A GRADIENT FLOW APPROACH TO THE BOLTZMANN EQUATION

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Abstract. We show that the spatially homogeneous Boltzmann equation evolves as the gradient flow of the entropy with respect to a suitable geometry on the space of probability measures which takes the collision process into account. This gradient flow structure allows to give a new proof for the convergence of Kac's random walk to the homogeneous Boltzmann equation, exploiting the stability of gradient flows.

1. Introduction

Since the pioneering work of Otto [17] it is known that many diffusion equations can be cast as gradient flows of entropy functionals in the space of probability measures. The relevant geometry is given by the L^2 Wasserstein distance. This approach has been used for a variety of equations as a powerful tool in the study of the trend to equilibrium, stability questions and construction of solutions. In each case – as a direct consequence of the gradient flow structure – the driving entropy functional is non-increasing along the solution. One of the most emblematic dissipative evolution equations is the Boltzmann equation modeling the evolution of a dilute gas under elastic collisions of the particles and Boltzmann's famous H-theorem asserts that the entropy is non-increasing along its solutions. However, uncovering a gradient flow structure for this equation has been an open problem since [17].

In this article we provide a solution and give a characterization of the spatially homogeneous Boltzmann equation as a gradient flow of the entropy. The crucial new insight is the identification of a novel geometry on the space of probability measures that takes the collision process between particles into account. Our main motivation to consider this gradient structure stems from the Kac program, in particular the propagation of chaos for Kac's stochastic many particle systems and its convergence to the homogeneous Boltzmann equation. We provide a new proof of this result by exhibiting a gradient flow structure also for the Kac system and showing that it Γ-converges to our gradient structure for the Boltzmann equation in the spirit of Sandier–Serfaty [18].

1.1. Homogenous Boltzmann equation and gradient flow structure.

We consider the spatially homogeneous Boltzmann equation

$$\partial_t f = Q(f) \ , \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a probability density and Q denotes the Boltzmann collision operator given by

$$Q(f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \left[f' f'_* - f f_* \right] B(v - v_*, \omega) \mathrm{d}v_* \mathrm{d}\omega \ . \tag{1.2}$$

Here B is the collision kernel and v, v_* and v', v'_* denote the pre- and post-collisional velocities respectively which are related according to

$$v' = v - \langle v - v_*, \omega \rangle \, \omega \ , \quad v'_* = v_* + \langle v - v_*, \omega \rangle \, \omega \ , \quad \omega \in S^{d-1} \ , \tag{1.3}$$

Key words and phrases. Boltzmann equation, spatially homogeneous, gradient flow, entropy.

and we will often use the notation $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, $f'_* = f(v'_*)$. We consider so-called pseudo-Maxwellian molecules, more precisely, we assume that B is bounded above and away from zero, see Assumption 2.1.

Boltzmann's H-theorem asserts that the entropy $\mathcal{H}(f) = \int f \log f$ is non increasing along solutions to the Boltzmann equation, more precisely, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f_t) = -D(f_t) \leq 0$, where

$$D(f_t) = \frac{1}{4}\int \log \frac{f'f'_*}{ff_*}(f'f'_* - ff_*)B(v - v_*, \omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}v \ . \tag{1.4}$$

Let us now give a heuristic description of the gradient flow structure of the Boltzmann equation. We recall that the gradient flow of a function E on a Riemannian manifold M is given as $\dot{x}_t = -\nabla E(x_t) = -K_{x_t}DE(x_t)$ with DE being the differential of E and $K_x : T^*_xM \to T_xM$ the canonical map from the cotangent to the tangent space induced by the Riemannian metric.

For the Boltzmann equation we formally take the manifold to be the set $\mathscr{P}(\mathbb{R}^d)$ of probability densities on $\mathbb{R}^d$ and the driving functional to be the entropy $\mathcal{H}$. Its differential $D\mathcal{H}(f)$ at f is given as $\log f = \frac{\delta\mathcal{H}}{\delta f}$ in the sense that for any tangent vector, i.e. a function s with $\int s(v)\mathrm{d}v = 0$, we have $\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[\mathcal{H}(f + \varepsilon s) - \mathcal{H}(f)\big] = D\mathcal{H}(f)[s] = \int \log f(v)s(v)\mathrm{d}v$. Identifying the gradient flow structure of the Boltzmann equation requires to identify the right geometry on the set $\mathscr{P}(\mathbb{R}^d)$ given in terms of a suitable map K. This is achieved by introducing the *Onsager operator* $\mathcal{K}^B_f$ given by

$$\mathcal{K}^B_f\varphi(v) = -\int \bar{\nabla}\varphi\Lambda(f)B(v - v_*, \omega)\mathrm{d}v_*\mathrm{d}\omega \ . \tag{1.5}$$

Here we have set $\bar{\nabla}\varphi = \varphi' + \varphi'_* - \varphi - \varphi_*$ and $\Lambda(f)$ is shorthand for $\Lambda\big(ff_*, f'f'_*\big)$, where $\Lambda(s,t) = (s-t)/(\log s - \log t)$ denotes the logarithmic mean. Now the Boltzmann equation can be written as

$$\partial_t f = Q(f) = -\mathcal{K}^B_f D\mathcal{H}(f) \ ,$$

giving the desired gradient flow structure.

This gradient flow interpretation of the Boltzmann equation can also be expressed by the following *variational characterization*. Denoting by $\langle\cdot,\cdot\rangle_f$ the Riemannian metric at f we have for any curve (f_t) of probability densities that

$$\mathcal{H}(f_T) - \mathcal{H}(f_0) = \int_0^T \langle\nabla\mathcal{H}(f_t), \partial_t f\rangle_{f_t}\,\mathrm{d}t \geq -\frac{1}{2}\int_0^T |\nabla\mathcal{H}(f_t)|^2_{f_t} + |\partial_t f|^2_{f_t}\mathrm{d}t \ .$$

Moreover, equality holds if and only if $\partial_t f = -\nabla\mathcal{H}(f_t)$, i.e. (f_t) is the gradient flow of the entropy, hence the solution to the Boltzmann equation. In this sense, the Boltzmann equation is a steepest descent flow decreasing the entropy *as fast as possible*.

Our first main result is a rigorous implementation of this variational characterization. To this end we replace the formal norm of the gradient and the speed of the curve with suitable notions.

Note that

$$|s|^2_f = \int \varphi\mathcal{K}^B_f\varphi = \frac{1}{4}\int |\bar{\nabla}\varphi|^2\Lambda(f)B(v - v_*, \omega)\mathrm{d}\omega\mathrm{d}v_*\mathrm{d}v \ ,$$

with φ such that $\mathcal{K}^B_f\varphi = s$ and where we have symmetrized over v, v_*, v', v'_*. In particular, the dissipation (1.4) takes the role of norm of the gradient, i.e. $|\nabla\mathcal{H}(f)|^2_f = \int \log f\mathcal{K}^B_f \log f = D(f)$.

In order to define the notion of speed of a curve $(f_t)_t$, we first consider the equation

$$\partial_t f(v) = \mathcal{K}^B_{f_t}\psi_t(v) = -\int \bar{\nabla}\psi_t\Lambda(f)B(v - v_*, \omega)\mathrm{d}v_*\mathrm{d}\omega \ . \tag{1.6}$$

We perform a change of variables, setting $U_t(v, v_*, \omega) = \bar{\nabla}\psi_t \Lambda(f) B(v - v_*, \omega)$ so that (1.6) becomes linear in (f, U) and reads for all test functions φ as:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi f_t = \frac{1}{4} \int \bar{\nabla}\varphi U_t \ . \tag{1.7}$$

This will be called *collision rate equation* since U governs the evolution of the density f by prescribing the rate at which collisions happen between the particles. Now, the quantity $\int_0^T |\partial_t f|_f^2 \mathrm{d}t$ will be replaced by the action

$$\mathcal{A}_T(f) := \inf \left\{ \frac{1}{4} \int_0^T \int \frac{|U_t|^2}{\Lambda(f_t) B} \, \mathrm{d}t \right\} \ , \tag{1.8}$$

where the infimum is over all $(U_t)_t$ satisfying the collision rate equation (1.7). See Section 3 for the precise construction where we study (1.7) and (1.8) in a natural measure valued setting. We then have the following variational characterization, see Theorem 4.3 below.

Theorem 1.1. *Let B satisfy Assumption 2.1. Then for any curve $(f_t)_{t\in[0,T]}$ of probability densities with $\mathcal{H}(f_0) < \infty$ and bounded second moment we have*

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2} \int_0^T D(f_t) \mathrm{d}t + \frac{1}{2} \mathcal{A}_T(f) \geq 0.$$

Moreover, we have $J_T(f) = 0$ if and only if $(f_t)_t$ is the solution to the homogeneous Boltzmann equation starting from f_0.

We remark that this result can be recast in the framework of gradient flows in metric spaces as developed in [1]. In particular it is possible to construct the Riemannian distance $\mathcal{W}_B$ on $\mathscr{P}(\mathbb{R}^d)$ associated with the Onsager operator $\mathcal{K}^B$. We explore this point of view in the appendix.

1.2. **Consistency for Kac's random walk.** A central motivation for considering the gradient flow structure just described is to give a new proof of the convergence of Kac's random walk to the solution of the spatially homogeneous Boltzmann equation. Kac introduced his random walk in the seminal work [11] as a probabilistic model for N colliding particles. It is a continuous time Markov chain on the set $\mathcal{X}_N$ of N velocities with fixed momentum and energy,

$$\mathcal{X}_N := \left\{ (v_1, \dots, v_N) \in \mathbb{R}^{dN} \mid \sum_{i=1}^N v_i = 0 \ , \ \sum_{i=1}^N |v_i|^2 = Nd \right\} \ .$$

In each step, two uniformly chosen particles i, j collide, i.e. $\boldsymbol{v}$ is updated to $R_{ij}^\omega \boldsymbol{v} = (v_1, \dots, v_i', \dots, v_j', \dots, v_N)$ where $v_i' = v_i - \langle v_i - v_j, \omega \rangle \omega$ and $v_j' = v_j + \langle v_i - v_j, \omega \rangle \omega$ with a random collision parameter $\omega \in S^{d-1}$ distributed according to $B(v_i - v_j, \cdot)$. The rate is chosen such that on average N collisions occur per unit of time. More precisely, the generator of the Markov chain is given by

$$Af(\boldsymbol{v}) = \frac{1}{2N} \int_{S^{d-1}} \sum_{i,j} \left[f(R_{ij}^\omega \boldsymbol{v}) - f(\boldsymbol{v}) \right] B(v_i - v_j, \omega) \mathrm{d}\omega \ . \tag{1.9}$$

The Markov chain is reversible with respect to the Hausdorff measure π_N on $\mathcal{X}_N$. If we denote by μ_t^N the law of the Markov chain starting from μ_0^N, then its density f_t^N w.r.t. π_N satisfies Kac's master equation $\partial_t f_t^N = A f_t^N$.
A natural way to study the convergence of Kac's random walk to the Boltzmann equation is via its empirical measures $L_N(\boldsymbol{v}) = \frac{1}{N} \sum_{i=1}^N \delta_{v_i} \in \mathscr{P}(\mathbb{R}^d)$. We will show the following:

Theorem 1.2. *Let B satisfy Assumption 2.1. For each N let $(\mu_t^N)_{t\geq 0}$ be the law of Kac's random walk starting form μ_0^N and denote by $c_t^N := (L_N)_\# \mu_t^N \in \mathscr{P}(\mathscr{P}(\mathbb{R}^d))$ the law of its empirical measures. Assume that μ_0^N is well-prepared for some $\nu_0 = f_0\mathcal{L} \in \mathscr{P}(\mathbb{R}^d)$ with $\mathcal{H}(\nu_0) < \infty$ in the sense that in the limit $N \to \infty$*

$$c_0^N \rightharpoonup \delta_{\nu_0} \ , \quad \frac{1}{N}\mathcal{H}(\mu_0^N|\pi_N) \to \mathcal{H}(\nu_0|M) \ .$$

Assume further that for some $p > 2$

$$\sup_N \langle \mathcal{E}_p^N, \mu_0^N \rangle < \infty \ , \qquad \mathcal{E}_p^N(\boldsymbol{v}) := \frac{1}{N}\sum_{i=1}^N |v_i|^p \ .$$

Then, for all $t > 0$, as $N \to \infty$ we have

$$c_t^N \rightharpoonup \delta_{\nu_t} \ , \quad \frac{1}{N}\mathcal{H}(\mu_t^N|\pi_N) \to \mathcal{H}(\nu_t|M) \ , \tag{1.10}$$

where $\nu_t = f_t\mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0.

Here $\mathcal{H}(\cdot|\pi_N)$ denotes the relative entropy w.r.t π_N and $\mathcal{H}(\cdot|M)$ the relative entropy w.r.t. the standard Gaussion density M in $\mathbb{R}^d$. Note that the well-preparedness assumption is satisfied for instance if the initial velocities are independent, i.e. $\mu_0^N = \nu_0^{\otimes N}$. An important feature of Kac's model is the *propagation of chaos*: if the initial distribution of velocities is asymptotically independent as $N \to \infty$ then the same holds for all times. One way of making this precise is the convergence (1.10), which is usually called entropic propagation of chaos. This is motivated by the fact that for a true product measure we have $\mathcal{H}(\nu^{\otimes N}) = N \cdot \mathcal{H}(\nu)$.

