N \psi_j(x) (A^{-1})_{ji} \phi_i(y)$$

we are done. \square

Assume now that Λ is a complete separable metric space. If $u_N(x) \geq 0$, then (2.40) is a probability measure on Λ^N and the map $\Lambda^N \ni (x_1, \dots, x_N) \rightarrow \sum_{j=1}^N \delta_{x_j}$ maps this to a point process ξ on Λ . It follows from proposition 2.3 and the identity (2.44) that ξ is a determinantal point process with correlation functions given by (2.43). Although (2.43) gives an explicit formula for the correlation kernel it is rather complicated. In particular, if we want to study a scaling limit as $N \rightarrow \infty$, we have to be able to find the inverse of the $N \times N$ -matrix A in a useful form. Sometimes it is possible to do row operations in the two determinants in (2.40) so that the matrix A becomes diagonal and hence trivial to invert.

Example 2.12. (*The orthogonal polynomial method*). Consider the GUE eigenvalue measure (2.30). The density can be written as

$$(2.47) \quad u_N(x_1, \dots, x_N) = \frac{1}{Z_N N!} \det(x_i^{j-1} e^{-x_i^2/2})_{1 \leq i, j \leq N}^2.$$

If $p_j(x)$ is an arbitrary polynomial of degree j , $j = 0, 1, \dots$, then by doing row operations in the determinant we see that

$$(2.48) \quad u_N(x_1, \dots, x_N) = \frac{1}{Z'_N N!} \det(p_{j-1}(x_i) e^{-x_i^2/2})_{1 \leq i, j \leq N}^2.$$

It now follows from proposition 2.11 that the GUE eigenvalue process has determinantal correlation functions. The elements in the matrix A are given by

$$(2.49) \quad a_{ij} = \int_{\mathbb{R}} p_{i-1}(x) p_{j-1}(x) e^{-x^2} dx.$$

It is clear that it is very natural to choose p_j to be the j th normalized Hermite polynomial so that $a_{ij} = \delta_{ij}$. The correlation kernel is then given by

$$(2.50) \quad K_N(x, y) = \sum_{j=0}^{N-1} p_j(x) p_j(y) e^{-(x^2+y^2)/2}.$$

We obtain the Airy kernel point process in the large N limit when we scale around the largest eigenvalue of an $N \times N$ random matrix from the GUE. More precisely, let $\lambda_1^{(N)} \geq \lambda_2^{(N)} \geq \dots \geq \lambda_N^{(N)}$ be the eigenvalues and set

$$x_j = \frac{\sqrt{2N}\lambda_j^{(N)} - 2N}{N^{1/3}},$$

$j \geq 1$. Under the GUE eigenvalue measure $\xi_N = \sum_{j=1}^N \delta_{x_j}$ becomes a point process and in the limit $N \rightarrow \infty$ this process converges to the Airy kernel point process. The proof is based on the fact that we can investigate the scaling limit of the correlation kernel (2.50) using asymptotics of Hermite polynomials. Also, if $\lambda_{\max}(N)$ ($= \lambda_1^{(N)}$) denotes the largest eigenvalue then proposition 2.9 can be used to show that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\frac{\sqrt{2N}\lambda_{\max}(N) - 2N}{N^{1/3}} \leq t \right] = F_{\text{TW}}(t).$$

From this example, which has several generalizations, we see that orthogonal polynomial asymptotics is important in studying the asymptotics of the eigenvalues in some random matrix ensembles.

2.3. Measures defined by products of several determinants. There is a useful extension of proposition 2.11 to the case when the measure is given by a product of several determinants. Later we will see that such measures arise naturally in interesting problems. Actually both our main models will be of this type. Let X be a complete separable metric space with a Borel measure μ and fix $m, n \geq 1$. Furthermore, let $\phi_{r,r+1} : X \times X \rightarrow \mathbb{C}$, $r = 1, \dots, m-1$ be given measurable *transition functions*, and $\phi_{0,1} : X_0 \times X \rightarrow \mathbb{C}$, $\phi_{m,m+1} : X \times X_{m+1} \rightarrow \mathbb{C}$ given *initial* and *final* transition functions. Here X_0 and X_{m+1} are some given sets, which could be X or $\{1, \dots, n\}$ for example. We will consider measures on $(X^n)^m$ of the form

$$p_{n,m}(\underline{x})d\mu(\underline{x}) = \frac{1}{(n!)^m Z_{n,m}} w_{n,m}(\underline{x})d\mu(\underline{x}) \quad (2.51)$$

$$\doteq \frac{1}{(n!)^m Z_{n,m}} \det(\phi_{0,1}(x_i^0, x_j^1)) \prod_{r=1}^{m-1} \det(\phi_{r,r+1}(x_i^r, x_j^{r+1})) \det(\phi_{m,m+1}(x_i^m, x_j^{m+1})) d\mu(\underline{x}),$$

where $\underline{x} = (x^1, \dots, x^m) \in (X^n)^m$, $x^r = (x_1^r, \dots, x_n^r)$, $d\mu(\underline{x}) = \prod_{r=1}^m \prod_{j=1}^n d\mu(x_j^r)$ and $x^0 \in X_0^n$, $x^{m+1} \in X_{m+1}^n$ are fixed points. Here,

$$Z_{n,m} = \frac{1}{(n!)^m} \int_{(X^n)^m} w_{n,m}(\underline{x})d\mu(\underline{x}) \quad (2.52)$$

and we assume that $Z_{n,m} \neq 0$. If $p_{n,m}(\underline{x}) \geq 0$ we get a probability measure on $(X^n)^m$. Set $\Lambda = \{1, \dots, m\} \times X$. If we map x_j^r to $(r, x_j^r) \in \Lambda$, \underline{x} gives $N = mn$ points in Λ with exactly n points in $\{r\} \times X$ for each r . In this way we get a point process ξ on Λ from the probability measure (2.51). Let ν denote counting measure on $\{1, \dots, m\}$. We will use $\lambda = \nu \times \mu$ as our reference measure on Λ . When $m = 1$ we can take $X_0 = X_{m+1} = \{1, \dots, n\}$, $\phi_{0,1}(i, x) = \phi_i(x)$ and $\phi_{m,m+1}(x, j) = \psi_j(x)$ to obtain the measure (2.40), so the present setting generalizes the one considered above. Our aim is to show that this more general setting also leads to a determinantal process. Variants of this type of setting have been developed in [11], [12], [13] and [21].

Given two transition functions ϕ, ψ we define their composition by $\phi * \psi(x, y) = \int_X \phi(x, z)\psi(z, y)d\mu(z)$. Set

$$\phi_{r,s}(x, y) = (\phi_{r,r+1} * \cdots * \phi_{s-1,s})(x, y),$$

when $r < s$ and $\phi_{r,s} \equiv 0$ if $r \geq s$. We assume that the transition functions are such that all functions $\phi_{r,s}$, $0 \leq r < s \leq m+1$ are well-defined. This will imply that the integral in (2.52) is convergent as can be seen by expanding the determinants. Set $A = (a_{ij})$, where

$$(2.53) \quad a_{i,j} = \phi_{0,m+1}(x_i^0, x_j^{m+1}).$$

Repeated use of the generalized Cauchy-Binet identity (2.39) gives $Z_{n,m} = \det A$. Since we assume that $Z_{n,m} \neq 0$, we see that A is invertible. Set

$$(2.54) \quad K_{n,m}(r, x; s, y) = \tilde{K}_{n,m}(r, x; s, y) - \phi_{r,s}(x, y),$$

where $r, s \in \{1, \dots, m\}$, $x, y \in X$ and

$$(2.55) \quad \tilde{K}_{n,m}(r, x; s, y) = \sum_{i,j=1}^n \phi_{r,m+1}(x, x_i^{m+1})(A^{-1})_{ij}\phi_{0,s}(x_j^0, y).$$

We can now formulate the main result for measures of the form (2.51).

Proposition 2.13. *We use the notation above. Let $g : \Lambda \rightarrow \mathbb{C}$ belong to $L^\infty(\Lambda, \lambda)$ with support in a Borel set $B \subseteq \Lambda$. Let $\psi(r, x; s, y) = \chi_B(x)\phi_{r,s}(x, y)g(s, y)$, $0 \leq r, s \leq m+1$, where we omit $\chi_B(x)$ if $r = 0$ and $g(s, y)$ if $s = m+1$. Assume that ψ defines a trace class operator, also denoted by ψ , on $L^2(\Lambda, \lambda)$ which satisfies $\text{Tr} \psi = \int_\Lambda \psi(z; z)\lambda(z)$. Then*

$$(2.56) \quad \begin{aligned} & \int_{(X^n)^m} \prod_{r=1}^m \prod_{j=1}^n (1 + g(r, x_j^r)) p_{n,m}(\underline{x}) d\mu(\underline{x}) = \det(I + \chi_B K_{n,m} g)_{L^2(\Lambda, \lambda)} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \prod_{j=1}^k g(z_j) \det(K_{n,m}(z_i; z_j))_{1 \leq i, j \leq k} d^k \lambda(z). \end{aligned}$$

It follows from proposition 2.3 that the point process ξ on Λ is determinantal with correlation kernel $K_{n,m}(z_1; z_2)$, $z_1, z_2 \in \Lambda$.

Proof. The kernel $\tilde{K}_{n,m}$ given by (2.55) has finite rank and by assumption ψ is trace class. Hence $\chi_B K_{n,m} g$ is trace class and the Fredholm determinant $\det(I + \chi_B K_{n,m} g)$ is well defined. Since also $\text{Tr}(\chi_B K_{n,m} g) = \int_\Lambda (\chi_B K_{n,m} g)(z; z) d\lambda(z)$, this Fredholm determinant has an expansion as given in the theorem, [14]. Write

$$Z_{n,m}[g] = \frac{1}{(n!)^m} \int_{(X^n)^m} \prod_{r=1}^m \prod_{j=1}^n (1 + g(r, x_j^r)) w_{n,m}(\underline{x}) d\mu(\underline{x}),$$

so that $Z_{n,m}[0] = Z_{n,m} = \det A$. Repeated use of the generalized Cauchy-Binet identity (2.39) gives

$$Z_{n,m}[g] = \det \left(\int_{X^m} \phi_{0,1}(x_i^0, t_1) \prod_{r=1}^m (1 + g(r, t_r)) \prod_{r=1}^{m-1} \phi_{r,r+1}(t_r, t_{r+1}) \phi_{m,m+1}(t_m, x_j^{m+1}) d^m \mu(t) \right)_{1 \leq i, j \leq n}.$$

We can write

$$\prod_{r=1}^m (1 + g(r, t_r)) = 1 + \sum_{\ell=1}^m \sum_{1 \leq r_1 < \dots < r_\ell \leq m} g(r_1, t_{r_1}) \dots g(r_\ell, t_{r_\ell})$$

and thus

$$(2.57) \quad Z_{n,m}[g] = \det \left(a_{ij} + \sum_{\ell=1}^m \sum_{1 \leq r_1 < \dots < r_\ell \leq m} \int_{X^\ell} d^\ell \mu(t) \phi_{0,r_1}(x_i^0, t_1) g(r_1, t_1) \right. \\ \left. \times \prod_{s=1}^{\ell-1} \phi_{r_s, r_{s+1}}(t_s, t_{s+1}) g(r_{s+1}, t_{s+1}) \phi_{r_\ell, m+1}(t_\ell, x_j^{m+1}) \right).$$

By definition $\phi_{r,s} = 0$ if $r \geq s$ and hence we can remove the ordering of the r_i 's in (2.57). We obtain

$$(2.58) \quad \frac{Z_{n,m}[g]}{Z_{n,m}[0]} = \det \left(\delta_{i,j} + \sum_{k=1}^n (A^{-1})_{ik} \sum_{\ell=1}^m \sum_{r_1, \dots, r_\ell}^m \int_{X^\ell} d^\ell \mu(t) \phi_{0,r_1}(x_k^0, t_1) g(r_1, t_1) \right. \\ \left. \times \prod_{s=1}^{\ell-1} \psi(r_s, t_s; r_{s+1}, t_{s+1}) \phi_{r_\ell, m+1}(t_\ell, j) \right) \\ = \det \left(\delta_{i,j} + \sum_{k=1}^n (A^{-1})_{ik} \int_{\Lambda} d\lambda(r, x) \int_{\Lambda} d\lambda(s, y) \phi_{0,r}(x_k^0, x) g(r, x) \right. \\ \left. \times \left(\sum_{\ell}^m \psi^{*(\ell-1)}(r, x; s, y) \right) \phi_{s, m+1}(y, x_j^{m+1}) \right),$$

where $\psi^{*0}(r, x; s, y) = \delta_{r,s} \delta(x - y)$ and recursively

$$\psi^{*\ell}(r, x; s, y) = \int_{\Lambda} \psi(r, x; u, t) \psi^{*(\ell-1)}(u, t; s, y) d\lambda(u, t)$$

for $\ell \geq 1$.