We point out that the previous theorem is well-known even for a larger class of collision kernels, see the references below. The contribution we make here is to provide a new angle of attack on this problem by exploiting the gradient flow structure. We will use the stability of gradient flows following the approach of Sandier–Serfaty [18]. It turns out that Kac's random walk is the gradient flow of the entropy $\mathcal{H}(\cdot|\pi_N)$ in $\mathscr{P}(\mathcal{X}_N)$ equipped with a suitable geometry, as we shall make precise in Section 5.1. In particular, the energy dissipation identity

$$J_T^N(\mu^N) = \mathcal{H}(\mu_t^N|\pi_N) - \mathcal{H}(\mu_0^N|\pi_N) + \frac{1}{2}\int_0^T D^N(\mu_t^N)\mathrm{d}t + \frac{1}{2}\mathcal{A}_T^N(\mu^N) = 0 \tag{1.11}$$

holds, where D^N is the dissipation of $\mathcal{H}(\cdot|\pi_N)$ along the master equation and $\mathcal{A}_T^N(\mu^N)$ is the action. This is based on results for general Markov chains and jump processes in [12, 14, 10]. To obtain the desired convergence to the Boltzmann equation it is sufficient together with some compactness to prove convergence (in fact only $\liminf$ estimates) for the constituent elements of the gradient flow structure, the entropy, dissipation and the action, which allow to pass to the limit in (1.11).

1.3. **Connection to the literature.** For an overview of results for the spatially homogeneous Boltzmann equation, we refer to the review by Desvillettes [8]. Modifications of the Wasserstein geometry have been studied recently in works by Maas [12] and Mielke [14] where gradient flow structures for finite Markov chains and reaction-diffusion equations have been found. The gradient flow structure for the homogenous Boltzamnn equation obtained here is related to the discrete framework of reaction equations in [14]. Formally, the homogeneous Boltzmann equation could be seen as a binary reaction equation with a continuum of species indexed by the velocity.

Theorem 1.2 on the convergence of Kac's random walk goes back to Kac who proved an analogue for a simplified model with one-dimensional velocities in [11]. The first proof of convergence to the homogeneous Boltzmann equation for the model considered here is

due to Sznitman [19]. In both cases more general collision kernels than in this article are considered, including in particular the case of hard spheres. Fine quantitative convergence results in Wasserstein distance were obtained later by Mischler–Mouhot [15] and Norris [16].

1.4. **Organization.** In Section 2 we collect necessary preliminaries, in particular we recall regularizing properties of the Ornstein–Uhlenbeck semigroup in the context of the Boltzmann equation. In Section 3 we introduce the collsion rate equation and the action of a curve. The characterization of the Boltzmann equation as entropic gradient flow is obtained in Section 4. In Section 5 we exibit a gradient flow structure for Kac's random walk and prove its convergence to the Boltzmann equation.
The appendices A, B, and C contain the construction of the distance associated to the Onsager operator, a reformulation of our results in the framework of gradient flows in metric spaces, and a variational approximation scheme for the Boltzmann equation based on the gradient structure.

Acknowledgment. The author is grateful for the remarks and suggestions of the referees which greatly improved the manuscript. He wishes to thank André Schlichting for fruitful discussions on this work and related topics. The author gratefully acknowledges support by the German Research Foundation through the Collaborative Research Center 1060 *The Mathematics of Emergent Effects* and the Hausdorff Center for Mathematics

2. Preliminaries

2.1. **Homogeneous Boltzmann equation, entropy and dissipation.** Let $d \geq 3$. We denote by $\mathscr{P}(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence in duality with bounded continuous functions. We denote by $\mathcal{H}(\mu)$ the *Boltzmann–Shannon entropy* defined for $\mu \in \mathscr{P}(\mathbb{R}^d)$ by

$$\mathcal{H}(\mu) = \int f(v) \log f(v) \mathrm{d}v \ ,$$

provided $\mu = f\mathcal{L}$ is absolutely continuous w.r.t. Lebesgue measure $\mathcal{L}$ and $\max(f \log f, 0)$ is integrable, otherwise we set $\mathcal{H}(\mu) = +\infty$. We will also write $\mathcal{H}(f)$ if $\mu = f\mathcal{L}$.
Let $\mathscr{P}_2(\mathbb{R}^d) = \{\mu \in \mathscr{P}(\mathbb{R}^d) \ : \ \int |v|^2 \mathrm{d}\mu(v) < \infty\}$ denote the set of probability measures with finite second moment. For $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ we define by

$$\mathcal{M}(\mu) := \int v \mathrm{d}\mu(v) \ , \quad \mathcal{E}(\mu) := \int |v|^2 \mathrm{d}\mu(v) \ , \tag{2.1}$$

the momentum and energy of μ. For $E > 0$ we let

$$\mathscr{P}_{2,E}(\mathbb{R}^d) := \{\mu \in \mathscr{P}_2(\mathbb{R}^d) \ : \ \mathcal{E}(\mu) \leq E\} \ , \tag{2.2}$$

the set of measures with energy less than E. Note that $\mathscr{P}_{2,E}(\mathbb{R}^d)$ is compact for the weak topology. For $m \in \mathbb{R}^d$ and $E > 0$ we let

$$M^{m,E}(v) = \frac{1}{(2\pi E)^{d/2}} \exp\left(-\frac{|v-m|^2}{2E}\right) \ ,$$

denote the Maxwellian or Gaussian density distribution with momentum m and energy Ed. The relative entropy w.r.t. $M^{m,E}$ of a probability measure $\mu = f\mathcal{L}$ is defined by

$$\mathcal{H}(\mu | M^{m,E}) = \int f(v) \log \frac{f(v)}{M^{m,E}(v)} \mathrm{d}v \ . \tag{2.3}$$

For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ we have

$$\mathcal{H}(\mu) = \mathcal{H}(\mu | M^{m,E}) - \frac{1}{2} \int \frac{1}{E} |v-m|^2 \mu(\mathrm{d}v) - \frac{d}{2} \log(2\pi E) \ . \tag{2.4}$$

By Jensen's inequality we have $\mathcal{H}(\cdot|M^{m,E}) \geq 0$. Hence, we see that $\mathcal{H}$ is bounded below on $\mathscr{P}_{2,E}(\mathbb{R}^d)$. Moreover, we have that $\mathcal{H}(\mu) = \mathcal{H}(\mu|M^\mu) + \mathcal{H}(M^\mu)$. Finally, we note that $\mathcal{H}$ is lower semicontinuous on $\mathscr{P}_2(\mathbb{R}^d)$ w.r.t. weak convergence. This follows from the corresponding property of $\mathcal{H}(\cdot|M^{m,E})$ and lower semicontinuity of moments.

We collect some well known results on existence and uniqueness and propagation of integrability for the homogeneous Boltzmann equation. Throughout this article we make the following assumption on the collision kernel.

Assumption 2.1. *$B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}_+$ is measurable, invariant under the transformation (1.3), and satisfies*

(i) *for any function $\xi \in C(S^{d-1})$ the map*
$$k \mapsto \int_{S^{d-1}} \xi(\omega) B(k,\omega) \mathrm{d}\omega$$
is continuous;

(ii) *there exist constants $c_B, C_B > 0$ such that for all $k \in \mathbb{R}^d, \omega \in S^{d-1}$:*
$$c_B \leq B(k,\omega) \leq C_B \ . \tag{2.5}$$

Theorem 2.2. *Let $f_0 : \mathbb{R}^d \to \mathbb{R}_+$ be such that*
$$\int_{\mathbb{R}^d} (1+|v|^2) f_0(v) \mathrm{d}v < \infty \ , \quad \int f_0(v) \log f_0(v) \mathrm{d}v < \infty \ .$$
Then there exists a unique solution $(f_t)_{t\geq 0}$ to the homogeneous Boltzmann equation (1.1). It conserves mass, momentum and energy, i.e.
$$\int \big(1, v, |v|^2\big) f_t(v) \mathrm{d}v = \int \big(1, v, |v|^2\big) f_0(v) \mathrm{d}v \quad \forall t \geq 0 \ .$$
Moreover, we have for all $t > 0$:
$$\mathcal{H}(f_t) - \mathcal{H}(f_s) \leq -\int_0^t D(f_r) \mathrm{d}r \leq 0 \ , \tag{2.6}$$
where
$$D(f) := \int_{\mathbb{R}^{2N}} \int_{S^{d-1}} \log \frac{f' f'_*}{f f_*} \big[f' f'_* - f f_*\big] B(v - v_*, \omega) \mathrm{d}v \mathrm{d}v_* \mathrm{d}\omega \ . \tag{2.7}$$

Proof. For the existence, uniqueness and conservation of mass, momentum and energy, we refer to [3, Prop. 1.1, 1.2]. (2.6) follows from the proof of [3, Thm. 2.1] taking into account the lower semicontinuity of the dissipation functional D, see Lemma 2.5 below. □

The quantity $D(f)$ is called the *entropy dissipation*. More generally, we define the entropy dissipation $D(\mu)$ for a probability measure μ by setting $D(\mu) = D(f)$, provided $\mu = f\mathcal{L}$ is absolutely continuous and $+\infty$ otherwise.

2.2. **Ornstein–Uhlenbeck regularization.** We recall that the (adjoint) Ornstein–Uhlenbeck semigroup $(S_t)_{t\geq 0}$ can be defined as a rescaled convolution with the standard Maxwellian distribution M. For $f \in L^1(\mathbb{R}^d)$ and $t \geq 0$ we have
$$S_t f = f_{e^{-2t}} * M_{1-e^{-2t}} \ ,$$
with the notation $g_\lambda(v) = \frac{1}{\lambda^{d/2}} g\Big(\frac{v}{\sqrt{\lambda}}\Big)$. Recall that $f_t := S_t f$ is the solution to the Fokker–Planck equation $\partial_t f = \nabla \cdot (\nabla f + f v)$, $f_0 = f$. We note that for any $f \in L^1$, $S_t f$ is C^∞ with the bounds
$$|S_t f| \leq C_t \ , \quad |\log S_t f|(v) \leq C_t(1+|v|^2) \ , \tag{2.8}$$
for a suitable constant C_t, see for instance [6].

For fixed $\omega \in S^{d-1}$ we will denote by T_ω the transformation $(v, v_*) \mapsto (v', v'_*)$ with v', v'_* given by (1.3). Note that T_ω is involutive and has unit Jacobian determinant. We will set

$$X = (v, v_*), \quad X' = (v', v'_*) = T_\omega X \ .$$

By abuse of notation we denote the standard Maxwellian distribution and the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^{2d}$ again by M and S_t. Note that $M(X) := M(v)M(v_*)$. For a function $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ we will set

$$T_\omega F(X) := F(T_\omega X) \ .$$

It is readily checked that the operations of scaling, convolution and the semigroup S_t behave well under tensorization. More precisely, if for a function $f : \mathbb{R}^d \to \mathbb{R}$ we set $F = f \otimes f$, i.e. $F(X) = f f_*$, then we have

$$F_\lambda = f_\lambda \otimes f_\lambda \ , F * M_\delta = (f * M_\delta) \otimes (f * M_\delta) \ , S_t F = (S_t f) \otimes (S_t f) \ .$$

The following commutation relation with the pre-post-collision change of variables will be crucial in the sequel. It can be found in [20, Prop. 4]. For the reader's convenience we will give the short proof.

Lemma 2.3. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$. Then, we have that for each $\omega \in S^{d-1}$ and any $\lambda, \delta > 0$:*

$$(T_\omega F)_\lambda = T_\omega(F_\lambda) \ , \quad (T_\omega F) * M_\delta = T_\omega(F * M_\delta) \ . \tag{2.9}$$

In particular, for each $t \geq 0$ we have that:

$$S_t(T_\omega F) = T_\omega(S_t F) \ . \tag{2.10}$$

If $F = f f_*$ we have for short $S_t(f' f'_*) = (S_t f)'(S_t f)'_*$.

Proof. Since $S_t F$ can be written as a composition of scaling of F and a convolution with (a scaling of) M, the commutation (2.10) is a direct consequence of (2.9). Commutation of T_ω with the scaling operation is readily checked. It remains to check commutation with convolution. First note that $M_\delta(T_\omega X) = M_\delta(X)$, since the relation between pre- and post-collisional velocities is such that $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. Using also the fact that T_ω is involutive with unit determinant, we find

$$\big((T_\omega F) * M_\delta\big)(X) = \int F(T_\omega Y) M_\delta(X - Y) \mathrm{d}Y = \int F(Y) M_\delta(X - T_\omega^{-1} Y) \mathrm{d}Y$$
$$= \int F(Y) M_\delta(T_\omega X - Y) \mathrm{d}Y = (F * M_\delta)(T_\omega X) \ .$$

□

2.3. **Integral functionals on measures.** We provide here basic results on integral functionals on measures that will be often used in the following.
Let X be locally compact Polish space. We denote by $\mathcal{M}(X : \mathbb{R}^n)$ the space of vector-valued Borel measures with finite variation on X. it will be endowed with the weak* topology of convergence in duality with $C_0(X; \mathbb{R}^n)$, i.e. continuous functions vanishing at infinity.
Let $f : \mathbb{R}^d \to [0, \infty]$ be a convex, lower semicontinuous, and positively 1-homogeneous function and define on $\mathcal{M}(X; \mathbb{R}^n)$ the functional

$$\mathcal{F}(\gamma) = \int_X f\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma \ ,$$

where σ is any non-negative finite Borel measure on X such that γ is absolutely continuous w.r.t. σ. Note that the definition is independend of the choice of σ by homogeneity of f.

Lemma 2.4.