Set

$$b(i; r, x) = \sum_{k=1}^n (A^{-1})_{ik} \phi_{0,r}(x_k^0, x) g(r, x) \\ c(r, x; j) = \int_{\Lambda} \left(\sum_{\ell}^m \psi^{*(\ell-1)}(r, x; s, y) \right) \psi_{s, m+1}(y, x_j^{m+1}) d\lambda(s, y)$$

and let $b : L^2(\Lambda, \lambda) \rightarrow \ell^2(n)$, $c : \ell^2(n) \rightarrow L^2(\Lambda, \lambda)$ denote the corresponding operators. Then, by (2.58),

$$\frac{Z_{n,m}[g]}{Z_{n,m}[0]} = \det(\delta_{ij} + (bc)(i, j))_{1 \leq i, j \leq n} = \det(I + cb)_{L^2(\Lambda, \lambda)}.$$

Now,

$$cb = \left(\sum_{\ell=1}^m \psi^{*(\ell-1)} \right) (\chi_B \tilde{K} g).$$

(The insertion of the χ_B does not change anything.) By assumption ψ is a trace class operator and using $\phi_{r,s} \equiv 0$ if $r \geq s$, we see that it is nilpotent, $\psi^{*\ell} \equiv 0$ if

$\ell \geq m$. Hence $\det(I - \psi) = 1$. Consequently,

$$\begin{aligned} \frac{Z_{n,m}[g]}{Z_{n,m}[0]} &= \det(I - \psi) \det\left(I + \left(\sum_{\ell=0}^{m-1} \psi^{*(\ell)}\right)(\chi_B \tilde{K}g)\right) \\ &= \det(I - \psi + \chi_B \tilde{K}g) = \det(I + \chi_B Kg), \end{aligned}$$

and we are done. \square

3. NON-INTERSECTING PATHS AND THE AZTEC DIAMOND

3.1. Non-intersecting paths and the LGV theorem. A natural way to obtain measures of the form (2.40) and (2.51) is from non-intersecting paths. This is a consequence of the Lindström-Gessel-Viennot theorem in the discrete setting, [35], and the Karlin-McGregor theorem in the case of non-colliding continuous Markov processes in one-dimension. In our applications below we will use the discrete setting so we will concentrate on that.

Let $\mathcal{G} = (V, E)$ be a directed acyclic graph with no multiple edges. A *directed path* π from a vertex u to a vertex v in \mathcal{G} is a sequence of vertices x_1, \dots, x_m in \mathcal{G} such that $x_i x_{i+1}$, the edges in the path, are directed edges in \mathcal{G} , $x_1 = u$ and $x_m = v$. The set of all directed paths from u to v will be denoted by $\Pi(u, v)$. If u_1, \dots, u_n and v_1, \dots, v_n are vertices in \mathcal{G} , then $\Pi(\mathbf{u}, \mathbf{v})$, $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, denotes the set of all directed paths (π_1, \dots, π_n) , where π_i is a directed path from u_i to v_i , $1 \leq i \leq n$. We say that two directed paths *intersect* if they share a common vertex. The families of paths in $\Pi(\mathbf{u}, \mathbf{v})$ that do not have any intersections with each other is denoted by $\Pi_{n.i.}(\mathbf{u}, \mathbf{v})$, and those that have at least one intersection by $\Pi_{w.i.}(\mathbf{u}, \mathbf{v})$. If $\sigma \in S_n$ is a permutation of $\{1, \dots, n\}$ we will write \mathbf{v}_σ for $(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)})$.

Let $w : E \rightarrow \mathbb{C}$ be a given function, called the *weight function*, $w(e)$ is the *weight* of the edge e in \mathcal{G} . The weight of a path is $w(\pi) = \prod_{e \in \pi} w(e)$, i.e. the product of the weights over all edges in the path. The weight of paths (π_1, \dots, π_n) from \mathbf{u} to \mathbf{v} is $w(\pi_1, \dots, \pi_n) = w(\pi_1) \dots w(\pi_n)$. If $S \subseteq \Pi(\mathbf{u}, \mathbf{v})$, then the weight of the set S is

$$(3.1) \quad W(S) = \sum_{(\pi_1, \dots, \pi_n) \in S} w(\pi_1, \dots, \pi_n).$$

The total weight of all paths between two vertices u and v will be denoted by

$$(3.2) \quad \phi(u, v) = W(\Pi(u, v)) = \sum_{p \in \Pi(u, v)} w(p).$$

We will call $\phi(u, v)$ the *transition weight* from u to v . Here we are assuming that the sum in the right hand side of (3.2) is convergent. We could also regard the weights as formal variables in some ring and (3.3) as an identity in that ring.

We can now formulate the Lindström-Gessel-Viennot (LGV) theorem which relates weights of non-intersecting paths and determinants.

Theorem 3.1. *Let \mathcal{G} be a directed, acyclic graph and $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$ two n -tuples of vertices in \mathcal{G} such that $\Pi_{n.i.}(\mathbf{u}, \mathbf{v}_\sigma) \neq \emptyset$ only if $\sigma = id$. Then,*

$$(3.3) \quad W(\Pi_{n.i.}(\mathbf{u}, \mathbf{v})) = \det(\phi(u_i, v_j))_{i,j=1}^n.$$

Proof. ([35]). Expand the determinant in the right hand side of (3.3). By (3.1) and (3.2),

$$\begin{aligned}
\det(\phi(u_i, v_j)) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \phi(u_1, v_{\sigma(1)}) \dots \phi(u_n, v_{\sigma(n)}) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n W(\Pi(u_i, v_{\sigma(i)})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n \left(\sum_{\pi_i \in \Pi(u_i, v_{\sigma(i)})} w(\pi_i) \right) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{\pi \in \Pi(\mathbf{u}, \mathbf{v}_\sigma)} w(\pi_1, \dots, \pi_n) \\
&= \sum_{\sigma \in S_n} \sum_{\pi \in \Pi_{\text{n.i.}}(\mathbf{u}, \mathbf{v}_\sigma)} \operatorname{sgn}(\sigma) w(\pi_1, \dots, \pi_n) + \sum_{\sigma \in S_n} \sum_{\pi \in \Pi_{\text{w.i.}}(\mathbf{u}, \mathbf{v}_\sigma)} \operatorname{sgn}(\sigma) w(\pi_1, \dots, \pi_n) \\
&\doteq S_1 + S_2.
\end{aligned}$$

By assumption $\Pi_{\text{n.i.}}(\mathbf{u}, \mathbf{v}_\sigma) = \emptyset$ unless $\sigma = \text{id}$, and hence $S_1 = W(\Pi_{\text{n.i.}}(\mathbf{u}, \mathbf{v}))$. It remains to show that $S_2 = 0$.

Choose a fixed total order of the vertices, and let ω denote the first vertex in this order which is a point of intersection between the paths π_1, \dots, π_n . Let π_i and π_j be the two paths with smallest indices which intersect ω . Define a map

$$(3.4) \quad (\sigma, \pi_1, \dots, \pi_n) \rightarrow (\sigma', \pi'_1, \dots, \pi'_n)$$

as follows. Set $\pi'_k = \pi_k$ for $k \neq i, j$, and if

$$\begin{aligned}
\pi_i &= u_i x_1 \dots x_\alpha \omega x_{\alpha+1} \dots x_\beta v_{\sigma(i)} \\
\pi_j &= u_j y_1 \dots \gamma \omega y_{\gamma+1} \dots y_\delta v_{\sigma(j)}
\end{aligned}$$

then

$$\begin{aligned}
\pi_i &= u_i x_1 \dots x_\alpha \omega y_{\gamma+1} \dots y_\delta v_{\sigma(j)} \\
\pi_j &= u_j y_1 \dots \gamma \omega x_{\alpha+1} \dots x_\beta v_{\sigma(i)}.
\end{aligned}$$

Also, we set $\sigma' = \sigma \circ (i, j)$, where (i, j) denotes the transposition of i and j . Clearly, $w(\pi_1, \dots, \pi_n) = w(\pi'_1, \dots, \pi'_n)$ and $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma')$.

If we can show that (3.4) is an involution, then $S_2 = 0$ follows since

$$\begin{aligned}
S_2 &= \sum_{\sigma' \in S_n} \sum_{\pi' \in \Pi_{\text{w.i.}}(\mathbf{u}, \mathbf{v}_\sigma)} \operatorname{sgn}(\sigma') w(\pi'_1, \dots, \pi'_n) \\
&= - \sum_{\sigma \in S_n} \sum_{\pi \in \Pi_{\text{w.i.}}(\mathbf{u}, \mathbf{v}_\sigma)} \operatorname{sgn}(\sigma) w(\pi_1, \dots, \pi_n) = -S_2.
\end{aligned}$$

That (3.4) is an involution is clear if (π'_1, \dots, π'_n) has the same first intersection point as (π_1, \dots, π_n) . Since only π_i and π_j were changed a new intersection point has to occur between them. Assume that x_m is the new intersection point which is smallest in the total ordering. We must then have $x_m = x_\ell$, where one of m and ℓ lies in $\{1, \dots, \alpha\}$ and the other in $\{\alpha + 1, \dots, \beta\}$, say m lies in the first set. But then $x_m \dots x_\ell$ in π_i is a cycle, which is impossible since we assumed that \mathcal{G} is acyclic. Hence (3.4) defines an involution. \square

We will see below that a combination of proposition 2.11 or proposition 2.13 with the LGV-theorem will lead us to interesting determinantal point processes in certain models.

3.2. The Aztec diamond. In this section we will discuss random domino tilings of a region called the Aztec diamond. The model can equivalently be thought of as a dimer model on a certain graph, [24]. A typical tiling of the Aztec diamond has the interesting feature that parts of it are completely regular, whereas the central part looks more or less random. In fact there is a well defined random curve which separates the regular regions from the irregular region, and it is this curve that will be our main interest. It turns out that the tiling can be described using non-intersecting paths in a certain directed graph, and these paths will lead to a description of a random tiling by a determinantal point process using the results of the previous section.

The *Aztec diamond*, A_n , of size n is the union of all lattice squares $[m, m+1] \times [\ell, \ell+1]$, $m, \ell \in \mathbb{Z}$, that lie inside the region $\{(x_1, y_1); |x_1| + |y_1| \leq n+1\}$. A *domino* is a closed 1×2 or 2×1 rectangle in \mathbb{R}^2 with corners in \mathbb{Z}^2 , and a *tiling* of A_n by dominos is a set of dominos whose interiors are disjoint and whose union is A_n . Let $\mathcal{T}(A_n)$ denote the set of all domino tilings of the Aztec diamond. The basic coordinate system used here will be referred to as coordinate system I (CS-I).

We can color the unit squares in the Aztec diamond in a checkerboard fashion so that the leftmost square in each row in the top half is white. Depending on how a domino covers the colored squares we can distinguish four types of dominos. A horizontal domino is an *N-domino* if its leftmost square is white, otherwise it is an *S-domino*. Similarly, a vertical domino is a *W-domino* if its upper square is white, otherwise it is an *E-domino*. Two dominos are *adjacent* if they share an edge of a square, and a domino is *adjacent to the boundary* if it shares an edge with ∂A_n . We can now define four regions where the tiling has a regular brick wall pattern. The *north polar region* (NPR) is defined to be the union of those N-dominos that are connected with the boundary by a sequence of adjacent N-dominos, the last one of which is adjacent to the boundary.

Let $T \in \mathcal{T}(A_n)$ be a tiling of the Aztec diamond and let $v(T)$ denote the number of vertical dominos in T . We define the weight of T by letting vertical dominos have weight a and horizontal dominos weight 1, so that the total weight is $a^{v(T)}$. If $a > 0$, which we assume, we get a probability measure on $\mathcal{T}(A_n)$ by normalizing this weight. When $a = 1$ we pick the tiling uniformly at random.

A tiling of the Aztec diamond with dominos can be described by a family of non-intersecting paths. These paths can be obtained by drawing paths on the different types of dominos. On an N-domino we draw no path. On a W-domino placed so that it has corners at $(0, 0)$ and $(1, 2)$ we draw a line from $(0, 1/2)$ to $(1, 3/2)$, and on an E-domino in the same position we draw a line from $(0, 3/2)$ to $(1, 1/2)$. Finally, on an S-domino, placed so that it has its corners at $(0, 0)$ and $(2, 1)$, we draw a line from $(0, 1/2)$ to $(2, 1/2)$. It is straightforward to see that these paths form a family of non-intersecting paths from $A_r = (-n-1+r, -r+1/2)$ to $B_r = (n+1-r, -r+1/2)$, $r = 1, \dots, n$. The top path, from A_1 to B_1 , can be viewed as a function $t \rightarrow X_n(t)$, $|t| \leq n$, and we will call it the *NPR-boundary process*, since the north polar region is exactly the part of the domino tiling that lies completely above $X_n(t)$, see fig.1.

These non-intersecting paths do not immediately, using the LGV-theorem, lead to a measure of the form (2.51). In order to obtain a measure of this form we have to transform the paths. We will only outline how this is done, see [22] for all the details. Introduce a new coordinate system (CS-II) with origin at $(-n, -1/2)$ and

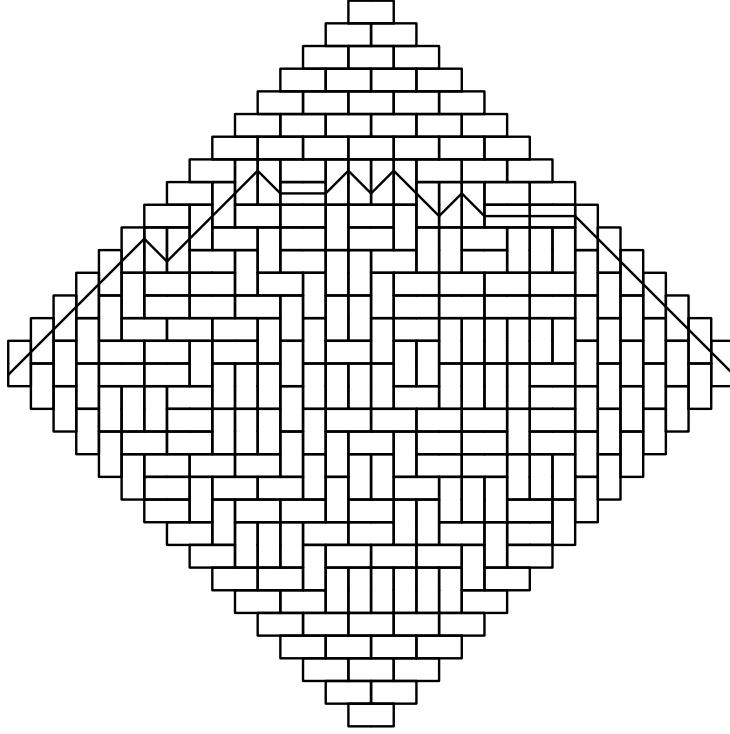


FIGURE 1. An NPR-boundary process.

axes $e_{II} = (1, 1)$, $f_{II} = (-1, 1)$ in CS-I, which gives the coordinate transformation

$$(3.5) \quad \begin{cases} x_1 = x_2 - y_2 - n \\ y_1 = x_2 + y_2 - 1/2. \end{cases}$$

In CS-II the non-intersecting paths go from $A_j = (0, -j+1)$ to $B_j = (n+1-j, -n)$, $1 \leq j \leq n$, and have three types of steps $(1, 0)$, $(0, -1)$ and $(1, -1)$, see fig.2. We can view them as non-intersecting paths in an appropriate directed graph \mathcal{G} . The weight on the domino tiling can be transported to a weight on the non-intersecting paths by letting the steps $(1, 0)$, $(0, -1)$ have weight a and the step $(1, -1)$ weight 1. Take $N \geq n$ and set $A_j = (0, 1-j)$ and $C_j = (n, -n+1-j)$, $1 \leq j \leq N$, see fig. 3. It is not so difficult to see that if π_1, \dots, π_N are non-intersecting paths from A_1, \dots, A_N to C_1, \dots, C_N , then π_k has to go through B_k , $1 \leq k \leq n$. Furthermore the paths from B_k to C_k , $1 \leq k \leq n$, and the paths from B_k to C_k , $n < k \leq N$, only have steps $(1, -1)$. Hence adding the paths from A_k to C_k , $n < k \leq N$, has no effect on the correspondence with domino tilings in the Aztec diamond or the weight, and we can just as well consider this extended system of paths.

Each path π_k from A_k to C_k has a first and a last vertex, which could coincide, on each vertical line $x_2 = k$, $1 \leq k \leq n$. In order to get a measure of the type we want we have to double the vertical lines so that the first and the last vertices on each vertical line ends up on different vertical lines. These first and last vertices will form the point process we are interested in. We can also shift the paths so that the initial and final points, which are fixed, end up at the same height. The

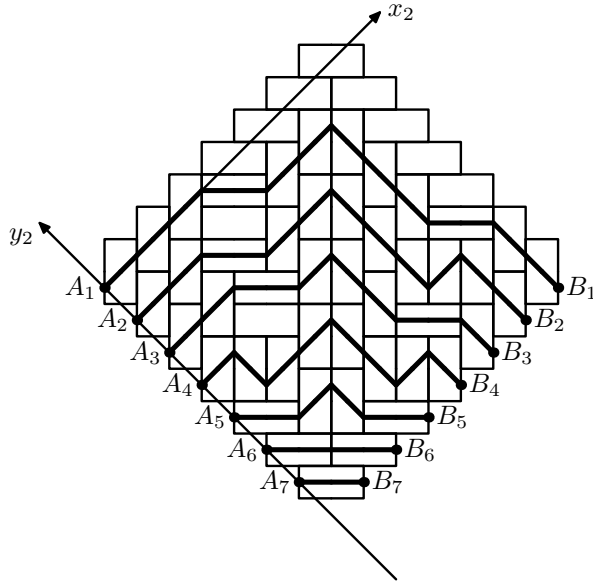


FIGURE 2. CS-II and non-intersecting paths describing the tiling.

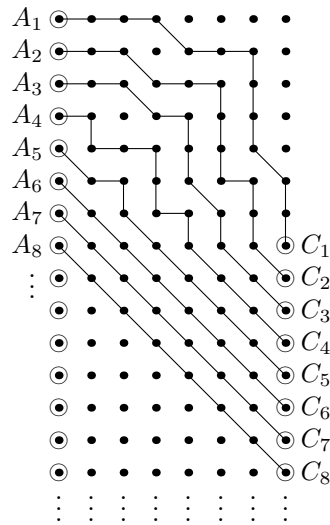


FIGURE 3. The non-intersecting paths in the graph \mathcal{G} .

result can be seen in fig. 4. These non-intersecting paths which connect $(0, 1 - j)$ to $(2n, 1 - j)$, $1 \leq j \leq N$, lie in a new directed graph \mathcal{G}' . The steps from even to odd columns are $(1, 0)$ with weight 1 or $(1, 1)$ with weight a , the steps from odd to even columns are $(1, 0)$ with weight 1, and we also have steps $(0, -1)$ with weight a in the even columns. With this choice of weights we still have a weight preserving bijection with the original domino tiling of A_n . The associated particles, which we

think of as a point process, are indicated in fig. 4. The NPR-boundary process corresponds to the top path in this picture.

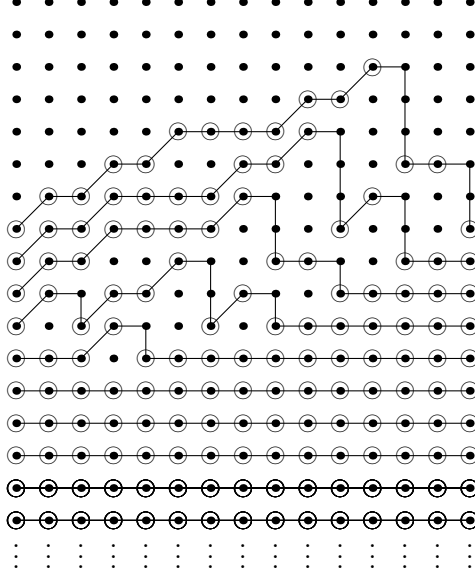


FIGURE 4. The non-intersecting paths in the graph \mathcal{G}' corresponding to the tiling in figure 3.2. The particles in the determinantal process are the circled dots.

The paths π_1, \dots, π_N just described connecting $(0, j-1)$ to $(2n, 1-j)$, $1 \leq j \leq N$, see fig. 4, can be thought of as being built up from $2n$ transition steps. We have points x_1^r, \dots, x_N^r on line r which connect via non-intersecting paths to points $x_1^{r+1}, \dots, x_N^{r+1}$ on line $r+1$. Let $\phi_{r,r+1}(x, y)$ be the transition weight to go from x on the line r to y on the line $r+1$. It follows from the discussion above that

$$(3.6) \quad \phi_{2i,2i+1}(x, y) = \begin{cases} a & \text{if } y - x = 1 \\ 1 & \text{if } y - x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(3.7) \quad \phi_{2i-1,2i}(x, y) = \begin{cases} a^{-(y-x)} & \text{if } y - x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

From the LGV-theorem we see that the weight of all non-intersecting paths from $x^r \in \mathbb{Z}^n$ on line r to $x^{r+1} \in \mathbb{Z}^n$ on line $r+1$ is

$$(3.8) \quad \det(\phi_{r,r+1}(x_i^r, x_j^{r+1}))_{1 \leq i, j \leq n}.$$

Note that the initial and final configurations are fixed $x_i^0 = 1 - i = x_i^{2n}$, $1 \leq i \leq N$. The weight of the whole configuration of non-intersecting paths is then

$$(3.9) \quad \prod_{r=0}^{2n-1} \det(\phi_{r,r+1}(x_i^r, x_j^{r+1}))_{1 \leq i, j \leq n}.$$

and normalizing we obtain a probability measure of the form (2.51). We know that the associated point process with points (r, x_j^r) , $1 \leq r \leq 2n - 1$, $1 \leq j \leq N$, has determinantal correlation functions with correlation kernel given by (2.54). For reasons that will become clear below we will call this correlation kernel the *extended Krawtchouk kernel*. In order to make use of this kernel for asymptotic computations we have to be able to compute the inverse matrix A^{-1} in (2.55) in some way, and obtain a more useful formula. This will be discussed in the beginning of the section 4. The fluctuations of the NPR-boundary process are described by the fluctuations of the particles x_1^1, \dots, x_1^{2n-1} , which are the last particles on each vertical line. If we consider a particular line, say line r , then the points x_1^r, \dots, x_N^r form a determinantal point process and we can obtain the distribution function for x_1^r , the last particle, from proposition 2.9. We will discuss the limit theorem that can be obtained in the section 4.

3.3. Relations to other models. The north polar region can be investigated in a different way, which is related to the corner growth model that will be studied in section 4. This is based on the so called shuffling algorithm, [10], [16], which is an algorithm for generating a random tiling of A_n where vertical tiles have weight a and horizontal tiles weight 1. Here is a description of the shuffling procedure following [16]. For a proof that it actually works, see [10]. The shuffling algorithm generates a random tiling of A_n starting from a random tiling of A_{n-1} . We can tile A_1 by either two vertical dominos, with probability $q = a^2/(1 + a^2)$, or two horizontal dominos, with probability $1 - q = 1/(1 + a^2)$. Assume now that we have generated a random tiling T of A_{n-1} according to the probability measure where the probability of T is proportional to $a^{v(T)}$. Two horizontal dominos sharing a side of length two form a *bad* pair if the lower one is an N-domino and the upper one an S-domino, two vertical dominos sharing a side of length two are a bad pair if the left one is an E-domino and the right one a W-domino. Start by removing all bad pairs in A_{n-1} . Next, move all remaining N-, S-, E- and W-dominos one step up, down, right and left respectively. After these steps what remains to fill A_n are 2×2 -blocks. In the final step we fill each of these 2×2 -blocks with a vertical pair with probability q and a horizontal pair with probability $1 - q$. This procedure will generate a random tiling of A_n where each vertical domino has weight a and horizontal domino weight 1. If we draw the non-intersecting paths in a somewhat different way to what was done above, this shuffling algorithm can be translated into a certain multilayer polynuclear growth (PNG) model, see [20].

How does the north polar region evolve during the shuffling algorithm? It is clear from the description of the algorithm that it can only grow. The growth will be directly related to the so called *corner growth model* which we first define. We will return to this model in section 5. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a *partition*, i.e. λ_i , $i \geq 1$ are non-negative integers, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and there is an $\ell \geq 0$ such that $\lambda_i = 0$ if $i \geq \ell$. The smallest such ℓ is called the *length*, $\ell(\lambda)$, of the partition. We say that λ is a partition of the integer $N = \lambda_1 + \lambda_2 + \dots$, written $\lambda \vdash N$. To the partition λ we associate the following set of integer points in the first quadrant, the *shape* of λ ,

$$(3.10) \quad S(\lambda) = \{(i, j) \in \mathbb{Z}_+^2; 1 \leq i \leq \lambda_j, j \geq 1\}.$$

We can also define the *filled-in shape* of λ

$$(3.11) \quad \overline{S}(\lambda) = S(\lambda) + [-1, 0]^2$$

a subset of $[0, \infty)^2$. The shape $S(\lambda)$ is one way of drawing the Young or Ferrer diagram associated to λ . The set $\overline{S}(\lambda) \cup (\mathbb{R}^2 \setminus (0, \infty)^2)$ has *corners* in certain places. These are positions where you can add a unit square to $\overline{S}(\lambda)$ so that it still corresponds to a partition. Starting with the empty shape corresponding to $\lambda = (0, 0, 0, \dots)$ we grow larger shapes by adding at successive times $1, 2, \dots$ unit squares independently at each corner with probability $1 - q$, where $0 < q < 1$ is fixed. We call this growth model the *corner growth model*, [31], [16],[18]. Let $S_{CG}(n)$ denote the random shape obtained at time n .