(i) *$\mathcal{F}$ is convex and sequentially lower semicontinuous w.r.t. weak* convergence.*

(ii) *If Y is another locally compact Polish space and $T : X \to Y$ is Borel measurable, then we have that $\mathcal{F}(T_\# \gamma) \le \mathcal{F}(\gamma)$ for all γ, where $\mathcal{F}$ is defined analogously on $\mathcal{M}(Y;\mathbb{R}^n)$.*

Proof. (i) This is proven in [5, Thm. 3.4.3].
(ii) Let $\bar\gamma^i = T_\#\gamma^i$ and $\bar\sigma = T_\#\sigma$. Let $(\sigma_y)_{y\in Y}$ be a desintegration of σ w.r.t. $\bar\sigma$. I.e. σ_y are measures on X such that $y \mapsto \sigma(E)$ is Borel measurable for all Borel sets $E \subset X$, $\sigma_y(E) = \sigma_y(E \cap T^{-1}(y))$, $\sigma_y(X) = \sigma(X)$ for all y, and we have $\sigma(E) = \int \sigma_y(E)\mathrm{d}\bar\sigma(y)$. Write $\lambda = \rho\sigma$, and note that we have $\bar\lambda = \bar\rho\bar\sigma$ with $\bar\rho(y) := \int \rho(x)\sigma_y(\mathrm{d}x)$. Now put $\rho_y(x) = \rho(x)/\bar\rho(y)$. Then we have

$$\begin{aligned}\mathcal{F}(T_\#\gamma) &= \int_Y \alpha\big[\bar\rho\big]\mathrm{d}\bar\sigma = \int_Y \alpha\Big[\int_X \rho_y \mathrm{d}\sigma_y \bar\rho(y)\Big]\bar\sigma(\mathrm{d}y)\\ &\le \int_Y\int_X \alpha\big[\rho_y(x)\bar\rho(y)\big]\sigma_y(\mathrm{d}x)\bar\sigma(\mathrm{d}y) = \int \alpha\big[\rho\big]\mathrm{d}\sigma = \mathcal{F}(\gamma)\ ,\end{aligned}$$

where we have used Jensen's inequality due to the convexity and homogeneity of α. □

As a first consequence we obtain

Lemma 2.5 (Lower semicontinuity of dissipation)**.** *For any sequence (μ_n) in $\mathscr{P}(\mathbb{R}^d)$ converging weakly to μ we have that*

$$D(\mu) \le \liminf_n D(\mu_n)\ . \tag{2.11}$$

Proof. Consider the convex, lower semicontinuous, and 1-homogeneous function $G(s,t) = \frac{1}{4}(t-s)(\log t - \log s)$. For $\mu \in \mathscr{P}(\mathbb{R}^d)$ define non-negative measures $\mu^1, \mu^2 \in \mathcal{M}_+(G)$ by

$$\mu^1(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) := B(v - v_*, \omega)\mu(\mathrm{d}v)\mu(\mathrm{d}v_*)\mathrm{d}\omega\ , \quad \mu^2 := T_\#\mu^1\ ,$$

where T is the change of variables $(v, v_*, \omega) \mapsto (T_\omega(v, v_*), \omega)$ between pre- and post-collisional variables defined in (1.3). We note that

$$D(\mu) = \mathcal{G}(\mu^1, \mu^2) := \int G\left(\frac{\mathrm{d}\mu^1}{\mathrm{d}\sigma}, \frac{\mathrm{d}\mu^2}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma\ ,$$

where σ is any measure such that $\mu^1, \mu^2 \ll \sigma$. Note that by the Assumption 2.1 on the collision kernel B, the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}(G)$ for $i = 1, 2$. Now the claim follows immediately from Lemma 2.4. □

3. Collision rate equation and action

In this section, we rigorously define the notion of speed of a curve $(f_t)_t$ associated to the formal Onsager operator $\mathcal{K}^B$. In the next subsection we study the collsion rate equation (1.7) in a measure-valued framework replacing f_t with probability measures μ_t and U_t with a family of signed measures on $\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}$. In Subsection 3.2 we study the action functional (1.8) on measures and define the action of a curve.

3.1. The collision rate equation.

Let us set

$$G \ = \ \mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}$$

and denote by $\mathcal{M}(G)$ the space of signed Borel measures with finite variation on G equipped with the weak* topology in duality with continuous functions vanishing at infinity. Recall that $\mathscr{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence in duality with bounded continuous functions.
We define solutions to the collision rate equation in the following way.

Definition 3.1 (Collision rate equation)**.** *We denote by $\mathcal{CRE}_T$ the set of all pairs $(\mu, \mathcal{U})$ satisfying the following conditions:*

(i) $\mu : [0,T] \to \mathscr{P}(\mathbb{R}^d)$ *is weakly continuous;*
(ii) $(\mathcal{U}_t)_{t\in[0,T]}$ *is a Borel family of measures in* $\mathcal{M}(G)$*;*
(iii) $\int_0^T |\mathcal{U}_t|\,(G)\mathrm{d}t \;<\; \infty$*;*
(iv) *for any* $\varphi \in C_b(\mathbb{R}^d)$ *we have in the sense of distributions:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \mathrm{d}\mu_t = \frac{1}{4}\int \bar{\nabla}\varphi \mathrm{d}\mathcal{U}_t \;. \tag{3.1}$$

Moreover, we will denote by $\mathcal{CRE}_T(\bar{\mu}_0,\bar{\mu}_1)$ *the set of pairs* $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ *satisfying in addition:* $\mu_0 = \bar{\mu}_0,\ \mu_1 = \bar{\mu}_1$*. Further,* $\mathcal{CRE}_T^E$ *denotes the set of pairs* $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ *such that* $\mathcal{E}(\mu_t) \le E$ *for all* $t \in [0,T]$*.*

Note that the integrability condition (ii) ensures that the right hand side in (iv) is well-defined. The measures $\mathcal{U}_t$ will be called *collision rates.*

Remark 3.2. If $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$, then for any $\varphi \in C_b(\mathbb{R}^d)$ and $0 \le t_0 \le t_1 \le T$ we have

$$\int \varphi \mathrm{d}\mu_{t_1} - \int \varphi \mathrm{d}\mu_{t_0} \;=\; \frac{1}{4}\int_{t_0}^{t_1}\int \bar{\nabla}\varphi \mathrm{d}\mathcal{U}_t \mathrm{d}t \;. \tag{3.2}$$

This follows readily from (iv) together with the continuity of $t \mapsto \mu_t$ in (i).
The curve $(\mu_t)_{t\in[0,T]}$ is also absolutely continuous w.r.t. the total variation norm. Indeed, from (3.2) we infer

$$\Big|\int \varphi \mathrm{d}(\mu_{t_1} - \mu_{t_0})\Big| \le |\varphi|_\infty \int_{t_0}^{t_1} |\mathcal{U}_t|(G)\mathrm{d}t \;,$$

and hence $||\mu_{t_1} - \mu_{t_0}||_{\mathrm{TV}} \le \int_{t_0}^{t_1} |\mathcal{U}_t|\mathrm{d}t$. Moreover, the distribution $\partial_t\mu_t$ on $[0,T]\times\mathbb{R}^d$ is actually a signed measure with total variation bounded by $\int_0^T |\mathcal{U}_t|(G)\mathrm{d}t$.

Remark 3.3. The continuity equation can sometimes be tested against more general test functions. For instance, let $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ and let $\mathcal{U}$ satisfy the stronger integrability condition

$$(iii') \qquad \int_0^T\int \big[1 + |v| + |v_*|\big]\mathrm{d}\,|\mathcal{U}_t|\,\mathrm{d}t \;<\; \infty \;. \tag{3.3}$$

Then (3.2) holds for all $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous with at most linear growth, i.e. $|\varphi(v)| \le c(1+|v|)$. This follows immediately by approximation with functions in C_b and the trivial estimate $|v'| + |v'_*| \le 3|v| + 3|v_*|$. If μ_t has density f_t w.r.t. Lebesgue measure, we infer as above that

$$\Big|\int (1+|v|)\varphi(v)\big(f_{t_1}(v) - f_{t_0}(v)\big)\mathrm{d}v\Big| \le |\varphi|_\infty \int_{t_0}^{t_1}\int \big[1+|v|+|v_*|\big]\mathrm{d}|\mathcal{U}_t|\mathrm{d}t \;,$$

and hence $t \mapsto (1+|v|)f_t$ is absolutely continuous in L^1.

Next, we note that being a solution to the collision rate equation is invariant under the action of the Ornstein–Uhlenbeck semigroup.
Given $\mu \in \mathscr{P}(\mathbb{R}^d)$, the action of the Ornstein–Uhlenbeck semigroup is given by $S_t\mu = \mu_{e^{-2t}} * M_{1-e^{-2t}}$, where μ_λ is the image of μ under the map $v \mapsto \sqrt{\lambda}v$. Given $\mathcal{U} \in \mathcal{M}(\mathbb{R}^{2d}\times S^{d-1})$ we define its convolution $\mathcal{U} * M$ with the Maxwellian M in $\mathbb{R}^{2d}$ as the measure given by

$$(\mathcal{U} * M)(\mathrm{d}X,\mathrm{d}\omega) = \int_{\mathbb{R}^{2d}} M(X-Y)\mathcal{U}(\mathrm{d}Y,\mathrm{d}\omega)\mathrm{d}X \;.$$

The action of the semigroup S_t is defined via $S_t\mathcal{U} = \mathcal{U}_{e^{-2t}} * M_{1-e^{-2t}}$, where $\mathcal{U}_\lambda$ is the image of $\mathcal{U}$ under the map $(X,\omega) \mapsto (\sqrt{\lambda}X,\omega)$.

Lemma 3.4. *Let $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ and set $\mu_t^s := S_s\mu_t$, $\mathcal{U}_t^s := S_s\mathcal{U}_t$ for $s \geq 0$ and $t \in [0,T]$. Then we have $(\mu^s,\mathcal{U}^s) \in \mathcal{CRE}_T$.*

Proof. It suffices to check that being a solution to the collision rate equation is stable under scaling and convolution with M. One readily checks that $(\mu_\lambda,\mathcal{U}_\lambda) \in \mathcal{CRE}_T$ for all $\lambda \geq 0$. To check stability under convolution fix a test function φ and set $\Phi(X) := \varphi(v) + \varphi(v_*)$. Then, using (2.9), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \mathrm{d}(\mu_t * M) &= \frac{\mathrm{d}}{\mathrm{d}t}\int (\varphi * M)\mathrm{d}\mu_t = \int \bar{\nabla}(\varphi * M)\mathrm{d}\mathcal{U}_t \\ &= \int (\Phi * M)(T_\omega X) - (\Phi * M)(X)\mathrm{d}\mathcal{U}_t(X,\omega) \\ &= \int ((T_\omega \Phi) * M)(X) - (\Phi * M)(X)\mathrm{d}\mathcal{U}_t(X,\omega) \\ &= \int \Phi(T_\omega X) - \Phi(X)\mathrm{d}(\mathcal{U}_t * M)(X,\omega) = \int \bar{\nabla}\varphi \mathrm{d}(\mathcal{U}_t * M)\ ,\end{aligned}$$

which shows that $(\mu * M,\mathcal{U} * M) \in \mathcal{CRE}_T$. □

3.2. **The action functional.** Let us first recall the definition of the logarithmic mean $\Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Lambda(s,t) \ = \ \int_0^1 s^\alpha t^{1-\alpha}\mathrm{d}\alpha \ = \ \frac{s-t}{\log s - \log t}\ ,$$

the latter expression being valid for positive $s \neq t$. Note that Λ is concave and positively homogeneous, i.e. $\Lambda(\alpha s, \alpha t) = \alpha\Lambda(s,t)$ for all $\alpha \geq 0$. Moreover it is easy to check that

$$\Lambda(s,t) \ \leq \ \frac{s+t}{2} \quad \forall s,t \geq 0\ . \tag{3.4}$$

Given a function $f : \mathbb{R}^d \to \mathbb{R}_+$ we will often write

$$\Lambda(f)(v,v_*,\omega) \ = \ \Lambda(ff_*, f'f'_*)\ .$$

We can now define a function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to [0,\infty]$ by setting

$$\alpha(s,t,u) := \begin{cases} \frac{u^2}{4\Lambda(s,t)}\ , & \Lambda(s,t) \neq 0\ , \\ 0\ , & \Lambda(s,t) = 0 \text{ and } u = 0\ , \\ +\infty\ , & \Lambda(s,t) = 0 \text{ and } u \neq 0\ . \end{cases} \tag{3.5}$$

The function α is lower semicontinuous, convex and positively homogeneous, i.e. for all $u \in \mathbb{R}$, $s,t \geq 0$, and $r > 0$ we have $\alpha(rs,rt,ru) \ = \ r\alpha(s,t,u)$. Indeed, this is easily checked using homogeneity and concavity of Λ and the convexity of the function $(u,y) \mapsto \frac{u^2}{y}$ on $\mathbb{R} \times (0,\infty)$.

We will now define an action functional on pairs of measures $(\mu,\mathcal{U})$ where $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(G)$ generalizing (1.8). For later reference, we work first in a more general setting. We consider the following integral functional associated with the function α on the space $\mathcal{M}(X;R^3)$ of vector-valued Borel measures with finite variation on a locally compact Polish space X:

$$\mathcal{F}_\alpha(\lambda) := \int \alpha\left(\frac{\mathrm{d}\lambda^1}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\lambda^2}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\lambda^3}{\mathrm{d}|\lambda|}\right)\mathrm{d}|\lambda|\ , \tag{3.6}$$

where $|\lambda|$ denotes the variation of λ.

Definition 3.5 (Action)**.** *For $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(G)$ the* action *is defined by*

$$\mathcal{A}(\mu,\mathcal{U}) := \mathcal{F}_\alpha(\mu^1,\mu^2,\mathcal{U})\ , \tag{3.7}$$

where μ^1, μ^2 are non-negative measures in $\mathcal{M}_+(G)$ given by

$$\mu^1(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) := B(v - v_*, \omega)\mu(\mathrm{d}v)\mu(\mathrm{d}v_*)\mathrm{d}\omega \ , \quad \mu^2 := T_\# \mu^1 \ , \tag{3.8}$$

where T is the change of variables $(v, v_*, \omega) \mapsto (T_\omega(v, v_*), \omega)$ between pre- and post-collisional variables defined in (1.3).