Consider now the evolution of the NPR under the shuffling algorithm. Put a point in the center of each N-domino in the NPR in A_n . Viewed from the coordinate system (CS-III) with origin at $(0, n + 3/2)$ and axes $e_{III} = (-1, -1)$, $f_{III} = (1, -1)$ in CS-I, these points form a random shape $S_{Az}(n)$ of some partition λ . Analysis of the shuffling algorithm shows that $S_{CG}(n)$ and $S_{Az}(n)$ are equal in law if $q = a^2/(1 + a^2)$ as above. Hence results for the NPR in the Aztec diamond can be translated into results about the random shape in the corner growth model and vice-versa.

Note that the waiting time before a corner point is added is a geometric random variable starting at 1. Let $G^*(M, N)$ denote the time when the point $(M, N) \in \mathbb{Z}_+^2$ is added to the square. It follows from the corner geometry that before the point (M, N) is added to the shape the points $(M - 1, N)$ and $(M, N - 1)$ must already have been added. Let $w(i, j)$, $i, j \geq 1$ denote independent geometric random variables starting at 0,

$$(3.12) \quad \mathbb{P}[w(i, j) = m] = (1 - q)q^m,$$

$m \geq 0$, and define recursively

$$(3.13) \quad G(M, N) = \max(G(M - 1, N), G(M, N - 1)) + w(M, N).$$

It follows from the lack of memory property of the geometric distribution that

$$(3.14) \quad G^*(M, N) = G(M, N) + M + N - 1.$$

Consequently, we have

$$(3.15) \quad S_{CG}(n) = \{(M, N) \in \mathbb{Z}_+^2; G(M, N) + M + N - 1 \leq n\}.$$

There is a translation of the growth rule (3.13) into a PNG-type model, see [30], [20], [21]. This polynuclear growth model is not the same as the one coming from the shuffling procedure above.

There is also a translation of the corner growth model into a totally asymmetric simple exclusion process (TASEP) in discrete time, [31], [16], [18]. This connection was used in [16] to prove the so called arctic circle theorem. The complement of the four polar regions in a tiling of the Aztec diamond is called the *temperate region*, and is the region where the tiling is more or less random. The *arctic circle theorem* says that, in the case of a uniform random tiling of A_n , the boundary of the temperate region scaled down by n , converges almost surely to a circle. When the weight $a \neq 1$ we get an ellipse instead. This can be translated to a limit of $G(M, N)/N$ as $M, N \rightarrow \infty$, $M/N \rightarrow \text{const} > 0$ (*time constant*). In section 4 we will see how the representation of the random tiling as a determinantal process allows us to get precise information about the fluctuations of the arctic ellipse or, equivalently, of the NPR-boundary process. This then also leads to precise limit theorems for $G(M, N)$.

The quantity $G(M, N)$ also has a *last-passage time* interpretation, which follows from the recursion (3.13). An *up/right path* π from $(1, 1)$ to (M, N) is a sequence of points (i_k, j_k) , $0 \leq k \leq M+N-1$, such that $(i_0, j_0) = (1, 1)$, $(i_{M+N-1}, j_{M+N-1}) = (M, N)$ and $(i_{k+1}, j_{k+1}) - (i_k, j_k) = (1, 0)$ or $(0, 1)$. Viewed from a coordinate system rotated 45° it is a simple random walk path. We have that

$$(3.16) \quad G(M, N) = \max_{\pi} \sum_{(i,j) \in \pi} w(i, j),$$

where the maximum is over all up/right paths from $(1, 1)$ to (M, N) .

There is an interesting limit of $G(N, N)$ as $N \rightarrow \infty$ if we choose the parameter $q = \alpha/N^2$. With this choice of q it is not hard to see, [19], that among the $w(i, j)$, $1 \leq i, j \leq N$, in each row and column there will be at most a single 1, and in the whole square all numbers will be ≤ 1 with probability $\rightarrow 1$ as $N \rightarrow \infty$. In the limit $N \rightarrow \infty$ the set $\{(i, j); 1 \leq i, j \leq N, w(i, j) \geq 1\}$ scaled down by N will converge to a Poisson process in $[0, 1]^2$ with intensity α . The number n of points in $[0, 1]^2$ is a $\text{Po}(\alpha)$ random variable. Let $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$ be the x - and y -coordinates of these points. The coordinates of the points in the Poisson process can then be written $(x_j, y_{\sigma(j)})$, $j = 1, \dots, n$, where $\sigma \in S_n$ is a permutation of $\{1, \dots, n\}$. The uniformity of the Poisson process implies that σ will be a uniform random permutation from S_n . Denote by $L(\alpha)$ the length of the *longest increasing subsequence* in this permutation. We say that $\sigma(i_1), \dots, \sigma(i_\ell)$ is an increasing subsequence in σ if $i_1 < \dots < i_\ell$ and $\sigma(i_1) < \dots < \sigma(i_\ell)$. If we look back at (3.15) we see that we should have

$$(3.17) \quad G(N, N) \rightarrow L(\alpha)$$

in distribution as $N \rightarrow \infty$. Hence, we may be able to use results for random tilings of the Aztec diamond also to study the problem of the distribution of the length of the longest increasing subsequence in a random permutation.

4. ASYMPTOTICS

4.1. Double contour integral formula for the correlation kernel. In order for the formulas (2.54) and (2.55) for the correlation kernel of the determinantal process defined by (2.51) to be useful we have to find a different representation. In particular we need some way of computing the inverse matrix A^{-1} . When A is a Toeplitz matrix it may be possible to do this, at least approximately, by using a Wiener-Hopf factorization of the symbol for A . For the models that we will consider A actually is a Toeplitz matrix and we will be able to find nice formulas.

The space X in (2.51) will now be \mathbb{Z} , and we will also take $X_0 = X_{m+1}$. Hence $\underline{x} \in (\mathbb{Z}^n)^m$. Let $f_r(z)$, $z = e^{i\theta}$, be a function in $L^1(\mathbb{T})$ with Fourier coefficients $\hat{f}_r(n)$, $n \in \mathbb{Z}$. Assume that the transition weights $\phi_{r,r+1}$ in (2.51) are given by

$$(4.1) \quad \phi_{r,r+1}(x, y) = \hat{f}_r(y - x),$$

$0 \leq r \leq m$, $x, y \in \mathbb{Z}$. Then, for $r < s$,

$$(4.2) \quad \phi_{r,s}(x, y) = \hat{f}_{r,s}(y - x),$$

where

$$(4.3) \quad f_{r,s}(z) = \prod_{\ell=r}^{s-1} f_\ell(z).$$

We see that $a_{ij} = \hat{f}_{0,m+1}(x_j^{m+1} - x_i^0) = \hat{f}_{0,m+1}(i - j)$, since $x_j^{m+1} = x_j^0 = 1 - j$. The matrix A is thus a Toeplitz matrix with symbol $a(z) = f_{0,m+1}(z)$, $A = T_n(a)$, where $T_n(a) = (\hat{a}(i - j))_{1 \leq i, j \leq n}$. A computation, [21], shows that

$$(4.4) \quad \sum_{x, y \in \mathbb{Z}} \tilde{K}_{n,m}(r, x; s, y) z^x w^{-y} = \frac{z}{w} f_{r,n+1}\left(\frac{1}{z}\right) f_{0,s}\left(\frac{1}{w}\right) \sum_{i, j=1}^n z^{-i} (T_n^{-1}(a))_{ij} w^j.$$

Let $T(a) = (\hat{a}(i - j))_{i, j \geq 1}$ denote the infinite Toeplitz matrix with symbol a . We say that $a \in L^1(\mathbb{T})$ has a Wiener-Hopf factorization, if it can be written $a = a_+ a_-$ on \mathbb{T} , where $a^+(z) = \sum_{n=0}^{\infty} a_n^+ z^n$, $a^-(z) = \sum_{n=0}^{\infty} a_n^- z^{-n}$, $(a_n^+), (a_n^-) \in \ell^1$. We can extend a_+ to $|z| \leq 1$ and a_- to $\{|z| \geq 1\} \cup \{\infty\}$. We also require that a_+ and a_- have no zeros in these regions and that a has winding number zero with respect to the origin. Also, suppose that

$$(4.5) \quad \sum_{n \in \mathbb{Z}} |n|^\alpha |\hat{a}_n| < \infty,$$

for some $\alpha > 0$. Then, $T_n(a)$ is invertible for n sufficiently large and

$$(4.6) \quad |(T_n^{-1}(a))_{jk} - (T(a_+^{-1})T(a_-^{-1}))_{jk}| \leq C \min((n+1-k)^{-\alpha}, (n+1-j)^{-\alpha})$$

for some constant C , $1 \leq j, k \leq n$.

If f_r is analytic in $1 - \epsilon < |z| < 1 + \epsilon$ for some $\epsilon > 0$, and has a Wiener-Hopf factorization as defined above, then $a(z) = f_{0,m+1}(z)$ will have a Wiener-Hopf factorization and (4.5) will be satisfied. Hence we can use (4.6) to compute the inverse of the Toeplitz matrix. Combined with (4.4) this yields

$$(4.7) \quad \lim_{n \rightarrow \infty} \sum_{x, y \in \mathbb{Z}} \tilde{K}_{n,m}(r, x; s, y) z^x w^{-y} = \frac{z}{z-w} F(z, w),$$

for $1 - \epsilon < |w| < 1 < |z| < 1 + \epsilon$, where

$$(4.8) \quad F(z, w) = \prod_{t=r}^m f_t^-\left(\frac{1}{z}\right) \prod_{t=0}^{s-1} f_t^+\left(\frac{1}{w}\right) \prod_{t=0}^{r-1} f_t^+\left(\frac{1}{z}\right)^{-1} \prod_{t=s}^m f_t^-\left(\frac{1}{w}\right)^{-1}.$$

Hence,

$$(4.9) \quad \begin{aligned} \lim_{n \rightarrow \infty} \tilde{K}_{n,m}(r, x; s, y) &= \tilde{K}_m(r, x; s, y) \\ &\doteq \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{w^y}{z^x} \frac{z}{z-w} F(z, w), \end{aligned}$$

where γ_r is a circle with radius r and center 0, $1 - \epsilon < r_1 < r_2 < 1 + \epsilon$. We obtain

$$(4.10) \quad \lim_{n \rightarrow \infty} K_{n,m}(r, x; s, y) = -\phi_{r,s}(x, y) + \tilde{K}_m(r, x; s, y) \doteq K_m(r, x; s, y).$$

From (4.2) and (4.8) it follows that

$$(4.11) \quad \phi_{r,s}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)\theta} F(e^{i\theta}, e^{i\theta}) d\theta.$$

The integral formula (4.9) gives $K_m(r, x; s, y)$ if $r \geq s$. Using (4.11) and the residue theorem we see that it also gives $K_m(r, x; s, y)$ for $r < s$ if we take $1 - \epsilon < r_2 < r_1 < 1 + \epsilon$. These integral formulas are good representations of the correlation kernel if we want to investigate its asymptotics.

4.2. Asymptotics for the Aztec diamond. In our discussion of the Aztec diamond we saw that adding more paths from A_k to C_k for $k > n$ had no effect on the probability measure since all these paths are fixed. Hence, we can take the limit $N \rightarrow \infty$ in the correlation kernel $K_{N,n}(r, x; s, y)$ coming from the measure (3.9) with transition functions (3.6), (3.7), without changing anything. We get a determinantal process on $2n - 1$ copies of \mathbb{Z} where the configuration is always frozen below the level $-(n - 1)$. We will thus be able to use the formula (4.9). Assume that $0 < a < 1$. The case $a = 1$ can be handled by considering the limit $a \rightarrow 1$ and using continuity. Set $f_{2i}(z) = az + 1$ and $f_{2i+1}(z) = (1 - a/z)^{-1}$. Then (4.1) holds and all the conditions on f_r above are satisfied, $f_{2i}^+(z) = az + 1$, $f_{2i}^-(z) = f_{2i+1}^+(z) = 1$ and $f_{2i+1}^-(z) = (1 - a/z)^{-1}$. Denote the generating function in (4.8) by $F_{n,r,s}(z, w)$ to indicate the dependence on n, r, s , $0 < r, s < 2n$. We obtain,

$$(4.12) \quad F_{n,2r-\epsilon_1,2s-\epsilon_2}(z, w) = \frac{(1 - aw)^{n-s+\epsilon_2}(1 + a/w)^s}{(1 - az)^{n-r+\epsilon_1}(1 + a/z)^r},$$

with $\epsilon_i \in \{0, 1\}$. The Aztec diamond particle process is a determinantal process with kernel

$$(4.13) \quad K_{\text{Kr},n}(r, x; s, y) = -\phi_{r,s}(x, y) + \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{w^y}{z^x} \frac{z}{z - w} F_{n,r,s}(z, w),$$

where $a < r_2 < 1/a$, $0 < r_1 < r_2$. Here, by (4.11),

$$(4.14) \quad \phi_{r,s}(x, y) = \frac{1}{(2\pi i)} \int_{\gamma_r} \frac{dz}{z} z^{y-x} F_{n,r,s}(z, z)$$

We will call the kernel (4.13) the *extended Krawtchouk kernel*. The reason for this name is that it can be expressed in terms of Krawtchouk polynomials, see [22] for all the details. Here we will only consider the case $r = s$. Let $p_k(x; q, n)$ be the normalized *Krawtchouk polynomial*, i.e. it is a polynomial of degree k satisfying the orthogonality condition

$$\sum_{x=0}^n p_j(x; q, n) p_k(x; q, n) \binom{n}{x} q^x (1 - q)^{n-x} = \delta_{jk},$$

$0 \leq j, k \leq n$, on $\{0, \dots, n\}$. Define the *Krawtchouk kernel*,

$$(4.15) \quad K_{\text{Kr},n,r,q}(x, y) = \sum_{k=0}^{r-1} p_k(x; q, n) p_k(y; q, n) \left[\binom{n}{x} q^x (1 - q)^{n-x} \binom{n}{y} q^y (1 - q)^{n-y} \right]^{1/2}.$$

It can then be shown, [22], using the contour integral formula for $p_j(x; q, n)$ that

$$(4.16) \quad K_{\text{Kr},n}(2(n - r) + 1, x - r + 1; 2(n - r) + 1, y - r + 1) = K_{\text{Kr},n,r,q}(x, y),$$

where $q = a^2/(1 + a^2)$.

Note the similarity between the Krawtchouk kernel and the Hermite kernel (2.50). The same argument that showed that the point process defined by (2.30) has determinantal correlation functions with kernel (2.50) shows that the probability measure, the *Krawtchouk ensemble*,

$$(4.17) \quad u_{n,r}(x_1, \dots, x_r) = \frac{1}{Z_{n,r}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^2 \prod_{j=1}^n \binom{n}{x_j} q^{x_j} (1 - q)^{n-x_j}$$

on $\{0, \dots, n\}^r$ defines a determinantal process on $\{0, \dots, n\}$ with correlation kernel (4.15). Probability measures of the form (2.30) and (4.17), i.e. a Vandermonde determinant squared times a product of single particle weights are called *orthogonal polynomial ensembles* and go back to the early work of Gaudin and Mehta, [26]. The Krawtchouk ensemble is an example of a discrete orthogonal polynomial ensemble, [19].

The NPR-boundary process is related to the top particles $x_{\max}^r = \max x_j^r$ in the Aztec diamond particle process. If we look first at a single line r we see that x_{\max}^r is the last particle in a point process on \mathbb{Z} given by the x_i^r . It follows from proposition 2.9 and (4.16) that

$$(4.18) \quad \mathbb{P}[x_{\max}^{2(n-r)+1} \leq t - r + 1] = \det(I - K_{\text{Kr}, r, n, q} \chi_{(t, \infty)})_{\ell^2(\mathbb{Z})}$$

for any $t \in \mathbb{R}$. Looking at the geometry of the Aztec diamond and how the non-intersecting paths were defined we see that the NPR-boundary process $X_n(t)$ is obtained by joining the points (in CS-I) $Q(j) = (2j - x_{\max}^{2j} - n, x_{\max}^{2j} - 1/2)$, $P(j) = (2j - x_{\max}^{2j} - n, x_{\max}^{2j-1} - 1/2)$, $1 \leq j \leq n$ with straight lines. Hence, (4.18) can be used to investigate $X_n(t)$.

We see that asymptotics for the Krawtchouk kernel will give us asymptotics for the NPR-boundary process. The relation between the NPR-boundary process and the corner growth model discussed above yields

$$(4.19) \quad \mathbb{P}[G^*(M, N) \leq n] = \mathbb{P}_{\text{Kr}, n}[x_{\max}^{2(n-M)+1} \leq n + 1 - M - N].$$

Combining (4.18) and (4.19) and using (3.14) we find

$$(4.20) \quad \mathbb{P}[G(M, N) \leq t] = \det(I - K_{\text{Kr}, M, t+M+N-1, q})_{\ell^2(\{t+M+1, t+M+2, \dots\})}$$

$t \in \mathbb{Z}$, see also [20].

We will not discuss the asymptotic analysis in any detail, but only give the results and outline the main structure of the proofs. From (4.13), (4.16) and Cauchy's theorem we see that the Krawtchouk kernel has the representation

$$(4.21) \quad K_{\text{Kr}, r, n, q}(x, y) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \frac{dz}{z} \int_{\gamma} \frac{dw}{w} \frac{z}{z-w} \frac{z^{n-x}}{w^{n-y}} \frac{(1-aw)^r (w+a)^{n-r+1}}{(1-az)^r (z+a)^{n-r+1}},$$

where Γ is given by $t \rightarrow \alpha_2 + it$, $t \in \mathbb{R}$ and γ is a circle with radius α_1 centered at the origin, $0 < \alpha_1 < \alpha_2 < 1/a$.

We want to show that close to the rightmost particle the Krawtchouk kernel, appropriately scaled, converges to the Airy kernel given by (2.37) so that in the scaling limit close to the edge of the Krawtchouk ensemble we get the Airy kernel point process. The last particle in the Krawtchouk ensemble, and hence the last particle in the Aztec diamond point process restricted to a line, will in the limit fluctuate according to the Tracy-Widom distribution. In order to show this it is useful to have a double contour integral representation of the Airy kernel. Using the formula

$$(4.22) \quad \text{Ai}(x) = \frac{1}{2\pi} \int_{\text{Im } z = \eta} e^{iz^3/3 + ixz} dz,$$

with $\eta > 0$, for the Airy function in (2.37) we obtain

$$(4.23) \quad A(x, y) = \frac{1}{(2\pi i)^2} \int_{\text{Im } z = \eta} dz \int_{\text{Im } w = \eta} dw \frac{e^{ixz + iyw + i(z^3 + w^3)/3}}{i(z+w)}.$$

In discussing asymptotic results for the Krawtchouk kernel we will for simplicity only consider the case $a = 1$ and the part of the NPR-boundary that lies above a neighbourhood of $x_1 = 0$.

Theorem 4.1. *Set $\beta = 2^{-3/2}(\sqrt{2} + 1)$ and $\gamma = 2^{-3/2}(\sqrt{2} - 1)$. If $r = \gamma n$, $x = \beta n + 2^{-5/6}n^{1/3}\xi$ and $y = \beta n + 2^{-5/6}n^{1/3}\eta$, then*

$$(4.24) \quad 2^{-5/6}n^{1/3}(\sqrt{2} - 1)^{x-y}K_{Kr,r,n,1/2}(x, y) \rightarrow A(\xi, \eta)$$

as $n \rightarrow \infty$, uniformly for ξ, η in a compact set in \mathbb{R} . Also, if X_{\max} denotes the rightmost particle in the Krawtchouk ensemble with $q = 1/2$, $r = \gamma n$, then

$$(4.25) \quad \mathbb{P}[X_{\max} \leq \beta n + 2^{-5/6}n^{1/3}\xi] \rightarrow F_{TW}(\xi)$$

as $N \rightarrow \infty$ for every ξ .

Proof. (Sketch) [22]. The integral (4.21) can be written ($a = 1$, which corresponds to $q = 1/2$),

$$(4.26) \quad K_{Kr,r,n,1/2}(x, y) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \frac{dz}{z} \int_{\gamma} \frac{dw}{w} \frac{z}{z-w} e^{nf(z)-nf(w)} \frac{g(z)}{g(w)},$$

where $e^{nf(z)}$ represents the leading order behaviour of the integrand and

$$f(z) = (1 - \beta) \log z - \gamma \log(1 - z) - (1 - \gamma) \log(z + 1).$$

We want to apply a steepest descent argument to this integral. The saddle point condition $f'(z) = 0$ gives $\beta z^2 + (2\gamma - 1)z + 1 - \beta = 0$, which has a double root if $(\gamma - 1/2)^2 + \beta^2 - \beta = 0$. This can also be written,

$$(4.27) \quad (1 - \beta - \gamma)^2 + (\beta - \gamma)^2 = 1/2.$$

We see that the β, γ in the theorem satisfy (4.27) and hence we have an Airy-type steepest descent problem in (4.26). If we carry out the argument in detail we find that

$$\begin{aligned} & \frac{2^{-5/6}n^{1/3}(\sqrt{2} - 1)^{x-y}}{(2\pi i)^2} \int_{\Gamma} \frac{dz}{z} \int_{\gamma} \frac{dw}{w} \frac{z}{z-w} e^{nf(z)-nf(w)} \frac{g(z)}{g(w)} \\ & \rightarrow \frac{1}{(2\pi i)^2} \int_{\text{Im } z = \eta} dz \int_{\text{Im } w = \eta} dw \frac{e^{ixz + iyw + i(z^3 + w^3)/3}}{i(z + w)} \end{aligned}$$

as $N \rightarrow \infty$. Together with (4.23) this gives (4.24).

We can also use (4.26) to derive estimates of the Krawtchouk kernel. By proposition 2.9, the left hand side of (4.25) can be written as a Fredholm expansion, and (4.24) together with the estimates can be used to show that this Fredholm expansion converges to the Fredholm expansion for the TW-distribution (2.38). \square

The equation (4.27) is actually the equation for the arctic circle. If we look at the relation (4.18) between $x_{\max}^{2(n-r)+1}$ and the position of the last particle in the Krawtchouk ensemble, and translate this back to our original coordinate system (CS-I) we obtain $x_1^2 + y_1^2 = 1/2$, which is the equation for the arctic circle. If we instead of $a = 1$ we took $0 < a < 1$, the equation for the arctic ellipse is $x_1^2/p + y_1^2/q = 1$, $q = a^2/(1 + a^2)$, $p = 1 - q$, which can be obtained similarly.

From (4.18) and the relation between x_{\max}^r and the NPR-boundary process we obtain

$$(4.28) \quad \mathbb{P}[X_n(0) \leq n/\sqrt{2} + 2^{-5/6}n^{1/3}\xi] \rightarrow F_{TW}(\xi)$$

as $n \rightarrow \infty$. We will consider a generalization of (4.28) in theorem 4.3.

From (4.26) and (4.20) (up to some technical details) we also obtain, in the case $q = 1/2$,

$$(4.29) \quad \lim_{N \rightarrow \infty} \mathbb{P}[G(N, N) \leq 2(\sqrt{2} + 1)N + 2^{1/6}(\sqrt{2} + 1)^{4/3}N^{1/3}\xi] = F_{\text{TW}}(\xi).$$

We will discuss a more general result in section 5. The full result theorem 5.3 can also be proved starting from (4.20), (4.21) and (4.23).

Since we have the formula (4.13) for the extended Krawtchouk kernel we should also be able to derive simultaneous distributions for the last particles x_{\max}^r for several lines, and hence study the convergence of the NPR-boundary process to a limiting stochastic process. Using (4.16) the limit (4.24) can be translated into a limit formula for $K_{K_r, n}(2r + 1, x; 2s + 1, y)$ when $r = s$. We now want to generalize this to the case $r \neq s$. The limiting correlation kernel will be the so called *extended Airy kernel*. It is defined by

$$(4.30) \quad A(\tau, \xi; \sigma, \eta) = \begin{cases} \int_0^\infty e^{-\lambda(\tau-\sigma)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & \text{if } \tau \geq \sigma \\ -\int_{-\infty}^0 e^{-\lambda(\tau-\sigma)} \text{Ai}(\xi + \lambda) \text{Ai}(\eta + \lambda) d\lambda, & \text{if } \tau < \sigma. \end{cases}$$

Note that $A(\tau, \xi; \tau, \eta) = A(\xi, \eta)$. Using (4.22) it is again possible to rewrite this as a double contour integral. In fact,

$$(4.31) \quad A(\tau, x; \sigma, y) = \frac{1}{(2\pi i)^2} \int_{\text{Im } z = \eta} dz \int_{\text{Im } w = \eta} dw \frac{e^{ixz + iyw + i(z^3 + w^3)/3}}{\sigma - \tau + i(z + w)}.$$

where $\eta > 0$ and $2\eta + \tau - \sigma < 0$ in the case $\sigma > \tau$.

It can be shown, [30], [21], that there is a stochastic process, the *Airy process*, $\tau \rightarrow \mathcal{A}(\tau)$ with continuous sample paths almost surely such that

$$(4.32) \quad \mathbb{P}[\mathcal{A}(\tau_1) \leq \xi_1, \dots, \mathcal{A}(\tau_m) \leq \xi_m] = \det(I - fAf)_{L^2(\{\tau_1, \dots, \tau_m\} \times \mathbb{R})},$$

where A is the extended Airy kernel and $f(\tau_j, x) = \chi_{(\xi_j, \infty)}(x)$, $1 \leq j \leq m$.

We can now show the following theorem. The proof is similar to that of theorem 4.1 but somewhat more involved. It is based on (4.13) and (4.31), see [22].

Theorem 4.2. *Define the rescaled variables ξ, η, τ, σ by $2r = n(1 + 1/\sqrt{2}) + 2^{-1/6}\tau n^{2/3}$, $2s = n(1 + 1/\sqrt{2}) + 2^{-1/6}\sigma n^{2/3}$, $x = n/\sqrt{2} + 2^{-5/6}(\xi - \tau^2)n^{1/3}$ and $y = n/\sqrt{2} + 2^{-5/6}(\eta - \tau^2)n^{1/3}$. Take $a = 1$. Then,*

$$(4.33) \quad \lim_{n \rightarrow \infty} (\sqrt{2})^{x-y+2(s-r)} e^{\xi\tau - \eta\sigma - \tau^3/3 + \sigma^3/3} K_{K_r, n}(2r, x; 2s, y) = A(\tau, \xi; \sigma, \eta),$$

uniformly for ξ, η, τ, σ in compact sets.