If the measure μ is absolutely continuous w.r.t. the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}^d$, the next lemma shows that we recover (1.8). For this we denote by $\mathcal{B} \in \mathcal{M}(G)$ the measure given by

$$\mathcal{B}(\mathrm{d}v, \mathrm{d}v_*, \mathrm{d}\omega) = B(v - v_*, \omega)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\omega \ .$$

Lemma 3.6. *Let $\mu = f\mathcal{L} \in \mathscr{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(G)$ be such that $\mathcal{A}(\mu, \mathcal{U}) < \infty$. Then there exists a Borel function $U : G \to \mathbb{R}$ such that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ and we have*

$$\mathcal{A}(\mu, \mathcal{U}) \ = \ \frac{1}{4}\int |U(v, v_*, \omega)|^2 \, \Lambda(f) B(v - v_*, \omega)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\omega \ . \tag{3.9}$$

Proof. Note that $\mu^i = \rho^i\mathcal{B}$, $i = 1, 2$ with $\rho^1(v, v_*, \omega) = f(v)f(v_*)$ and $\rho^2(v, v_*, \omega) = f(v')f(v'_*)$. Choose $\sigma \in \mathcal{M}(G)$ such that $\mathcal{B} = h\sigma$ and $\mathcal{U} = \widetilde{U}\sigma$ are both absolutely continuous w.r.t. σ and denote by $\widetilde{\rho}^i$ the density of μ^i w.r.t σ. Now by homogeneity of α

$$\mathcal{A}(\mu, \mathcal{U}) \ = \ \int \alpha(\widetilde{\rho}^1, \widetilde{\rho}^2, \widetilde{U})\mathrm{d}\sigma \ < \ \infty \ . \tag{3.10}$$

Let $A \subset G$ be such that $\int_A \Lambda(\rho^1, \rho^2)\mathrm{d}\mathcal{B} = 0$. Homogeneity of Λ yields

$$0 \ = \ \int_A \Lambda(\rho^1, \rho^2)\mathrm{d}\mathcal{B} \ = \ \int_A \Lambda(\widetilde{\rho}^1, \widetilde{\rho}^2)\mathrm{d}\sigma \ ,$$

i.e. $\Lambda(\widetilde{\rho}^1, \widetilde{\rho}^2) = 0$ σ-a.e. on A. Now the finiteness of the integral in (3.10) implies that $\widetilde{U} = 0$ σ-a.e. on A. Thus $|\mathcal{U}|(A) = 0$ and hence $\mathcal{U}$ is absolutely continuous w.r.t. the measure $\Lambda(f)\mathcal{B}$. Formula (3.9) now follows immediately from the homogeneity of α. □

In view of the previous lemma, given a pair of functions $f : \mathbb{R}^d \to \mathbb{R}_+$ and $U : G \to \mathbb{R}$ we will define their action via $\mathcal{A}(f, U) := \mathcal{A}(\mu, \mathcal{U})$ with $\mu = f\mathcal{L}$ and $\mathcal{U} = U\Lambda(f)\widetilde{\mathcal{B}}$.
Next, we establish lower semicontinuity of the action w.r.t. convergence of μ and $\mathcal{U}$.

Lemma 3.7 (Lower semicontinuity of the action)**.** *Assume that $\mu_n \rightharpoonup \mu$ weakly in $\mathscr{P}(\mathbb{R}^d)$ and $\mathcal{U}_n \rightharpoonup^* \mathcal{U}$ weakly* in $\mathcal{M}(G)$. Then*

$$\mathcal{A}(\mu, \mathcal{U}) \ \leq \ \liminf_n \mathcal{A}(\mu_n, \mathcal{U}_n) \ .$$

Proof. Note that by the Assumption 2.1 on the collision kernel B, the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}(G)$ for $i = 1, 2$. Now the claim follows immediately from Lemma 2.4. □

The next estimate will be useful at several points in the paper. For later reference, we formulate it in the general context of (3.6).

Lemma 3.8 (Integrability estimate)**.** *For any Borel function $\Psi : X \to \mathbb{R}_+$ and $\lambda \in \mathcal{M}(X; \mathbb{R}^3)$ with $\mathcal{F}_\alpha(\lambda) < \infty$ and λ^1, λ^2 non-negative measures we have*

$$\int \Psi \mathrm{d}\left|\lambda^3\right| \leq \sqrt{2\mathcal{F}_\alpha(\lambda)} \left(\int \Psi^2 \mathrm{d}(\lambda^1 + \lambda^2)\right)^{\frac{1}{2}} \ . \tag{3.11}$$

Proof. Let us write $\lambda^i = \rho^i|\lambda|$. Since $\mathcal{F}_\alpha(\lambda)$ is finite, the set $A = \{\alpha(\rho^1,\rho^2,\rho^3) = \infty\}$ has zero measure with respect to $|\lambda|$. We can now estimate:

$$\begin{aligned}\int \Psi \mathrm{d}\left|\lambda^3\right| \;&\le\; \int \Psi \left|\rho^3\right| \mathrm{d}|\lambda| = \; 2\int_{A^c} \Psi\sqrt{\Lambda(\rho^1,\rho^2)}\sqrt{\alpha(\rho^1,\rho^2,\rho^3)}\mathrm{d}|\lambda| \\ &\le\; 2\left(\int \alpha(\rho^1,\rho^2,\rho^3)\mathrm{d}|\lambda|\right)^{\frac12}\left(\int_{A^c}\Psi^2\Lambda(\rho^1,\rho^2)\mathrm{d}|\lambda|\right)^{\frac12} \\ &\le\; \sqrt{2\mathcal{F}_\alpha(\lambda)}\left(\int \Psi^2 \mathrm{d}(\lambda^1+\lambda^2)\right)^{\frac12}\;,\end{aligned}$$

where last inequality follows from the estimate (3.4). □

Corollary 3.9. *Let $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ be such that $A := \int_0^T \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t$ and $E := \int_0^T \mathcal{E}(\mu_t)\mathrm{d}t$ are finite. Then the integrability condition (3.3) is satisfied, precisely*

$$\int_0^T \int \left[1+|v|+|v_*|\right]\mathrm{d}\left|\mathcal{U}_t\right|\mathrm{d}t \le 6\sqrt{AC_B(T+E)}\;.$$

Proof. Let $\mu^i,\mathcal{U} \in \mathcal{M}(G\times[0,T])$ be given by $\mathrm{d}\mu^i = \mathrm{d}\mu^i_t\mathrm{d}t$ and $\mathrm{d}\mathcal{U} = \mathrm{d}\mathcal{U}_t\mathrm{d}t$ and note that

$$\int_0^T \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t = \int_0^T \mathcal{F}_\alpha(\mu^1_t,\mu^2_t,\mathcal{U}_t)\mathrm{d}t = \mathcal{F}_\alpha(\mu^1,\mu^2,\mathcal{U})\;.$$

Then, one concludes by Lemma 3.8, choosing $\Psi(v,v_*,\omega,t) = 1+|v|+|v_*|$. □

Note that for a given curve $(\mu_t)_{t\in[0,T]}$ there will be several compatible collisions rates $(\mathcal{U}_t)_t$ such that $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$. For instance, when $\mathcal{V}_t$ is symmetric under the transformation $(v,v_*,\omega) \mapsto (v',v'_*,\omega)$ we have $\int \bar\nabla\varphi\mathrm{d}\mathcal{V}_t = 0$ for any test function φ. Hence, $(\mu,\mathcal{U}+\mathcal{V}) \in \mathcal{CRE}_T$ whenever $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$. Thus, we define the action of a curve as the minimal action of all compatible collision rates.

Definition 3.10 (Action of a curve)**.** *Given a curve $(\mu_t)_{t\in[0,T]}$ in $\mathscr{P}(\mathbb{R}^d)$ its* action *is defined by*

$$\mathcal{A}_T(\mu) := \inf\left\{\int_0^T \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t \;:\; (\mu,\mathcal{U}) \in \mathcal{CRE}_T\right\}\;. \tag{3.12}$$

If there is no $\mathcal{U}$ with $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$, we set $\mathcal{A}_T(\mu) = +\infty$.

The next result shows that under additional control on the energy of the curve the infimum above is attained by an optimal collision rate.

Proposition 3.11 (Optimal collision rate)**.** *Let $(\mu_t)_{t\in[0,T]}$ be a curve in $\mathscr{P}(\mathbb{R}^d)$ such that*

$$\mathcal{A}_T(\mu) < \infty\;, \qquad E := \int_0^T \mathcal{E}(\mu_t)\mathrm{d}t < \infty\;. \tag{3.13}$$

Then, there exists a family $(\mathcal{U}_t)_t$ with $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ attaining the infimum in (3.12).

Proof. Let $(\mathcal{U}^n_t)_t$ be a minimizing sequence of collision rates for (3.12) and define the measures $\mathcal{U}^n \in \mathcal{M}(G\times[0,T])$ given by $\mathrm{d}\mathcal{U}^n = \mathrm{d}\mathcal{U}^n_t\mathrm{d}t$. By Lemma 3.8, for every measurable function Ψ on $\mathbb{R}^{2d}\times S^{d-1}\times[0,T]$ we have

$$\begin{aligned}&\sup_n \int \Psi \mathrm{d}\left|\mathcal{U}^n\right| \\ &\le \sqrt{2A}\left(\int \left(\Psi^2+\Psi^2\circ T\right)B(v-v_*,\omega)\mathrm{d}\omega\mathrm{d}\mu_t(v)\mathrm{d}\mu_t(v_*)\mathrm{d}t\right)^{\frac12}\;,\end{aligned} \tag{3.14}$$

with $A = \sup_n \int_0^T \mathcal{A}(\mu_t,\mathcal{U}^n_t)\mathrm{d}t < \infty$. Choosing $\Psi = \mathbf{1}_{G\times I}$ and using Assumption 2.1, we obtain $\left|\mathcal{U}^n\right|(G\times I) \le 2\sqrt{C_B A\cdot\mathcal{L}(I)}$. Hence, $\mathcal{U}^n$ has uniformly bounded variation and we

have that up to extracting a subsequence $\mathcal{U}^n \rightharpoonup^* \mathcal{U}$ in $\mathcal{M}(G\times[0,T])$. Moreover, we see that $\mathcal{U}$ can be disintegrated w.r.t. Lebesgue measure on $[0,T]$ and we can write $\mathcal{U} = \int_0^T \mathcal{U}_t \mathrm{d}t$ for a Borel family $(\mathcal{U}_t)$ still satisfying (iii) in Definition 3.1.
To see that $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$, it suffices to show that for any test functions $a \in C\big([0,T]\big)$ and $\varphi \in C_b(\mathbb{R}^d)$ we have

$$\int a(t)\bar{\nabla}\varphi \mathrm{d}\mathcal{U}_t^n \mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int a(t)\bar{\nabla}\varphi \mathrm{d}\mathcal{U}_t \mathrm{d}t \ . \tag{3.15}$$

This follows from a straightforward argument, approximating $\bar{\nabla}\varphi$ with compactly supported continuous functions G once we establish the following tightness estimate for $\mathcal{U}^n$: Denoting by B_R the ball of radius R in $\mathbb{R}^{2d}$ and $M_R := B_R^c \times S^{d-1} \times [0,T]$ we have

$$|\mathcal{U}^n|\,(M_R) \leq 2\sqrt{AC_B}\left(\int_0^T \int_{B_{R/2}^c} \mathrm{d}\mu_t(v)\mathrm{d}\mu_t(v_*)\mathrm{d}t\right)^{\frac{1}{2}} \leq \ 2\frac{\sqrt{AC_BTE}}{R} \ ,$$

which goes to zero uniformly in n as $R \to \infty$. This estimate follows again from (3.14), noting that if (v,v_*) or (v',v'_*) lies outside B_R, then (v,v_*) lies ouside of $B_{R/2}$, and further using the estimate $\mu_t(\{|v| \geq R\}) \leq \int \frac{|v|^2}{R^2}\mathrm{d}\mu_t(v)$, and the upper bound on the energy in (3.13). Finally, we conclude that $\int_0^T \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t = \mathcal{A}(\mu)$ by noting that $\int_0^T \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t = \mathcal{F}_\alpha(\mu^1,\mu^2,\mathcal{U})$ and using the lower semicontinuity of $\mathcal{F}_\alpha$ given by Lemma 2.4. □

4. Variational characterization of the homogeneous Boltzmann equation

In this section we establish the variational characterization of the homogeneous Boltzmann equation stated in Theorem 1.1. The crucial ingredient is a chain rule allowing to take derivatives of the entropy along suitable curves of finite action.
Recall that $\mathcal{E}(\mu)$ denotes the energy of μ, see (2.1).