We can now give a theorem that says that the appropriately rescaled NPR-boundary process converges to the Airy process.

Theorem 4.3. *Let $X_n(t)$ be the NPR-boundary process and $\mathcal{A}(\tau)$ the Airy process, and let the weight a for vertical dominos = 1, so that we have a uniform random tiling of the Aztec diamond. Then,*

$$(4.34) \quad \frac{X_n(2^{-1/6}n^{2/3}\tau) - n/\sqrt{2}}{2^{-5/6}n^{1/3}} \rightarrow \mathcal{A}(\tau) - \tau^2,$$

as $n \rightarrow \infty$, in the sense of convergence of finite-dimensional distributions.

Proof. (Sketch), [22]. The joint distribution of the left hand side of (4.34) for different times τ can be expressed in terms of the joint distribution of $\max_{k \geq 1} x_k^r$ with appropriate r . The fact that x_k^r form a determinantal process whose kernel is the extended Krawtchouk kernel gives that the joint distribution is a Fredholm determinant involving this kernel. The limit (4.33) and some estimates can be used to show that this Fredholm determinant converges to a Fredholm determinant like (4.32) involving the extended Airy kernel. This Fredholm determinant will give the joint distribution of the right hand side of (4.34). \square

4.3. Asymptotics for random permutations. As above we let $L(\alpha)$ denote the length of the longest increasing subsequence in a uniform random permutation from S_N where N is an independent Poisson(α) random variable. From (3.17) we know that $G(N, N) \rightarrow L(\alpha)$ in distribution as $N \rightarrow \infty$ if $q = \alpha/N^2$. Thus by (4.20),

$$(4.35) \quad \begin{aligned} \mathbb{P}[L(\alpha) \leq n] &= \lim_{N \rightarrow \infty} \mathbb{P}[G(N, N) \leq n] \\ &= \lim_{N \rightarrow \infty} \det(I - K_{\text{Kr}, N, n+2N-1, \alpha/N^2})_{\ell^2(\{n+N, n+N+1, \dots\})}. \end{aligned}$$

Let a_N be given by $a_N^2/(1+a_N^2) = \alpha/N^2$, so that essentially $a_N = \sqrt{\alpha}/N$. Then, by (4.21),

$$\begin{aligned} &K_{\text{Kr}, N, n+2N-1, \alpha/N^2}(x+N, y+N) \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{z}{z-w} \frac{w^{y+1}}{z^{x+1}} \frac{(1-a_N w)^N (1+a_N/w)^{N+n}}{(1-a_N z)^N (1+a_N/z)^{N+n}}, \end{aligned}$$

where $a_N < r_1 < r_2 < 1/a_N$. Here we can let $N \rightarrow \infty$ and obtain

$$(4.36) \quad \begin{aligned} &\lim_{N \rightarrow \infty} K_{\text{Kr}, N, n+2N-1, \alpha/N^2}(x+N, y+N) \\ &= \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{1}{1-w/z} \frac{w^{y+1}}{z^{x+1}} e^{-\sqrt{\alpha}(w-1/w) + \sqrt{\alpha}(z-1/z)} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k+y+1)\theta - 2\sqrt{\alpha}i \sin \theta} d\theta \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k+x+1)\theta + 2\sqrt{\alpha}i \sin \theta} d\theta \right) \\ &= \sum_{k=1}^{\infty} J_{x+k}(2\sqrt{\alpha}) J_{y+k}(2\sqrt{\alpha}) \doteq B^\alpha(x, y). \end{aligned}$$

The second equality follows by expanding $(1-w/z)^{-1}$ in a geometric series. The kernel $B^\alpha(x, y)$ on $\ell^2(\mathbb{Z})$ is called the *discrete Bessel kernel*. The limits (4.35) and (4.36) and some estimates of the Krawtchouk and discrete Bessel kernels now yield the following theorem, [5], [19].

Theorem 4.4. *Let N be a Poisson(α) random variable and pick independently a permutation σ from S_N with the uniform distribution. Denote by $L(\alpha)$ the length of the longest increasing subsequence in σ . Then,*

$$(4.37) \quad \mathbb{P}[L(\alpha) \leq n] = \det(I - B^\alpha)_{\ell^2(\{n, n+1, \dots\})}.$$

The formula (4.37) can be used to prove a limit theorem for $L(\alpha)$. This theorem was first proved in [1] using a completely different approach.

Theorem 4.5. *With $L(\alpha)$ as in the previous theorem we have*

$$(4.38) \quad \mathbb{P}\left[\frac{L(\alpha) - 2\sqrt{\alpha}}{\alpha^{1/6}} \leq t\right] = F_{TW}(t)$$

as $\alpha \rightarrow \infty$.

Proof. (Sketch). The Bessel functions have the following asymptotics

$$(4.39) \quad \alpha^{1/6} J_{2\sqrt{\alpha} + \xi\alpha^{1/6}}(2\sqrt{\alpha}) \rightarrow \text{Ai}(\xi)$$

uniformly for ξ in a compact interval as $\alpha \rightarrow \infty$. This, together with appropriate estimates, gives

$$(4.40) \quad \alpha^{1/6} B^\alpha(2\sqrt{\alpha} + \xi\alpha^{1/6}, 2\sqrt{\alpha} + \eta\alpha^{1/6}) \rightarrow \int_0^\infty \text{Ai}(\xi + t)\text{Ai}(\eta + t)dt = A(\xi, \eta)$$

as $\alpha \rightarrow \infty$. Thus, by (4.37),

$$(4.41) \quad \lim_{\alpha \rightarrow \infty} \mathbb{P}\left[\frac{L(\alpha) - 2\sqrt{\alpha}}{\alpha^{1/6}} \leq t\right] = \det(I - A)_{L^2(t, \infty)} = F_{TW}(t).$$

□

If $\ell_N(\sigma)$ denotes the length of the longest increasing subsequence in a uniform random permutation from S_N , a de-Poissonization argument, [17], [1], makes it possible to deduce

$$(4.42) \quad \lim_{N \rightarrow \infty} \mathbb{P}\left[\frac{\ell_N(\sigma) - 2\sqrt{N}}{N^{1/6}} \leq t\right] = F_{TW}(t),$$

from the previous theorem.

5. THE CORNER GROWTH MODEL

5.1. Mapping to non-intersecting paths. In this section we will give another approach to the corner growth model by mapping it to non-intersecting paths in a different way than that related to the Aztec diamond.

Consider a *right/down path* λ from $(0, L)$ to $(K, 0)$, i.e. a sequence of points in \mathbb{Z}^2 , $P_j(\lambda) = (x_j, y_j)$, $j = 0, \dots, K + L$, such that $P_0(\lambda) = (0, L)$, $P_{K+L}(\lambda) = (K, 0)$ and $P_{j+1}(\lambda) - P_j(\lambda) = (1, 0)$ or $(0, -1)$. We use the same notation, λ , as for a partition since there is a unique associated partition, $\lambda_k = \max\{x_j; y_j = k - 1\}$. Note that $\lambda_1 = K$ and $\ell(\lambda) = L$.

Let $w(i, j)$, $(i, j) \in \mathbb{Z}_+^2$, be independent geometric random variables with parameter $a_i b_j$, $0 \leq a_i, b_j < 1$,

$$(5.1) \quad \mathbb{P}[w(i, j) = m] = (1 - a_i b_j)(a_i b_j)^m,$$

$m \geq 0$. Also, we set $w(0, j) = w(j, 0) = 0$, $j \geq 0$. Define as previously, (3.13),

$$(5.2) \quad G(i, j) = \max(G(i - 1, j), G(i, j - 1)) + w(i, j),$$

$(i, j) \in \mathbb{Z}_+^2$, where $G(i, 0) = G(0, i) = 0$, $i \geq 0$.

Given a partition, or right/down path λ , set $W(\lambda) = (w(i, j))_{(i, j) \in S(\lambda)}$, where $S(\lambda)$ is the shape of λ as defined previously. If $\lambda = (M, \dots, M, 0, \dots)$ with $\ell(\lambda) = N$, then $W(\lambda)$ is the $M \times N$ -matrix $(w(i, j))_{1 \leq i \leq M, 1 \leq j \leq N}$. We want to map $W(\lambda)$ to a family of non-intersecting paths in a weighted graph $\mathcal{G}(\lambda)$, in such a way that the top path gives the values $(G(P_j(\lambda)))_{j=0}^{K+L}$ of $G(i, j)$ along the right/down path.

Corresponding to (5.1) we define the *weight* of $W(\lambda)$ by

$$(5.3) \quad \prod_{(i,j) \in S(\lambda)} (a_i b_j)^{w(i,j)}.$$

We also want the mapping to the non-intersecting paths to be weight-preserving so that we can use the paths to study $G(M, N)$.

The directed graph $\mathcal{G}(\lambda)$ is defined as follows. The vertices are $\{-L, -L+1, \dots, K\} \times \mathbb{Z}$, and the undirected edges connect $(i, j), (i+1, j)$ for $i = -L, \dots, K-1, j \in \mathbb{Z}$ (horizontal edges) and $(i, j), (i, j+1)$ for $i = -L+1, \dots, K, j \in \mathbb{Z}$ (vertical edges). The step $P_j(\lambda)P_{j+1}(\lambda)$ in the right/down path λ is a *right-step* if $P_j(\lambda) = (i-1, x)$ and $P_{j+1}(\lambda) = (i, x)$. In that case the vertical edges with first coordinate $-L+j+1$ are directed from $(-L+j+1, k)$ to $(-L+j+1, k+1), k \in \mathbb{Z}$, i.e upwards, and are given the weight a_i (*up-edges*). The step $P_j(\lambda)P_{j+1}(\lambda)$ is a *down-step* if $P_j(\lambda) = (x, i)$ and $P_{j+1}(\lambda) = (x, i-1)$. In that case the vertical edges with first coordinate $-L+j+1$ are directed from $(-L+j+1, k)$ to $(-L+j+1, k-1), k \in \mathbb{Z}$, i.e downwards, and are given the weight b_i (*down-edges*). All horizontal edges are directed to the right.

To a path π in $\mathcal{G}(\lambda)$ from $(-L, 1-j)$ to $(K, 1-j)$, for some $j \geq 1$, we can associate points $Q_i(\pi) = (i-L, x_i), 0 \leq i \leq K+L$. We let $Q_i(\pi)$ be the last vertex in the directed path with first coordinate $-L+i$. If the edges on the vertical line $x = -L+i$ are up-edges then $x_i \geq x_{i-1}$, if they are down-edges, then $x_i \leq x_{i-1}$. We have $x_0 = x_{L+L} = 1-j$.

We can now formulate a theorem which gives the mapping from $W(\lambda)$ to non-intersecting paths in $\mathcal{G}(\lambda)$. To our knowledge this theorem has not appeared in its present form in the literature so we will give a proof in section 5. For an investigation of the behaviour along right/down paths in the Poissonian case see [6].

Theorem 5.1. *Let λ be a partition giving a right/down path $(P_j(\lambda))_{j=0}^{K+L}, L = \ell(\lambda), K = \lambda_1$, from $(0, L)$ to $(K, 0)$. There is a one-to-one weight preserving mapping from $W(\lambda)$ with weight (5.3) to non-intersecting paths (π_1, π_2, \dots) in the weighted directed graph $\mathcal{G}(\lambda)$, where π_j goes from $(-L, 1-j)$ to $(K, 1-j), j \geq 1$. The path π_j consists only of horizontal edges if $j \geq \min(K, L)$. If $Q_i(\pi_1) = (i-L, x_i), 0 \leq i \leq K+L$, are the points associated to the top path π_1 , then $x_i = G(P_i(\lambda))$.*

To the paths π_1, π_2, \dots we can associate a point configuration $(r, x_j^r), -L \leq r \leq K, j \geq 1$, by letting $Q_i(\pi_j) = (i-L, x_j^{i-L})$. Note that $x_j^{-L} = x_j^K = 1-j, j \geq 1$ are fixed. Also, $x_j^r = 1-j, -L \leq r \leq K$, if $j > \min(K, L)$. Hence theorem 5.1 maps $W(\lambda)$ with probability measure (5.1) to a point process in $\{-L+1, \dots, K-1\} \times \mathbb{Z}$. By the general formalism presented above this will be a determinantal point process.