Proposition 4.1 (Chain rule)**.** *Let $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ such that $\mathcal{E}(\mu_t) \leq E$ for all t, $\mathcal{H}(\mu_t)$ is finite for some $t \in [0,T]$ and*

$$\int_0^T \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t < \infty \ , \quad \int_0^T \sqrt{D(\mu_t)}\sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t < \infty \ . \tag{4.1}$$

Then $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0,T]$ and we have that

$$\mathcal{H}(\mu_t) - \mathcal{H}(\mu_s) = \int_s^t \frac{1}{4}\int \bar{\nabla}\log f_r \mathrm{d}\mathcal{U}_r \mathrm{d}r \quad \forall 0 \leq s \leq t \leq T \ , \tag{4.2}$$

where f_r is the density of μ_r. In particular, the map $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\mu_t) = \frac{1}{4}\int \bar{\nabla}\log f_t \mathrm{d}\mathcal{U}_t \quad \text{for a.e. } t \ . \tag{4.3}$$

Proof. Note that by (4.1) and Lemma 3.6 we have $\mu_r = f_r\mathrm{d}v$, $\mathcal{U}_r = U_r\mathrm{d}X\mathrm{d}\omega$ for a.e. r and suitable densities f_r, U_r. We will now proceed in several steps.
Step 1: Regularization.
We will perform a three-fold regularization procedure. First, we regularize the curve by the Ornstein–Uhlenbeck semigroup. For $\delta > 0$ we set $\mu_t^\delta = S_\delta\mu_t$, and $\mathcal{U}_t^\delta = S_\delta\mathcal{U}_t$. Then we perform a convolution in time. For a standard mollifier η on $\mathbb{R}$ supported in $[-1,1]$ and $\gamma > 0$ we define

$$\mu_t^{\delta,\gamma} = \int \eta(t')\mu_{t-\gamma t'}^\delta \mathrm{d}t' \ , \quad \mathcal{U}_t^{\delta,\gamma} = \int \eta(t')\mathcal{U}_{t-\gamma t'}^\delta \mathrm{d}t' \ .$$

(For this the curves are assumed to be extended trivially by $\mu_0^\delta, \mathcal{U}_0^\delta$ on $[-\gamma, 0]$ and similarly on $[T, T+\gamma]$.) By Lemma 3.4 we have that $(\mu^\delta, \mathcal{U}^\delta) \in \mathcal{CRE}_T$ and by linearity of the collision rate equation also $(\mu^{\delta,\gamma}, \mathcal{U}^{\delta,\gamma}) \in \mathcal{CRE}_T$.
Finally, let g be a probability density in $\mathscr{P}_{2,E}(\mathbb{R}^d)$ such that

$$|\log g(v)| \leq C(1 + |v|) \,, \tag{4.4}$$

for some constant C (for instance choose $g(v)$ proportional to $e^{-\alpha|v|}$ for suitable $\alpha > 0$). Then we set for $\varepsilon > 0$, $\mu^{\delta,\gamma,\varepsilon} := (1+\varepsilon)^{-1}(\mu^{\delta,\gamma} + \varepsilon g \mathcal{L})$, and $\mathcal{U}^{\delta,\gamma,\varepsilon} = (1+\varepsilon)^{-1}\mathcal{U}^{\delta,\gamma}$ and note that $(\mu^{\delta,\gamma,\varepsilon}, \mathcal{U}^{\delta,\gamma,\varepsilon}) \in \mathcal{CRE}_T$. Let f^δ, U^δ denote the densities of $\mu^\delta, \mathcal{U}^\delta$ and similarly with γ and ε.
Step 2: Estimates for the regularized curve.
Note that that the second moment of μ_r^δ is bounded by $e^{-2\delta}E + (1 - e^{-2\delta})d$, hence we have

$$\int |v|^2 \mathrm{d}\mu_r^{\delta,\gamma,\varepsilon}(v) \leq E + d \tag{4.5}$$

for all $r \in [0, T], \delta, \gamma, \varepsilon > 0$.
Next, we look at the behaviour of the action and dissipation under the regularization.

Claim 4.2. Put (dropping the time-parameter r from the notation) $F(X) = f f_*$ and

$$L_1(X, \omega) = \log \frac{T_\omega F}{F}\big(T_\omega F - F\big)(X) \,, \;\; L_2(X, \omega) = \frac{|U(X, \omega)|^2}{\Lambda(F, T_\omega F)(X)} \,.$$

and $K_1 = L_1 B$, $K_2 = L_2/B$ with $B(X, \omega) = B(v - v_*, \omega)$. Then we have

$$\frac{|U^\delta|^2}{\Lambda(f^\delta)} \leq S_\delta L_2 \leq C_B S_\delta K_2 \,, \quad |\bar{\nabla} \log f^\delta|^2 \Lambda(f^\delta) \leq S_\delta L_1 \leq \frac{1}{c_B} S_\delta K_1 \,, \tag{4.6}$$

$$\mathcal{A}(\mu^\delta, \mathcal{U}^\delta) \leq \frac{C_B}{c_B} \mathcal{A}(\mu, \mathcal{U}) \,, \quad D(\mu^\delta) \leq \frac{C_B}{c_B} D(\mu) \,. \tag{4.7}$$

Proof of Claim 4.2. We consider the action. Note that

$$\frac{|U^\delta|^2}{\Lambda(f^\delta)}(X, \omega) = \frac{|S_\delta U|^2}{\Lambda(S_\delta F, T_\omega(S_\delta F))}(X)$$

Using the commutation relation $T_\omega(S_\delta F) = S_\delta(T_\omega F)$ from Lemma 2.3 and Jensen's inequality applied to the convex function $(u, x, y) \mapsto |u|^2/\Lambda(x, y)$ we obtain the first inequality in (4.6). Then (2.5) yields the second one. The first estimate in (4.7) follows from (4.6) by noting that

$$4\mathcal{A}(\mu^\delta, \mathcal{U}^\delta) = \int_G \frac{|U^\delta|^2}{\Lambda(f^\delta) B} \leq \frac{C_B}{c_B} \int_G S_\delta K_2 = \frac{C_B}{c_B} \int_G K_2 = 4 \frac{C_B}{c_B} \mathcal{A}(\mu, \mathcal{U}) \,.$$

The remaining estimates for the dissipation follow similarly, using the convexity of the function $(x, y) \mapsto \log(x/y)(x - y)$. □

A similar convexity argument gives that

$$\int_0^T \mathcal{A}(\mu_r^{\delta,\gamma}, \mathcal{U}_r^{\delta,\gamma}) \mathrm{d}r \leq \frac{C_B}{c_B} \int_0^T \mathcal{A}(\mu_r, \mathcal{U}_r) \mathrm{d}r \,. \tag{4.8}$$

Taking into account Corollary 3.9 and (4.5) we obtain

$$\int_0^T \int \big[1 + |v| + |v_*|\big] \mathrm{d}|\mathcal{U}_r^{\delta,\gamma}| \mathrm{d}r \leq C \,, \tag{4.9}$$

uniformly in $\delta, \gamma > 0$.
Step 3: Integrated chain rule for regularized curve.

Now, we claim that

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{H}(\mu_r^{\delta,\gamma,\varepsilon}) = \int_{\mathbb{R}^d} \log f_r^{\delta,\gamma,\varepsilon}\partial_r f_r^{\delta,\gamma,\varepsilon} = \frac{1}{4}\int_G \bar{\nabla}\log f_r^{\delta,\gamma,\varepsilon} U_r^{\delta,\gamma,\varepsilon}\ , \tag{4.10}$$

where the integral over G is w.r.t. the measure $\mathrm{d}X\mathrm{d}\omega$. Indeed, to justify the first identity in (4.10) we use convexity of $r \mapsto r\log r$ and (4.4) to estimate

$$\begin{aligned}&\frac{1}{h}\big|f_{r+h}^{\delta,\gamma,\varepsilon}\log f_{r+h}^{\delta,\gamma,\varepsilon} - f_r^{\delta,\gamma,\varepsilon}\log f_r^{\delta,\gamma,\varepsilon}\big| \le \frac{1}{h}\big|f_{r+h}^{\delta,\gamma,\varepsilon} - f_r^{\delta,\gamma,\varepsilon}\big|C(1+|v|)\\ \le& C(1+|v|)\left\|\eta'\right\|_\infty \int_0^T \big(f_t^\delta + \varepsilon g\big)\mathrm{d}t\ .\end{aligned}$$

Since $(f_t)_t$ has uniformly bounded second moment, by dominated convergence we can take the time derivative inside the integral. The second identity in (4.10) follows by applying the collision rate equation, using (4.4) and Remark 3.3.
Integrating (4.10) between s and t we obtain

$$\mathcal{H}(f_t^{\delta,\gamma,\varepsilon}) - \mathcal{H}(f_s^{\delta,\gamma,\varepsilon}) = \int_s^t \frac{1}{4}\int_G \bar{\nabla}\log f_r^{\delta,\gamma,\varepsilon} U_r^{\delta,\gamma,\varepsilon}\mathrm{d}r\ . \tag{4.11}$$

Step 4: Passing to the limit.
We will now pass to the limit in (4.11) to obtain (4.2) letting $\gamma \to 0$, $\varepsilon \to 0$ and $\delta \to 0$ in this order. Consider first the right hand side.

a) RHS, $\gamma \to 0$.
Using the bound $|\log f_r^{\delta,\gamma,\varepsilon}| \le c(\delta,\varepsilon)(1+|v|)$ ensured by (4.4) which is uniform in γ for fixed δ,ε and the integrability condition (4.9) for $U^{\delta,\gamma}$, we can pass to the limit as $\gamma \to 0$ and obtain

$$(1+\varepsilon)^{-1}\int_s^t \frac{1}{4}\int_G \bar{\nabla}\log(f_r^\delta + \varepsilon g)U_r^\delta \mathrm{d}r\ . \tag{4.12}$$

b) RHS, $\varepsilon \to 0$.
We can pass to the limit as $\varepsilon \to 0$ in the integral over G in (4.12) via dominated convergence, using the estimate (dropping time parameter r in the notation):

$$\begin{aligned}&|\bar{\nabla}\log(f^\delta + \varepsilon g)U^\delta| \le \frac{1}{2}|\bar{\nabla}\log(f^\delta+\varepsilon g)|^2\Lambda(f^\delta+\varepsilon g) + \frac{1}{2}\frac{|U^\delta|^2}{\Lambda(f^\delta+\varepsilon g)}\\ &\le \frac{1}{2}\Big|\bar{\nabla}\log(f^\delta+\varepsilon g)\Big|\cdot\Big|(f^\delta+\varepsilon g)\big((f^\delta)_* + \varepsilon g_*\big) - \big((f^\delta)' + \varepsilon g'\big)\big((f^\delta)'_* + \varepsilon g'_*\big)\Big|\\ &\quad + \frac{1}{2}\frac{|U^\delta|^2}{\Lambda(f^\delta)}\ .\end{aligned} \tag{4.13}$$

Here, in the second inequality we have used the definition of Λ and the monotonicity of the logarithmic mean. The first term is integrable thanks to the bound (2.8) and the fact that f^δ and g have finite second moment. The argument in Claim 4.2 yields that the second term in (4.13) is integrable for a.e. r.
To pass to the limit in the time integral in (4.12) it suffices to exhibit in a similar way a majorant for the space integral:

$$\left|\int_G \bar{\nabla}\log(f^\delta+\varepsilon g)U^\delta\right| \le \left(\int_G 2R\right)^{\frac{1}{2}}\left(\int_G \frac{|U_r^\delta|^2}{\Lambda(f_r^\delta)}\right)^{\frac{1}{2}} \le 2\sqrt{C_\delta C_B}\sqrt{\mathcal{A}(\mu_r,\mathcal{U}_r)}\ ,$$

where R stands for the first summand in (4.13) and we used again the bound (2.8) and Claim 4.2. Summarizing, we can pass to the limit as $\varepsilon \to 0$ in (4.12) and obtain

$$\int_s^t \frac{1}{4} \int_G \bar{\nabla} \log f_r^\delta U_r^\delta \mathrm{d}r \ . \tag{4.14}$$

c) RHS, $\delta \to 0$.
Note that $\bar{\nabla} \log f_r^\delta U_r^\delta$ converges pointwise to $\bar{\nabla} \log f_r U_r$ as $\delta \to 0$ at every r where the densities of $\mu_r, \mathcal{U}_r$ exist. To pass to the limit in the integral over G it suffices to exhibit a sequence of majorants converging in $L^1(G)$. We estimate (dropping the time parameter r in the notation)

$$|\bar{\nabla} \log f^\delta U^\delta| \leq \sqrt{|\bar{\nabla} \log f^\delta|^2 \Lambda(f^\delta)} \sqrt{\frac{|U^\delta|^2}{\Lambda(f^\delta)}} \leq \frac{C_B}{c_B} \sqrt{S_\delta K_1} \sqrt{S_\delta K_2} \ ,$$

with the notation from Claim 4.2. Note that $\int_G K_1 = 4D(\mu_r)$ and $\int_G K_2 = 4\mathcal{A}(\mu_r, \mathcal{U}_r)$. By assumption these quantities are finite for a.e. $r \in [0,T]$. Thus, for a.e. r we have that $S_\delta K_i$ converges to K_i in $L^1(G)$. Hence, also our majorant $\sqrt{S_\delta K_1 S_\delta K_2}$ converges to $\sqrt{K_1 K_2}$ in $L^1(G)$. Finally, to pass to the limit in the time integral, we use the already established almost everywhere in time convergence of the space integral and exhibit a majorant similar as above:

$$\int_G \bar{\nabla} \log f_r^\delta U_r^\delta \mathrm{d}r \leq \frac{C_B}{c_B} \left(\int_G S_\delta K_1 \right)^{\frac{1}{2}} \left(\int_G S_\delta K_2 \right)^{\frac{1}{2}} = \frac{4C_B}{c_B} \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)} \ .$$

Recall that the last expression is integrable by assumption.