Theorem 5.2. *Let λ be a partition, a right/down path from $(L, 0)$ to $(0, K)$, $L = \ell(\lambda), K = \lambda_1$. The probability measure (5.1) on $W(\lambda)$ can be mapped to a determinantal point process on $\{-L+1, \dots, K-1\} \times \mathbb{Z}$. Set*

$$(5.4) \quad f_r^+(z) = \frac{1-a_i}{1-a_i z} \quad , \quad f_r^-(z) = 1$$

if the edges on $x = r+1$ are up-edges with weight a_i , and

$$(5.5) \quad f_r^-(z) = 1 \quad , \quad f_r^+(z) = \frac{1-b_i}{1-b_i/z}$$

if the edges on $x = r + 1$ are down-edges with weight b_i , $-L \leq r < K$. The correlation kernel is then given by

$$(5.6) \quad K_\lambda(r, x; s, y) = -\phi_{r,s}(x, y) + \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{w^y}{z^x} \frac{z}{z-w} F_\lambda(z, w),$$

where $\max(b_i) < r_1 < r_2 < \min(1/a_i)$,

$$(5.7) \quad F_\lambda(z, w) = \prod_{t=r}^{K-1} f_t^-\left(\frac{1}{z}\right) \prod_{t=-L}^{s-1} f_t^+\left(\frac{1}{w}\right) \prod_{t=-L}^{r-1} f_t^+\left(\frac{1}{z}\right)^{-1} \prod_{t=s}^{K-1} f_t^-\left(\frac{1}{w}\right)^{-1},$$

$\phi_{r,s} \equiv 0$ if $r \geq s$ and

$$(5.8) \quad \phi_{r,s}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(y-x)} F_\lambda(e^{i\theta}, e^{i\theta}) d\theta$$

if $r < s$.

Proof. (Sketch) The transition weight to go from (r, x) to $(r + 1, y)$ in $\mathcal{G}(\lambda)$ is

$$(5.9) \quad \phi_{r,r+1}(x, y) = \begin{cases} a_i^{y-x} & \text{if } y \geq x \\ 0 & \text{if } y < x \end{cases}$$

if the vertical edges on the line $r + 1$ are up-edges with weight a_i , and

$$(5.10) \quad \phi_{r,r+1}(x, y) = \begin{cases} b_i^{x-y} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

if they are down-edges with weight b_i . It follows from theorem 5.1 and the LGV-theorem that the probability measure on the point configuration is given by (2.51). Hence it has determinantal correlation functions by proposition 2.13. We have $\phi_{r,r+1}(x, y) = \hat{f}_r(y - x)$, where $f_r = f_r^+ f_r^-$. The matrix A in (2.55) is a Toeplitz matrix and we can use the Wiener-Hopf factorization technique in section 4 to see that the correlation kernel is given by (5.6), compare (4.9). \square

5.2. The Schur and Plancherel measures. Take $\lambda = (N, \dots, N, 0, \dots)$, $\ell(\lambda) = N$. Then we have only up-edges to the left of the origin and down-edges to the right of the origin. Furthermore $G(N, N) = \max_{j \geq 1} x_j^0$. Restrict the attention to the point process x_j^0 , $j \geq 1$, above the origin. This is then a determinantal point process with correlation kernel

$$(5.11) \quad K_N(x, y) = \frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \frac{dz}{z} \int_{\gamma_{r_1}} \frac{dw}{w} \frac{w^y}{z^x} \frac{z}{z-w} \prod_{j=1}^N \frac{(1 - a_j/z)(1 - b_j w)}{(1 - a_j/w)(1 - b_j z)}.$$

Consider now the case $a_i = \sqrt{q}$, $1 \leq i \leq N$, $b_i = \sqrt{q}$, $1 \leq i \leq M$, $b_i = 0$, $M < i \leq N$, where $M \leq N$. The kernel can then be expressed in terms of Meixner polynomials, [18], and is called the *Meixner kernel*. If we scale appropriately around the last particle this kernel has the Airy kernel as its scaling limit and proceeding as we did for the Krawtchouk ensemble we can prove the following theorem, [18]

Theorem 5.3. *Let $0 < q < 1$ and consider $G(M, N)$ defined by (5.2), where $w(i, j)$ are i.i.d. geometric random variables with parameter q . Then, for $\gamma \geq 1$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}\left[\frac{G([\gamma N], N) - N\omega(\gamma, q)}{N^{1/3}\sigma(\gamma, q)} \leq s\right] = F_{TW}(s)$$

for any $s \in \mathbb{R}$, where

$$\omega(\gamma, q) = \frac{(1 + \sqrt{q\gamma})^2}{1 - q} - 1,$$

$$\sigma(\gamma, q) = \frac{q^{-1/6}\gamma^{-1/6}}{1 - q}(\sqrt{\gamma} + \sqrt{q})^{2/3}(1 + \sqrt{q\gamma})^{2/3}.$$

The probability measure on the points $x_1^0 > \dots > x_N^0$ ($M = N$) is the so-called *Schur measure*, [28]. The points $x_1^0 > \dots > x_N^0$ can be related to a partition μ by $x_i^0 = \mu_i - i + 1$ so we can also think about the Schur measure as a measure on partitions. This is the probability measure on partitions μ that we obtain if we map $(w(i, j))$, an $N \times N$ matrix, $w(i, j)$ as in (5.1), to a pair of semi-standard Young tableaux with shape μ using the RSK-correspondence, [32]. We will not prove this here. In the present case all the vertical edges to the left of the origin and at the origin in the directed graph are up-edges, whereas those to the right of the origin are down-edges. The non-intersecting paths pass through the points $\mu_i - i + 1$, $1 \leq i \leq N$, above the origin. By the LGV-theorem the Schur measure can thus be written

$$(5.12) \quad p_S(\mu) = \frac{1}{Z_N} \det(\phi(1 - j, \mu_i - i + 1)) \det(\psi(1 - j, \mu_i - i + 1)),$$

where

$$\phi(u, v) = \sum_{1 \leq i_1 < \dots < i_{v-u} \leq N} a_{i_1} \dots a_{i_{v-u}} = h_{v-u}(a_1, \dots, a_N),$$

$$\psi(u, v) = \sum_{1 \leq i_1 < \dots < i_{v-u} \leq N} b_{i_1} \dots b_{i_{v-u}} = h_{v-u}(b_1, \dots, b_N)$$

Here $h_k(a_1, \dots, a_N)$ is the k 'th complete symmetric polynomial. The symmetric polynomial

$$(5.13) \quad s_\mu(a_1, \dots, a_N) = \det(h_{\mu_i - i + j}(a_1, \dots, a_N))$$

is the *Schur polynomial* labelled by μ . The Schur measure can thus be written

$$(5.14) \quad p_S(\mu) = \frac{1}{Z_N} s_\mu(a_1, \dots, a_N) s_\mu(b_1, \dots, b_N),$$

which explains the name. The normalization is, [32],

$$(5.15) \quad Z_N = \sum_{\mu} s_\mu(a_1, \dots, a_N) s_\mu(b_1, \dots, b_N) = \prod_{i,j=1}^N \frac{1}{1 - a_i b_j}.$$

It follows from above that, under the Schur measure, $\sum_{i \geq 1} \delta_{\lambda_i - i}$ is a determinantal point process on \mathbb{Z} with correlation kernel (5.11).

If we restrict our attention to the case when $W = (w(i, j))_{1 \leq i, j \leq N}$ is a permutation matrix, then $G(N, N) = \ell_N(\sigma)$ is exactly the length of the longest increasing subsequence in the permutation corresponding to W . Restricting to a permutation matrix means that we want to have exactly one up-step of size 1 on each line with up-edges and exactly one down-step of size 1 on each line with down-edges. If the height configuration at the origin is $\mu_i - i + 1$, $i \geq 1$, $\ell_N(\sigma) = \mu_1$, the corresponding measure on partitions μ is

$$(5.16) \quad \mathbb{P}_{\text{Plan}, N}[\mu] = \frac{1}{N!} [a_1 \dots a_N b_1 \dots b_N] s_\mu(a) s_\mu(b),$$

where $[a_1 \dots a_N b_1 \dots b_N]$ means that we take the coefficient of the monomial $a_1 \dots a_N b_1 \dots b_N$ in $s_\mu(a)s_\mu(b)$. This measure is called the *Plancherel measure*, and is also given by $f_\mu^2/N!$, where f_μ is the number of standard Young tableaux with shape μ . We saw above that when studying the problem of the length of the longest increasing subsequence in a uniform random permutation from S_N it was natural to let N be a $\text{Poisson}(\alpha)$ random variable. We thus consider the *Poissonized Plancherel measure*,

$$(5.17) \quad \mathbb{P}_{\text{PP}}^\alpha[\mu] = \sum_{N=0}^{\infty} \frac{\alpha^N}{N!} \mathbb{P}_{\text{Plan},N}[\mu],$$

where $\mathbb{P}_{\text{Plan},N}[\mu] = 0$ if N is not a partition of N .

Theorem 5.4. ([5],[19]). *Under the map $\mu \rightarrow \sum_{i \geq 1} \delta_{\mu_i - i}$, the Poissonized Plancherel measure is mapped to a determinantal point process with correlation kernel $B^\alpha(x, y)$, the discrete Bessel kernel, given by (4.36).*

Proof. (Sketch). Suppose that $g : \mathbb{Z} \rightarrow \mathbb{C}$ has support in $[-L, \infty) \cup \mathbb{Z}$ for some $L \geq 0$. By (5.14) - (5.17),

$$(5.18) \quad \begin{aligned} & \sum_{\mu} \prod_{j=1}^{\infty} (1 + g(\mu_j - j)) \mathbb{P}_{\text{PP}}^\alpha[\mu] \\ &= \sum_{N=0}^{\infty} \frac{\alpha^N}{(N!)^2} [a_1 \dots a_N b_1 \dots b_N] \prod_{i,j=1}^N \frac{1}{1 - a_i b_j} \sum_{\mu} \prod_{j=1}^{\infty} (1 + g(\mu_j - j)) p_S(\mu). \end{aligned}$$

Now, since the Schur measure has determinantal correlation functions we know that

$$(5.19) \quad \sum_{\mu} \prod_{j=1}^{\infty} (1 + g(\mu_j - j)) p_S(\mu) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{x \in \mathbb{Z}^k} \prod_{j=1}^k g(x_j) \det(K_N(x_i + 1, x_j + 1))_{1 \leq i, j \leq k},$$

where K_N is given by (5.11). Inserting (5.19) into (5.18) a rather long computation, which we omit, gives

$$\sum_{\mu} \prod_{j=1}^{\infty} (1 + g(\mu_j - j)) \mathbb{P}_{\text{PP}}^\alpha[\mu] = \det(I + B^\alpha g)_{\ell^2(\mathbb{Z})},$$

which proves the theorem by proposition 2.3. \square

5.3. A discrete polynuclear growth model. Consider now the right/down path which is given by $\lambda = (N, N - 1, \dots, 1, 0, \dots)$ and choose $a_i = b_i = \sqrt{q}$. set $G(i + 1/2, j + 1/2) = G(i, j)$, $i, j \geq 0$, and

$$(5.20) \quad h(x, t) = G\left(\frac{t+x+1}{2}, \frac{t-x+1}{2}\right)$$

for $x \in \mathbb{Z}$, $t \geq 0$, $|x| \leq t$ and $h(x, t) = 0$ if $|x| > t$. We see that $h(x, N)$, $x = -N, \dots, N$ are exactly the values of $G(i, j)$ along the right down path λ . Set $\omega(x, t) = 0$ if $t - x$ is even or if $|x| > t$, and

$$(5.21) \quad \omega(x, t) = w\left(\frac{t+x+1}{2}, \frac{t-x+1}{2}\right)$$

otherwise. From (5.2) it follows that

$$(5.22) \quad h(x, t+1) = \max(h(x-1, t), h(x, t), h(x+1, t)) + \omega(x, t+1).$$

We think of $h(x, t)$ as the height above x at time t . This growth model is a *discrete polynuclear growth model*, [25]. It is different from the one related to the shuffling procedure in the Aztec diamond. It follows from theorem 5.2 that the height fluctuations above $[ct]$, $0 \leq c < 1$ at time t are of order $t^{1/3}$ for large t and are described by the Tracy-Widom distribution. But we actually know the whole extended kernel (5.6) for $\lambda = (N, N-1, \dots)$ and hence we can study $x \rightarrow h(x, t)$, $|x| \leq t$, as a process. This was done in [30] for the Poissonian limit of the model corresponding to random permutations, where we have an extended discrete Bessel kernel.

If we consider the process at time $2N-1$, and look at even x , we are studying the process $u \rightarrow G(N+u, N-u)$, $|u| < N$. The appropriate scaling limit of the kernel (5.6) will again be the extended Airy kernel, similarly to what we got when we studied the NPR-boundary process. We have the following theorem, [21], which we will not prove. Basically we have again to investigate the convergence of the extended kernel (5.6) to the extended Airy kernel using the saddle-point method. We define the rescaled process $H_N(t)$ by

$$(5.23) \quad G(N+u, N-u) = \frac{2\sqrt{q}}{1-\sqrt{q}}N + dN^{1/3}H_N\left(\frac{1-\sqrt{q}}{1+\sqrt{q}}dN^{-2/3}u\right),$$

where

$$(5.24) \quad d = \frac{(\sqrt{q})^{1/3}(1+\sqrt{q})^{1/3}}{1-q}.$$

We extend it to a continuous process defined for all times by linear interpolation.