d) LHS.
Let us turn to show convergence of the left hand side of (4.11). Appealing to the bound (4.4) for g we obtain the estimate

$$\left| \mathcal{H}(f_t^{\delta,\gamma,\varepsilon}) - \mathcal{H}(f_t^{\delta,\varepsilon}) \right| \leq C \int \left(1 + |v|\right) \left| f_{t-\gamma t'}^{\delta,\varepsilon} - f_t^{\delta,\varepsilon} \right| \eta(t') \mathrm{d}t' \ .$$

and we can pass to the limit as $\gamma \to 0$ by the continuity of $t \mapsto (1+|v|) f_t^{\delta,\varepsilon}$ in L^1, see Remark 3.3. The bound (2.8) allows to pass to the limit as $\varepsilon \to 0$ and we are left with $\mathcal{H}(f_t^\delta) - \mathcal{H}(f_s^\delta)$. Assume first that $\mathcal{H}(\mu_s)$ is finite. Recall that entropy is decreasing along the Ornstein–Uhlenbeck semigroup and lower semicontinuous. As $\delta \to 0$ we thus have that $\mathcal{H}(f_t^\delta)$ increases to $\mathcal{H}(\mu_t)$. Thus, $\mathcal{H}(f_t^\delta) - \mathcal{H}(f_s^\delta)$ converges to $\mathcal{H}(f_t) - \mathcal{H}(f_s)$ and $\mathcal{H}(\mu_t)$ is finite due to the boundedness of the right hand side of (4.2) in the limit. Since by assumption there exists s with $\mathcal{H}(\mu_s) < \infty$, this shows that $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0,T]$ and (4.2) is established.
Finally, using the estimate

$$\frac{1}{4} \int_G \bar{\nabla} \log f_r \mathrm{d}\mathcal{U}_r \leq \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)} \ , \tag{4.15}$$

that is obtained just as the one before for f_r^δ we see that $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and (4.3) follows. □

We can now prove the variational characterization of the homogeneous Boltzmann equation as the gradient flow of the entropy. For convenience we rephrase the statement here.
By a weak solution to the homogeneous Boltzmann equation we mean a weakly continuous family of probability densities $(f_t)_{t \geq 0}$ such that we have for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ in distribution sense:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \varphi f_t = -\frac{1}{4} \int \int_G \bar{\nabla} \varphi (f' f'_* - f f_*) B(v - v_*, \omega) \mathrm{d}v \mathrm{d}v_* \mathrm{d}\omega \mathrm{d}t \ . \tag{4.16}$$

Theorem 4.3. *For any curve $(f_t)_{t\in[0,T]}$ of probability densities such that*

$$\mathcal{H}(f_0) < \infty\,, \qquad t \mapsto \mathcal{E}(f_t) \text{ is bounded}\,, \tag{4.17}$$

we have that:

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2}\int_0^T D(f_t)\mathrm{d}t + \frac{1}{2}\mathcal{A}_T(f) \geq 0\,.$$

Moreover, we have $J_T(f) = 0$ if and only if $(f_t)_t$ is a weak solution to the homogeneous Boltzmann equation satisfying the integrability assumptions (4.17) and

$$\int_0^T D(f_t)\mathrm{d}t < \infty\,. \tag{4.18}$$

Assuming finite entropy and energy of the initial datum f_0, Theorem 2.2 gives existence and uniqueness of a classical solution $(f_t)_t$ to the homogeneous Boltzmann equation. It satisfies (4.17) and (2.6), in particular, (4.18) holds. Thus, there is actually only one curve such that $J_T(f) = 0$, namely the unique solution to the Boltzmann equation.

Proof of Theorem (4.3). Let $(f_t)_{t\in[0,T]}$ be a curve satisfying (4.17) and let E be a bound for the energy of f_t for $t \in [0,T]$. To show $J_T(f) \geq 0$ we can assume that $\mathcal{A}_T(f) < \infty$ and $\int_0^T D(f_t)\mathrm{d}t < \infty$, since otherwise $J_T(f) = +\infty$. Let $(\mathcal{U}_t)_t$ be optimal collision rates given by Proposition 3.11. But then $J_T(f) \geq 0$ follows immediately from Proposition 4.1 and the estimate (4.15).
We now show that any weak solution (f_t) satisfying (4.17) and (4.18) satisfies $J_T(f) = 0$. Let again E be a bound for the energy of f_t for $t \in [0,T]$. Setting $\mu_t = f_t\mathcal{L}$ and

$$\mathcal{U}_t = -\bar{\nabla}\log f_t\Lambda(f_t)\mathcal{B} = -\big[(f')_t(f'_*)_t - f_t(f_*)_t\big]\mathcal{B}\,,$$

we see by (4.16) that $(\mu,\mathcal{U})$ belongs to $\mathcal{CRE}$. Moreover, we have that $\mathcal{A}(\mu_t,\mathcal{U}_t) = D(f_t)$ and thus by (4.18) we can apply the chain rule (4.2) to obtain

$$\mathcal{H}(f_T) - \mathcal{H}(f_0) = -\int_0^T D(f_t)\mathrm{d}t = -\frac{1}{2}\int_0^T D(f_t)\mathrm{d}t - \frac{1}{2}\mathcal{A}_T(\mu)\,,$$

i.e. $J_T(f) = 0$.
Conversely, let us show that any curve $(f_t)_t$ with $J_T(f) = 0$ is a weak solution satisfying (4.18). From (4.17) we obtain that $\mathcal{H}(\mu_t) < \infty$ for all t and that $\mathcal{A}_T(f) < \infty$ and (4.18) holds. By Proposition 3.11 there exists a family $\mathcal{U}_t$ with $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$ such that $\int_0^T \mathcal{A}(f_t,\mathcal{U}_t)\mathrm{d}t = \mathcal{A}_T(f)$, in particular $t \mapsto f_t$ is weakly continuous. By Lemma 3.6 the measure $\mathcal{U}_t$ has a density $U_t\Lambda(f_t)B$. From the chain rule (4.2) and the Cauchy–Schwartz and Young inequalities we infer that

$$\begin{aligned}
\mathcal{H}(f_T) - \mathcal{H}(f_0) &= \int_0^T \frac{1}{4}\int_G \bar{\nabla}\log f_r U_r\Lambda(f_r)B\mathrm{d}r \\
&\geq -\int_0^T \left[\sqrt{\frac{1}{4}\int_G |\bar{\nabla}\log f_r|^2\Lambda(f_r)B}\sqrt{\frac{1}{4}\int_G |U_r|^2\Lambda(f_r)B}\right]\mathrm{d}r \\
&\geq -\frac{1}{2}\int_0^T \left[\frac{1}{4}\int_G |\bar{\nabla}\log f_r|^2\Lambda(f_r)B + \frac{1}{4}\int_G |U_r|^2\Lambda(f_r)B\right]\mathrm{d}r \\
&= -\frac{1}{2}\int_0^T D(f_r)\mathrm{d}r - \frac{1}{2}\mathcal{A}_T(f)\,.
\end{aligned}$$

Since $J_T(f) = 0$, we see that the two inequalities have to be identities. This implies that $\int_0^T\int |U_r + \bar{\nabla}\log f_r|^2\Lambda(f_r)B\mathrm{d}t = 0$, hence $U_r = -\bar{\nabla}\log f_r$ for a.e. r and a.e. (v,v_*,ω) with $\Lambda(f_r)(v,v_*,\omega) > 0$. Thus, the collision rate equation for $(\mu,\mathcal{U})$ turns into the weak formulation of the Boltzmann equation. □

5. Consistency with Kac's random walk

In this section we give a new proof of the convergence of Kac's random walk to the solution of the spatially homogeneous Boltzmann equation, see Theorem 1.2, exploiting that both evolutions have a gradient flow structure. We recall from Section 1.2 that Kac's random walk is the continuous time Markov chain on

$$\mathcal{X}_N := \left\{ (v_1, \dots, v_N) \in \mathbb{R}^{dN} \mid \sum_{i=1}^N v_i = 0 \, , \, \sum_{i=1}^N |v_i|^2 = Nd \right\} \, ,$$

with generator

$$Af(\boldsymbol{v}) = \frac{1}{N} \int_{S^{d-1}} \sum_{i<j} \left[f(R_{ij}^\omega \boldsymbol{v}) - f(\boldsymbol{v}) \right] B(v_i - v_j, \omega) \mathrm{d}\omega \, , \tag{5.1}$$

where $R_{ij}^\omega \boldsymbol{v} = (v_1, \dots, v_i', \dots, v_j', \dots, v_N)$, with $v_i' = v_i - \langle v_i - v_j, \omega \rangle \, \omega$ and $v_j' = v_j + \langle v_i - v_j, \omega \rangle \, \omega$. Let us denote by π_N the normalized Hausdorff measure on $\mathcal{X}_N$ and note that the Markov chain is reversible with respect to π_N. Denoting by μ_t^N the law of the chain starting in μ_0^N. Then its density f_t^N w.r.t. π_N satisfies Kac's master equation

$$\partial_t f_t^N = A f_t^N \, . \tag{5.2}$$

We recall the following result. For $\boldsymbol{v} \in \mathbb{R}^{Nd}$ and $p \geq 1$ we set

$$\mathcal{E}_p^N(\boldsymbol{v}) := \frac{1}{N} \sum_{i=1}^N |v_i|^p \, .$$

Lemma 5.1 (Propagation of moments for Kac's random walk, [15, Lem. 5.3])**.** *Let μ_0^N an initial condition with $\langle \mathcal{E}_p^N, \mu_0^N \rangle = \int \mathcal{E}_p^N \mathrm{d}\mu_0^N < \infty$. Then the law $(\mu_t^N)_{t \geq 0}$ of Kac's random walk satisfies*

$$\sup_{t \geq 0} \langle \mathcal{E}_p^N, \mu_t^N \rangle \leq \max\{C_p, \langle \mathcal{E}_p^N, \mu_0^N \rangle\} \, ,$$

for some constant C_p depending only on p.

We will first detail the gradient flow structure of the master equation.

5.1. **Gradient flow structure.** Kac's random walk possesses the structure of a gradient flow in $\mathscr{P}(\mathcal{X}_N)$ of the relative entropy $\mathcal{H}(\cdot|\pi_N)$ with respect to a suitable geometry on $\mathscr{P}(\mathcal{X}_N)$ as we shall now describe. For general Markov chains on finite state spaces a gradient flow structure has been discovered in [12, 14]. Here we briefly show how to extend this result to the present case of the continuous state space $\mathcal{X}_N$. The construction is similar as in Section 3, see also [10]. Let us stress however that for the purpose of showing consistency with the Boltzmann equation it will only be important to know that the solution $(f_t)_t$ to (5.2) satisfies the energy identity $J_T^N(f) = 0$, see (5.5) below.
We introduce a jump kernel on $\mathcal{X}_N$ by setting

$$J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u}) \;=\; \frac{1}{2N} \int_{S^{d-1}} \sum_{i,j=1}^N \delta_{R_{ij}^\omega \boldsymbol{v}}(\mathrm{d}\boldsymbol{u}) B(v_i - v_j, \omega) \mathrm{d}\omega \, .$$

Given a probability measure $\mu \in \mathscr{P}(\mathcal{X}_N)$ we define $\mu^1, \mu^2 \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ via

$$\mathrm{d}\mu^1(\boldsymbol{v}, \boldsymbol{u}) = J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u}) \mathrm{d}\mu(\boldsymbol{v}) \, , \quad \mathrm{d}\mu^2(\boldsymbol{v}, \boldsymbol{u}) = J(\boldsymbol{u}, \mathrm{d}\boldsymbol{v}) \mathrm{d}\mu(\boldsymbol{u}) \, . \tag{5.3}$$

For a pair $(\mu, \mathcal{V})$ with $\mu \in \mathscr{P}(\mathcal{X}_N)$ and $\mathcal{V} \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ we define the action

$$\mathcal{A}^N(\mu, \mathcal{V}) := 2\mathcal{F}_\alpha(\mu^1, \mu^2, \mathcal{V}) \, ,$$

where $\mathcal{F}_\alpha$ is defined in (3.6). We define a distance on $\mathscr{P}(\mathcal{X}_N)$ by setting

$$\mathcal{W}_N(\mu_0,\mu_1)^2 := \inf_{\mu,\mathcal{V}} \int_0^1 \mathcal{A}^N(\mu_t,\mathcal{V}_t)\mathrm{d}t \ ,$$

where the infimum is taken over all curves $(\mu_t)_{t\in[0,1]}$ connecting μ_0 to μ_1 and all $(\mathcal{V}_t)_{t\in[0,1]}$ subject to the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{X}_N} \varphi \mathrm{d}\mu_t - \frac{1}{2}\int_{\mathcal{X}_N^2} [\varphi(\boldsymbol{u})-\varphi(\boldsymbol{v})]\mathrm{d}\mathcal{V}_t(\boldsymbol{v},\boldsymbol{u}) = 0 \ , \quad \forall \varphi \in C_b(\mathcal{X}_N) \ .$$

It follows from the results in [10, Thm. 4.4, Prop. 4.3], by considering J as a jump kernel on the ambient space $\mathbb{R}^{dN}$, that $\mathcal{W}_N$ defines a distance and that the infimum in the definition is attained by an optimal pair $(\mu,\mathcal{V})$. For a curve $(\mu_t)_{t\in[0,T]}$ in $\mathscr{P}(\mathcal{X}_N)$ we define its action by

$$\mathcal{A}_T^N(\mu) := \inf\left\{\int_0^T \mathcal{A}_N(\mu_t,\mathcal{V}_t)\mathrm{d}t\right\} \ ,$$

where the infimum is taken over all $(\mathcal{V}_t)_t$ such that $(\mu,\mathcal{V})$ satisfy the continuity equation. There exists an optimal $\mathcal{V}$ attaining the infimum, see [10, Prop. 4.3]. In fact, for a.e. t, $\mathcal{A}^N(\mu_t,\mathcal{V}_t)$ equals the metric derivative of the curve w.r.t. $\mathcal{W}_N$. We define the *entropy dissipation* of $\mu \in \mathscr{P}(\mathcal{X}_N)$ by

$$D^N(\mu) = \frac{1}{4N}\int_{\mathcal{X}_N}\int_{S^{d-1}} \sum_{i,j}\Big[f(R_{ij}^\omega \boldsymbol{v}) - f(\boldsymbol{v})\Big] \\ \times\Big[\log f(R_{ij}^\omega \boldsymbol{v}) - \log f(\boldsymbol{v})\Big] B(v_i - v_j,\omega)\mathrm{d}\omega\mathrm{d}\pi_N(\boldsymbol{v}) \ ,$$

provided $\mu = f\pi_N$ and we set $D^N(\mu) = +\infty$ if μ is not absolutely continuous. Note that along any solution f_t to the master equation (5.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f_t|\pi_N) = -D^N(f_t) \ . \tag{5.4}$$

Proposition 5.2. *For any curve $(\mu_t)_{t\in[0,T]}$ in $\mathscr{P}(\mathcal{X}_N)$ with $\mathcal{H}(\mu_0|\pi_N) < \infty$ we have*

$$J_T^N(\mu) = \mathcal{H}(\mu_T|\pi_N) - \mathcal{H}_N(\mu_0|\pi_N) + \frac{1}{2}\int_0^T D^N(\mu_t)\mathrm{d}t + \frac{1}{2}\mathcal{A}_T^N(\mu) \geq 0 \ . \tag{5.5}$$

Moeover, $J_T^N(\mu) = 0$ holds if and only if $\mu_t = f_t\pi_N$ where f_t solves (5.2).