Theorem 5.5. *We have $H_N(t) \rightarrow \mathcal{A}(t) - t^2$, where $\mathcal{A}(t)$ is the Airy process, in the sense of convergence of finite-dimensional distributions.*

From (3.16) we know that $G(N, N)$ is a certain point-to-point last passage time. It is also natural to consider a *point-to-line last passage time*,

$$(5.25) \quad G_{\text{pl}}(N) = \max_{|u| < N} G(N+u, N-u),$$

i.e. we take the maximum of $\sum_{(i,j) \in \pi}$ over all up/right paths from $(1, 1)$ to the line $x+y=2N$. A maximal path will be like a directed polymer with one free end. From (5.23) we see that it is natural to consider the maximum of the process $H_N(t)$. To do so we need a stronger form of convergence than in theorem 5.5. The next theorem is proved in [21].

Theorem 5.6. *There is a continuous version of the Airy process $\mathcal{A}(t)$ and $H_N(t) \rightarrow \mathcal{A}(t) - t^2$ as $N \rightarrow \infty$ in the weak star topology of probability measures on $C(-T, T)$ for any fixed T .*

It is proved in [3] that

$$(5.26) \quad \mathbb{P}\left[\frac{G_{\text{pl}}(N) - 2\sqrt{q}(1-\sqrt{q})^{-1}N}{dN^{1/3}} \leq s\right] \rightarrow F_1(s)$$

as $N \rightarrow \infty$, where F_1 is the largest eigenvalue, Tracy-Widom law, for the Gaussian Orthogonal Ensemble (GOE). This law is different from F_{TW} , which is often denoted by F_2 , [37]. We will not give its explicit form here. If we combine this result

with theorem 5.6 we see that

$$(5.27) \quad F_1(s) = \mathbb{P}[\sup_{t \in \mathbb{R}} (\mathcal{A}(t) - t^2) \leq s].$$

It would be interesting to have a more direct approach to this result.

The maximal path in the point-to-line last passage problem is not necessarily unique so there could be several possible endpoints on the line. Set

$$(5.28) \quad K_N = \inf\{s; \sup_{t \leq s} H_N(t) = \sup_{t \in \mathbb{R}} H_N(t)\},$$

and similarly for the limiting process

$$(5.29) \quad K = \inf\{s; \sup_{t \leq s} (\mathcal{A}(t) - t^2) = \sup_{t \in \mathbb{R}} (\mathcal{A}(t) - t^2)\}.$$

If we could show that the process $t \rightarrow \mathcal{A}(t) - t^2$ has a unique point of maximum almost surely it would follow that $K_N \rightarrow K$, and the law of K would be the law of transversal fluctuations of the endpoint of a maximal path in a point-to-line problem. However, the above argument gives us no clue what this law could be. For all we know it could be Gaussian.

5.4. Proof of theorem 5.1. The procedure is very close to the Robinson-Schensted-Knuth correspondence. See [27] for a related analysis.

We have to define the paths π_1, π_2, \dots . Set $w^{(0)}(i, j) = w(i, j)$. Assume that we have defined $w^{(k)}(i, j)$ for some $k \geq 0$, $i, j \geq 0$. Set $G^{(k)}(0, j) = G^{(k)}(j, 0) = 0$, $j \geq 0$, and

$$(5.30) \quad G^{(k)}(i, j) = \max(G^{(k)}(i-1, j), G^{(k)}(i, j-1)) + w^{(k)}(i, j),$$

$i, j \geq 1$, so that $G^{(0)}(i, j) = G(i, j)$. Also, define

$$(5.31) \quad w^{(k+1)}(i, j) = \min(G^{(k)}(i-1, j), G^{(k)}(i, j-1)) - G^{(k)}(i-1, j-1).$$

This defines $w^{(k)}(i, j)$ and $G^{(k)}(i, j)$ recursively for all $k \geq 0$, $i, j \geq 1$.

The path π_k goes between the points $(-L + j, G^{(k-1)}(P_j(\lambda)) - k + 1)$, $j = 0, \dots, K + L$ and respects the direction of the edges in $\mathcal{G}(\lambda)$.

Claim 5.7. *We have that $w^{(k)}(i, j) = 0$ if $0 \leq i \leq k$ or $0 \leq j \leq k$.*

To prove the claim we use induction on k . The claim is true by definition if $k = 0$. Now, if $w^{(k)}(i, j) = 0$ for $0 \leq i \leq k$ or $0 \leq j \leq k$, then $G^{(k)}(i, j) = 0$ for $0 \leq i \leq k$ or $0 \leq j \leq k$. If $k \leq k+1$ or $j \leq k+1$, then $i-1 \leq k$ or $j-1 \leq k$. Hence, $G^{(k)}(i-1, j) = 0$ or $G^{(k)}(i, j-1) = 0$, so $w^{(k+1)}(i, j) = 0$ by (5.31). This proves claim 5.7.

Claim 5.8. *The paths π_1, π_2, \dots in $\mathcal{G}(\lambda)$ do not intersect.*

We first prove that

$$(5.32) \quad \min(G^{(k)}(i-1, j), G^{(k)}(i, j-1)) \geq G^{(k+1)}(i, j)$$

for all $i, j \geq 1$, $k \geq 0$. If $(i, j) = (1, 1)$, then (5.32) is clearly true since $G^{(k+1)}(1, 1) = 0$ for all $k \geq 0$ by claim 5.7. Assume (A) that (5.32) holds for $1 \leq i, j \leq m$. If $i = m+1$ and $j = 1$, then (5.32) holds because $G^{(k+1)}(m+1, 1) = 0$ by claim 5.7. Assume (B) that (5.32) holds for $i = m+2$, $1 \leq j < n$, where $n \leq m$. We want to

prove (5.32) for $i = m + 1, j = n$. By (5.30) and (5.31) the inequality (5.32) with $i = m + 1, j = n$ is equivalent to

$$\begin{aligned} \min(G^{(k)}(m + 1, n - 1), G^{(k)}(m, n)) &\geq \max(G^{(k+1)}(m, n), G^{(k+1)}(m + 1, n - 1)) \\ &+ \min(G^{(k)}(m + 1, n - 1), G^{(k)}(m, n)) - G^{(k)}(m, n - 1) \end{aligned}$$

i.e.

$$(5.33) \quad G^{(k)}(m, n - 1) \geq \max(G^{(k+1)}(m, n), G^{(k+1)}(m + 1, n - 1)).$$

By assumption (A), $G^{(k)}(m, n - 1) \geq G^{(k+1)}(m, n)$ and by assumption (B) $G^{(k)}(m, n - 1) \geq G^{(k+1)}(m + 1, n - 1)$, so (5.33) holds.

By induction (5.32) also holds for $i = m + 1, 1 \leq j \leq m$. A similar argument shows that (5.32) holds for $1 \leq i \leq m, j = m + 1$. It remains to consider the case $(i, j) = (m + 1, m + 1)$. The (5.32) is equivalent to

$$(5.34) \quad G^{(k)}(m, m) \geq \max(G^{(k+1)}(m, m + 1), G^{(k+1)}(m + 1, m))$$

by (5.30) and (5.31). Now $i = m, j = m + 1$ in (5.32) gives $G^{(k)}(m, m) \geq G^{(k+1)}(m, m + 1)$, and $i = m + 1, j = m$ in (5.32) gives $G^{(k)}(m, m) \geq G^{(k+1)}(m + 1, m)$. Hence (5.34) holds and we have shown that (5.32) holds for $1 \leq i, j \leq m + 1$ and hence holds for all $i, j \geq 1$ by induction.

We can use (5.32) to prove claim 5.8. Consider an up-step $P_j(\lambda)$ to $P_{j+1}(\lambda)$, $P_j(\lambda) = (i - 1, x)$, $P_{j+1}(\lambda) = (i, x)$. The path π_k then goes from $(-L + j, G^{(k-1)}(P_j(\lambda)) - k + 1)$ to $(-L + j + 1, G^{(k-1)}(P_{j+1}(\lambda)) - k + 1)$ via the points $(-L + j + 1, m)$, where $G^{(k-1)}(P_j(\lambda)) - k + 1 \leq m \leq G^{(k-1)}(P_{j+1}(\lambda)) - k + 1$. Non-intersection between π_k and π_{k+1} holds if $G^{(k-1)}(P_j(\lambda)) - k + 1 > G^{(k)}(P_{j+1}(\lambda)) - k$ or $G^{(k-1)}(i - 1, x) \geq G^{(k)}(i, x)$, which follows from (5.32). A down-step is treated analogously. This proves claim 5.8.

Claim 5.9. *The path π_k is horizontal if $k > \min(K, L)$.*

Assume that $K \leq L$ and $k > K$. the path π_k goes through the points $(-L + j, G^{(k-1)}(P_j(\lambda)) - k + 1)$. here $P_j(\lambda) = (x, y)$ where $x \leq K$. Now, $G^{(k-1)}(x, y) = 0$ if x or y is $< k$ by claim 5.7, so π_k goes through the points $(-L + j, -k + 1)$ and is horizontal.

Claim 5.10. *Set $W^{(k)}(\lambda) = (w^{(k)}(i, j))_{(i, j) \in S(\lambda)}$ and define the weight of $W^{(k)}(\lambda)$ to be*

$$\prod_{(i, j) \in S(\lambda)} (a_i b_j)^{w^{(k)}(i, j)}.$$

Then, the weight of $W^{(k-1)}(\lambda)$ is equal to the weight of $W^{(k)}(\lambda)$ times the weight of π_k in $\mathcal{G}(\lambda)$ for all $k \geq 1$.

Let $1 \leq i \leq K$. In the weight of $W^{(k-1)}(\lambda)$, a_i enters as

$$a_i^{\sum_{j=1}^m w^{(k-1)}(i, j)}$$

if the up-step with weight a_i is the step from $(i - 1, m)$ to (i, m) in the right/down path λ . The up-step has size $G^{(k-1)}(i, m) - G^{(k-1)}(i - 1, m)$ so we want to show that

$$(5.35) \quad G^{(k-1)}(i, m) - G^{(k-1)}(i - 1, m) + \sum_{j=1}^m w^{(k)}(i, j) = \sum_{j=1}^m w^{(k-1)}(i, j).$$

Now, by (5.30) and (5.31),

$$\begin{aligned} w^{(k-1)}(i, j) &= G^{(k-1)}(i, j) - \max(G^{(k-1)}(i-1, j), G^{(k-1)}(i, j-1)) \\ -w^{(k)}(i, j) &= G^{(k-1)}(i-1, j-1) - \min(G^{(k-1)}(i-1, j), G^{(k-1)}(i, j-1)). \end{aligned}$$

Adding these two inequalities and summing over j we obtain

$$\begin{aligned} \sum_{j=1}^m (w^{(k-1)}(i, j) - w^{(k)}(i, j)) &= \sum_{j=1}^m [G^{(k-1)}(i, j) - G^{(k-1)}(i-1, j)] \\ &\quad - \sum_{j=0}^{m-1} [G^{(k-1)}(i, j) - G^{(k-1)}(i-1, j)] = G^{(k-1)}(i, m) - G^{(k-1)}(i-1, m-1), \end{aligned}$$

so we have established (5.35). The argument for b_i is analogous and we have proved claim 5.10.

To prove the theorem it remains to show that the mapping from $W(\lambda)$ to (π_1, π_2, \dots) is one-to-one and weight preserving. That the map is weight preserving follows by claim 5.9 and repeated use of claim 5.10. We have to show that the map is invertible, i.e. given (π_1, π_2, \dots) we can reconstruct $W(\lambda)$.

If $k > \min(K, L)$, then $W^{(k)}(\lambda) = 0$ by claim 5.7. Assume that we know $W^{(k)}(\lambda)$ for some $k \geq 1$. We want to show that we can reconstruct $W^{(k-1)}(\lambda)$. Repeating this we eventually get $W^{(0)}(\lambda) = W(\lambda)$.

Let λ be a partition of N and let $\lambda = \lambda^0 > \lambda^1 > \dots > \lambda^{N-1} > \lambda^N = \emptyset$ be a sequence of partitions such that we get λ^k from λ^{k-1} by removing one point on the boundary. (We have a path from λ to \emptyset in the Young lattice, [32].) Assume that we know all the values of $G^{(k-1)}(m, n)$ for (m, n) along the boundary of λ^j . From π_k we know all the values of $G^{(k-1)}(m, n)$ along the boundary of $\lambda^0 = \lambda$. Let (m, n) be the point on the boundary of λ^j that we remove when we go to λ^{j+1} . Since λ^{j+1} is also a partition, $(m-1, n)$ and $(m, n-1)$ must also be points along the boundary of λ^j . Hence, we know $G^{(k-1)}(m-1, n)$ and $G^{(k-1)}(m, n-1)$. We can now get $w^{(k-1)}(m, n)$ from (5.30). Since we know $w^{(k)}(m, n)$ we can also get $G^{(k-1)}(m-1, n-1)$ from (5.31). Hence we know the values of $G^{(k-1)}(m, n)$ along the boundary of λ^{j+1} . Proceeding in this way we successively get the numbers $w^{(k-1)}(m, n)$ for $(m, n) \in S(\lambda)$. This completes the proof of theorem 5.1.

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