Proof. We will focus on showing that any solution $(\mu_t)_t$ to the master equation (5.2) satisfies $J_T^N(\mu) = 0$ since this will be used in the sequel. The other statements can be obtained by following a similar line of reasoning as in Section 4, namely establishing a chain rule for the entropy analogous to Proposition 4.1 via a regularization argument (in fact the situation is much simpler due to linearity of the master equation).
Let $\mu_t = f_t\pi^N$ be a solution to the master equation (5.2). Then the couple $(\mu_t,\mathcal{V}_t)$ solves the continuity equation if we choose

$$\mathrm{d}\mathcal{V}_t(\boldsymbol{v},\boldsymbol{u}) = \Psi_t(\boldsymbol{v},\boldsymbol{u})\Lambda(f_t(\boldsymbol{v}),f_t(\boldsymbol{u}))J(\boldsymbol{v},\mathrm{d}\boldsymbol{u})\pi^N(\mathrm{d}v)$$

with $\Psi_t(\boldsymbol{v},\boldsymbol{u}) = \log f_t(\boldsymbol{u}) - \log f_t(\boldsymbol{v})$. Note moreover that $\mathcal{A}(\mu_t,\mathcal{V}_t) = D_N(\mu_t)$. Thus, integrating (5.4) yields $J_T(\mu) = 0$. □

5.2. **Convergence to the Boltzmann equation.** In this section we will give a new proof that the distribution of the empirical measure of N particles evolving by Kac's random walk converges to the solution of the homogeneous Boltzmann equation as $N \to \infty$. For convenience let us recall the setup and the convegence statement.
Consider the map assigning to a configuration in $\mathcal{X}_N$ its empirical measure

$$L_N : \mathcal{X}_N \to \mathscr{P}(\mathbb{R}^d) \ , \quad \boldsymbol{v} \mapsto \frac{1}{N} \sum_{i=1}^N \delta_{v_i} \ .$$

Let us set

$$\mathscr{P}_*(\mathbb{R}^d) := \big\{ \mu \in \mathscr{P}(\mathbb{R}^d) : \mathcal{M}(\mu) = 0, \mathcal{E}(\mu) = d \big\} \ ,$$

the set of probability measures with zero momentum and energy d, recall (2.1). Note that for any $\boldsymbol{v} \in \mathcal{X}_N$ we have $L_N \boldsymbol{v} \in \mathscr{P}_*(\mathbb{R}^d)$. Let us denote by $M = M^{0,d}$ the standard Maxwellian distribution and by $\mathcal{H}(\mu|M)$ the relative entropy, see (2.3). We consider $\mathscr{P}_*(\mathbb{R}^d)$ as a subset of $\mathscr{P}(\mathbb{R}^d)$ equipped with the topology of weak convergence.

Theorem 5.3. *For each N let $(\mu_t^N)_{t\geq 0}$ be the law of Kac's random walk starting from μ_0^N and let $c_t^N := (L_N)_\# \mu_t^N$ be the law of the empirical measures. Assume that μ_0^N is well-prepared for some $\nu_0 = f_0 \mathcal{L} \in \mathscr{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\nu_0|M) < \infty$ in the sense that in the limit $N \to \infty$*

$$c_0^N \rightharpoonup \delta_{\nu_0} \ , \quad \frac{1}{N} \mathcal{H}(\mu_0^N|\pi_N) \to \mathcal{H}(\nu_0|M) \ .$$

Assume further that for some $p > 2$

$$\sup_N \big\langle \mathcal{E}_p^N, \mu_0^N \big\rangle < \infty \ .$$

Then, for all $t > 0$, as $N \to \infty$ we have

$$c_t^N \rightharpoonup \delta_{\nu_t} \ , \quad \frac{1}{N} \mathcal{H}(\mu_t^N|\pi_N) \to \mathcal{H}(\nu_t|M) \ , \tag{5.6}$$

where $\nu_t = f_t \mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0.

The strategy of the proof will be to pass to the limit in the variantional formulation of the master equation and obtain the variational formulation of the Boltzmann equation. The key ingredient to this will be to establish lim inf estimates relating the entropy, dissipation and action for the Kac walk and the Boltzmann equation. Although the proofs of the latter might seem long, the core argument is rather simple and boils down to the lower semicontinuity of integral functionals stated in Lemma 2.4. A non-trivial additional ingredient that we develop is a probabilistic representation result that allows to view certain curves in $\mathscr{P}(\mathscr{P}_*(\mathbb{R}^d))$ as superposition of curves in $\mathscr{P}_*(\mathbb{R}^d)$, see Propositon 5.5.
Let us now first give the proof of convergence theorem. Afterwards we will develop the necessary ingredients.

Proof of Theorem 5.3. By Proposition 5.2 we have that $(\mu_t^N)_{t\geq 0}$ satisfies

$$\mathcal{H}(\mu_T^N|\pi_N) - \mathcal{H}(\mu_0^N|\pi_N) + \frac{1}{2} \int_0^T D^N(\mu_t^N) \mathrm{d}t + \frac{1}{2} \mathcal{A}_T^N(\mu) = 0 \ . \tag{5.7}$$

Together with the convergence of $\mathcal{H}(\mu_0^N|\pi_N)/N$ this implies in particular

$$\sup_N \frac{1}{N} \mathcal{A}_T^N(\mu^N) < \infty \ .$$

The compactness result Lemma 5.4 then yields that up to a subsequence we have that $c_t^N \rightharpoonup c_t$ weakly for all t and a continuous curve $(c_t)_{t\geq 0}$ in $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$ with c_t concentrated on $\mathscr{P}_*(\mathbb{R}^d)$ for all t. A priori, c_t is not a Dirac measure. However, by the superposition

principle Proposition 5.5 the curve $(c_t)_{t\in[0,T]}$ can be represented as $c_t = (e_t)_\# \Theta$ for a probability measure Θ on $C\big([0,T], \mathscr{P}(\mathbb{R}^d)\big)$. Thanks to the lim inf-inequalities for the entropy, dissipation and action given by (5.29), (5.15) and (5.14), dividing by N in (5.7) and passing to the limes inferior we obtain

$$\int \left[\mathcal{H}(\eta_T) - \mathcal{H}(\eta_0) + \frac{1}{2}\int_0^T D(\eta_t)\mathrm{d}t + \frac{1}{2}\mathcal{A}_T(\eta) \right] \mathrm{d}\Theta(\eta) \leq 0 \,, \tag{5.8}$$

using also that $\mathcal{H}(\eta|M) = \mathcal{H}(\eta) + \mathcal{H}(M)$ for $\eta \in \mathscr{P}_*(\mathbb{R}^d)$ and that $\eta_0, \eta_T \in \mathscr{P}_*(\mathbb{R}^d)$ for Θ-a.e. η. By Theorem 4.3 the integrand is non-negative. Thus we have in fact equality in (5.8) and we infer that Θ is concentrated on gradient flow curves $(\eta_t)_t$, i.e. satisfying $J_T(\eta) = 0$. Since Θ-a.s. $\eta_0 = \nu_0$ and the unique gradient flow curve starting from ν_0 is given by $\nu_t = f_t\mathcal{L}$ with f_t the solution to the Boltzmann equation with initial datum f_0. Thus, we infer that $c_t = (e_t)_\#\Theta = \delta_{\nu_t}$ for all t and that the convergence of c_t^N to δ_{ν_t} holds for the full sequence. Finally, we prove (5.6). From the previous discussion we retain that

$$\begin{aligned} 0 \geq \liminf_N \frac{1}{N} J_T^N(\mu^N) - J_T(\nu) &= \liminf_N \frac{1}{N}\mathcal{H}(\mu_T^N|\pi_N) - \mathcal{H}(\nu_T|M) \\ &+ \frac{1}{2}\left[\liminf_N \frac{1}{N}\int_0^T D_N(\mu_t^N)\mathrm{d}t + \mathcal{A}_T^N(\mu^N) - \int_0^T D(\nu_t)\mathrm{d}t + \mathcal{A}_T(\nu) \right] \geq 0 \,. \end{aligned}$$

Using again (5.29), (5.14), (5.15), we infer that we have equality

$$\liminf_N \frac{1}{N}\mathcal{H}(\mu_t^N|\pi_N) = \mathcal{H}(\nu_t|M) \,.$$

Since by the same argument this must hold for any subsequence, we conclude the convergence (5.6) for the full sequence. □

We now develop the ingredients to the previous proof. We will first show that any sequence of curves in $\mathscr{P}(\mathcal{X}_N)$ with uniformly bounded action after passing to the empirical measure admits a limit curve in $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$. Then we will give a representation of this curve as a superposition of curves in $\mathscr{P}(\mathbb{R}^d)$ and establish lim inf inequalities for the action and dissipation of the limit curve. Finally, we prove the lim inf inequality for the entropy.

5.2.1. *Convergence to a limit curve.*

Lemma 5.4. *Let $(\mu_t^N)_{t\in[0,T]}$ be a sequence of curves in $\mathscr{P}(\mathcal{X}_N)$ such that*

$$\sup_N \frac{1}{N}\mathcal{A}_T^N(\mu^N) < \infty \,, \tag{5.9}$$

and put $c_t^N = (L_N)_\#\mu_t^N$. Then there exists a continuous curve $(c_t)_{t\in[0,T]}$ in $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$ such that up to a subsequence we have that $c_t^N \rightharpoonup c_t$ weakly for all $t \in [0,T]$. If we assume moreover that for some $p > 2$

$$\sup_N \sup_{t\in[0,T]} \langle \mathcal{E}_p^N, \mu_t^N \rangle < \infty \,, \tag{5.10}$$

then c_t is concentrated on $\mathscr{P}_*(\mathbb{R}^d)$ for all t.

Proof. We consider the set $\mathscr{P}_{2,E}(\mathbb{R}^d)$ of probability measures with energy less than E, with $E = d$, recall (2.2). Recall that $\mathscr{P}_{2,E}(\mathbb{R}^d)$ is compact w.r.t. weak convergence, hence also $\mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d))$ is compact. On $\mathscr{P}_{2,E}(\mathbb{R}^d)$, weak convergence is equivalent to convergence of the first moment, or convergence in the L^1-Wasserstein distance W_1. Let us denote by $\widetilde{W}_1$ the L^1-Wasserstein distance on $\mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d))$ induced by the L^1-Wasserstein distance

W_1 on $\mathscr{P}_{2,E}(\mathbb{R}^d)$. Since W_1 is bounded on $\mathscr{P}_{2,E}(\mathbb{R}^d)$, $(\mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d)), \widetilde{W}_1)$ is compact. We claim that

$$\mathcal{W}_N(\mu_s^N, \mu_t^N) \geq \frac{C}{\sqrt{N}} W_{1,d}(\mu_s^N, \mu_t^N) \geq C\sqrt{N}\widetilde{W}_1(c_s^N, c_t^N) \,, \tag{5.11}$$

for some universal constant $C > 0$. Indeed, the first inequality is given by [10, Prop. 4.5] where we view μ_s^N, μ_t^N as measures on $\mathbb{R}^{Nd}$ equipped with the distance $d(\boldsymbol{v}, \boldsymbol{u}) = \sum_i |v_i - u_i|$ and note that $\int_{\mathbb{R}^{Nd}} d(\boldsymbol{v}, \boldsymbol{u})^2 J(\boldsymbol{v}, \mathrm{d}\boldsymbol{u}) = CN$, and let $W_{1,d}$ denote the L^1-Wasserstein distance induced by d. The second inequality follows from the fact that the map L_N is $1/N$-Lipschitz from $(\mathbb{R}^{Nd}, d)$ to $(\mathscr{P}(\mathbb{R}^d), W_1)$. Together with (5.11), (5.9) implies that the curves $(c_t^N)_t$ are uniformly equicontinuous in $\mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d))$ w.r.t. the distance $\widetilde{W}_1$. Thus, the Arzela–Ascoli theorem yields that there exists a continuous curve $(c_t)_{t\in[0,T]}$ in $\mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d))$ such that up to extraction of a subsequence we have that $c_t^N \rightharpoonup c_t$ weakly for all $t \in [0, T]$.
Finally, assume in addition (5.10) and let us show that c_t is concentrated on $\mathscr{P}_*(\mathbb{R}^d)$ for all t. We need to show $c_t\big(\{\mathcal{M} = 0, \mathcal{E} = d\}\big) = 1$. Since $c_t^N\big(\{\mathcal{M} = 0\}\big) = 1$ and $\mathcal{M}$ is continuous on $\mathscr{P}_{2,E}(\mathbb{R}^d)$, and hence $\{\mathcal{M} = 0\}$ is closed, the weak convergence $c_t^N \rightharpoonup c_t$ implies that $c_t\big(\{\mathcal{M} = 0\}\big) = 1$. It remains to show that $c_t\big(\{\mathcal{E} = d\}\big) = 1$. Since c_t is concentrated on $\mathscr{P}_{2,E}(\mathbb{R}^d) = \{\mathcal{E} \leq d\}$, it suffices to show that $\langle \mathcal{E}, c_t\rangle = \lim_N \langle \mathcal{E}, c_t^N\rangle = d$. Set $\mathcal{E}_p(\eta) := \int |v|^p \mathrm{d}\eta(v)$, then (5.10) implies that for any t:

$$\sup_N \langle \mathcal{E}_p, c_t^N\rangle < \infty \,. \tag{5.12}$$

Note that $\mathcal{E}_2 = \mathcal{E}$. Since by Jensen's inequality we have $\mathcal{E}_2(\nu)^{p/2} \leq \mathcal{E}_p(\nu)$, (5.12) readily yields that $\mathcal{E}_2$ is uniformly integrable w.r.t. c_t^N. Moreover, $\sup_N c_t^N\big(\{\mathcal{E}_{2+\varepsilon} \geq R\}\big) \to 0$ as $R \to \infty$ for $\varepsilon < p-2$ and $\mathcal{E}_2$ is continuous on $\{\mathcal{E}_{2+\varepsilon} \leq R\}$. Thus the we obtain the desired convergence $\langle \mathcal{E}_2, c_t\rangle = \lim_N \langle \mathcal{E}_2, c_t^N\rangle$, see e.g. [1, Prop. 5.1.10]. □

5.2.2. *Superposition principle and limits for the action and dissipation.*

Proposition 5.5 (Superposition principle for the limit curve)**.** *Let $(\mu_t^N)_{t\in[0,T]}$ be a sequence of curves in $\mathscr{P}(\mathcal{X}_N)$ such that*

$$\sup_N \frac{1}{N}\mathcal{A}_T^N(\mu^N) < \infty \,, \tag{5.13}$$

put $c_t^N = (L_N)_\# \mu_t^N$, and let $(c_t)_{t\in[0,T]}$ be the limit curve of Lemma 5.4. Then, there exists a Borel probability measure Θ on $C\big([0,T], \mathscr{P}(\mathbb{R}^d)\big)$ and a Borel family of measures $(\mathcal{U}_t^\eta)_{t\in[0,T],\eta\in\mathscr{P}(\mathbb{R}^d)}$ such that the following hold:

- *$c_t = (e_t)_\#\Theta$ for all $t \in [0, T]$,*
- *for Θ-a.e. curve $(\eta_t)_{t\in[0,T]}$, the pair $(\eta_t, \mathcal{U}_t^{\eta_t})_{t\in[0,T]}$ belongs to $\mathcal{CRE}_T^E$, with $E = d$.*

Proposition 5.6 (lim inf-inequality for action and dissipation)**.** *In the setting of Proposition 5.5 we have*

$$\liminf_N \frac{1}{N}\mathcal{A}_T^N(\mu^N) \geq \int \mathcal{A}_T(\eta)\, \mathrm{d}\Theta(\eta) \,, \tag{5.14}$$

$$\liminf_N \frac{1}{N}\int_0^T D^N(\mu_t^N)\mathrm{d}t \geq \int \left[\int_0^T D(\eta_t)\mathrm{d}t\right] \mathrm{d}\Theta(\eta) \,, \tag{5.15}$$

where $\mathcal{A}_T(\eta)$, $D(\eta)$ are the action and dissipation defined in (3.12), (2.7).

In order to prove the superposition principle Proposition 5.5, we will describe curves in $\mathscr{P}(\mathscr{P}(\mathbb{R}^d)$ as curves in $\mathscr{P}(\mathbb{R}^\infty)$ by choosing a countable number of coordinates given by integrals against test functions. This allows to employ a superposition principle for

solutions to the continuity equation over $\mathbb{R}^\infty$ by Ambrosio and Trevisan [2]. Let us briefly recall this result.
Consider $\mathbb{R}^\infty = \mathbb{R}^\mathbb{N}$ and let $p_i : \mathbb{R}^\infty \to \mathbb{R}$ be the natural projections for $i \in \mathbb{N}$ and let $\pi_n = (p_1, \dots, p_n) : \mathbb{R}^\infty \to \mathbb{R}^n$. Equip $\mathbb{R}^\infty$ with the seperable and complete distance

$$d_\infty(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |p_n(x) - p_n(y)|\} \ .$$

In a similar way, $C\big([0,T], \mathbb{R}^\infty\big)$ can be equipped with a seperable and complete distance. We denote by $AC_w\big([0,T], \mathbb{R}^\infty\big)$ the subset of $C\big([0,T], \mathbb{R}^\infty\big)$ consisting of all γ such that $p_i \circ \gamma \in AC\big([0,T], \mathbb{R}\big)$ for all i.
A function $F : \mathbb{R}^\infty \to \mathbb{R}$ is called *smooth cylindrical*, if it is of the form

$$F(x) = \psi\big(p_1(x), \dots, p_n(x)\big) \ ,$$

for some $\psi \in C_b^1(\mathbb{R}^n)$ and $n \in \mathbb{N}$. Its gradient $\nabla F : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined by

$$\nabla F(x) = \big(\partial_1 \psi\big(\pi_n(x)\big), \dots, \partial_n \psi\big(\pi_n(x)\big), 0, 0, \dots\big) \ .$$

Then, we have the following representation result.

Theorem 5.7. [2, Thm. 7.1] *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be a Borel vector field and let $(\nu_t)_{t\in(0,T)}$ be a family of Borel probability measures on $\mathbb{R}^\infty$ continuous in duality with smooth cylinder functions satisfying*

$$\int_0^T |p_i(\boldsymbol{b}_t)| \mathrm{d}\nu_t \mathrm{d}t < \infty \quad \forall i \in \mathbb{N} \ , \tag{5.16}$$

and in the sense of distributions in $(0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int F \mathrm{d}\nu_t = \int (\boldsymbol{b}_t, \nabla F) \mathrm{d}\nu_t \quad \forall F \text{ smooth cylindrical} \, . \tag{5.17}$$

Then, there exists a Borel probability measure $\boldsymbol{\lambda}$ on $C\big([0,T], \mathbb{R}^\infty\big)$ satisfying $(e_t)_\# \boldsymbol{\lambda} = \nu_t$ for all t, concentrated on $\gamma \in AC_w\big([0,T], \mathbb{R}^\infty\big)$ solving the ODE $\dot\gamma = \boldsymbol{b}_t(\gamma)$ a.e. in $(0,T)$.

Proof of Proposition 5.5. We will proceed in 3 steps. Starting from a solution to the discrete continuity equation over $\mathcal{X}_N$ we pass to the emperical measure and obtain a limiting family of collision rates $\mathcal{U}_t^\eta$. Then, by choosing integrals against a collection of test functions as coordinates, we describe the limiting curve c via a continuity equation over $\mathbb{R}^\infty$ with a vector field determined by the collision rates $\mathcal{U}_t^\eta$. Finally, we apply the superposition principle for $\mathbb{R}^\infty$ and see that the obtained random curve in $\mathbb{R}^\infty$ is indeed the coordinate description of a random curve (η_t) in $\mathscr{P}(\mathbb{R}^d)$ solving the collision rate equation driven by the rates $\mathcal{U}_t^{\eta_t}$.
Step 1: Limiting collision rate. Recall from Section 5.1 that we can choose measures $\mathcal{V}_t^N \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ such that $\mathcal{A}_T^N(\mu^N) = \int_0^T \mathcal{A}_N(\mu_t^N, \mathcal{V}_t^N)\mathrm{d}t$. Let us define the measures $\mathcal{V}^N := \mathcal{V}_t^N \mathrm{d}t$ and $\mu^{N,k} := \mu_t^{N,k}\mathrm{d}t$, $k = 1, 2$, in $\mathcal{M}(\mathcal{X}^N \times \mathcal{X}^N \times [0,T])$. Note that by the stucture of the jump kernel J, for any $(\boldsymbol{v}, \boldsymbol{u})$ in the support of $\mu_t^{N,1}, \mu_t^{N,2}$, with $\boldsymbol{v} \neq \boldsymbol{u}$, there exist unique (i, j, ω) with $1 \le i < j \le N$, $\omega \in S^{d-1}$ such that $\boldsymbol{u} = R_{ij}^\omega(\boldsymbol{v})$ (when $\boldsymbol{v} = \boldsymbol{u}$, we pick $i = j$ and ω at random). We push forward $\mathcal{V}^N, \mu^{N,k}$ by the map $(\boldsymbol{v}, \boldsymbol{u}) \mapsto (L_N(\boldsymbol{v}), L_N(\boldsymbol{u}), v_i, v_j, \omega)$ with i, j, ω as above. This defines measures $\gamma^N, \beta^{N,k}$ on

$\mathscr{P}_{2,E}(\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \times S^{d-1} \times [0,T]$. We find that

$$\begin{aligned}
\mathrm{d}\beta^{N,1}(\eta,\eta',v,v_*,\omega,t) &= \frac{N}{2}\delta_{\eta^{N,v,v_*,\omega}}(\mathrm{d}\eta')B(v-v_*,\omega)\eta(\mathrm{d}v)\eta(\mathrm{d}v_*)\mathrm{d}\omega\mathrm{d}c_t^N(\eta)\mathrm{d}t \\
&= \frac{N}{2}\delta_{\eta^{N,v,v_*,\omega}}(\mathrm{d}\eta')\mathrm{d}\eta^1(v,v_*,\omega)\mathrm{d}c_t^N(\eta)\mathrm{d}t\ , \qquad (5.18)\\
\mathrm{d}\beta^{N,2}(\eta,\eta',v,v_*,\omega,t) &= \frac{N}{2}\delta_{\eta'^{N,T_\omega^{-1}(v,v_*),\omega}}(\mathrm{d}\eta)B(v-v_*,\omega) \\
&\qquad \mathrm{d}(T_\omega)_\# \eta'^{\otimes 2}(v,v_*)\mathrm{d}\omega\mathrm{d}c_t^N(\eta')\mathrm{d}t \\
&= \frac{N}{2}\delta_{\eta'^{N,T_\omega^{-1}(v,v_*),\omega}}(\mathrm{d}\eta)\mathrm{d}\eta'^2(v,v_*,\omega)\mathrm{d}c_t^N(\eta')\mathrm{d}t\ , \qquad (5.19)
\end{aligned}$$

where we set $\eta^{N,v,v_*,\omega} = \eta + \frac{1}{N}(\delta_{v'} + \delta_{v'_*} - \delta_v - \delta_{v_*})$ with v, v_*, v', v'_* related via (1.3) and recall that $c_t^N = (L_N)_\# \mu_t^N$ and recall the notation (3.8). To see this, note that $L_N(\boldsymbol{u}) = L_N(\boldsymbol{v})^{N,v_i,v_j,\omega}$ if $\boldsymbol{u} = R_{ij}^\omega(\boldsymbol{v})$ and that we can write

$$\sum_{i,j=1}^N f(v_i,v_j) = N^2 \int f(v,v_*)L_N(\boldsymbol{v})(\mathrm{d}v)L_N(\boldsymbol{v})(\mathrm{d}v_*)\ .$$

To obtain the expression for $\beta^{N,2}$, note further, that if $\boldsymbol{v} = R_{i,j}^\omega(\boldsymbol{u})$, we have that $(v_i,v_j) = T_\omega(u_i,u_j)$.

From the weak convergence of c_t^N to c_t for all t granted by Lemma 5.4, we infer that as $N \to \infty$ we have $\frac{2}{N}\beta^{N,k} \rightharpoonup \beta^k$ in duality with C_b where

$$\mathrm{d}\beta^k(\eta,\eta',v,v_*,\omega,t) = \delta_\eta(\mathrm{d}\eta')\mathrm{d}\eta^k(v,v_*,\omega)\mathrm{d}c_t(\eta)\mathrm{d}t\ . \qquad (5.20)$$

From Lemma 2.4 (ii) we infer that

$$\mathcal{F}_\alpha\left(\frac{2}{N}\beta^{N,1},\frac{2}{N}\beta^{N,2},\frac{2}{N}\gamma^N\right) \le \frac{2}{N}\mathcal{F}_\alpha(\mu^{N,1},\mu^{N,2},\mathcal{V}^N) = \frac{1}{N}\mathcal{A}_T^N(\mu^N)\ ,$$

and the last expression is bounded by assumption. From Lemma 3.8 we infer as in the proof of Proposition 3.11 that $\frac{2}{N}\gamma^N$ has uniformly bounded variation and hence converges weakly* up to a further subsequence to a limit γ. This can be improved to convergence in duality with bounded continuous functions using again Lemma 3.8 and the fact that $\beta^{N,k}$ converge in duality with bounded continuous functions. By lower semicontinuity and homogeneity of $\mathcal{F}_\alpha$ we find

$$\mathcal{F}_\alpha(\beta^1,\beta^2,\gamma) \le \liminf_N \frac{1}{N}\mathcal{A}_T^N(\mu^N)\ . \qquad (5.21)$$

As in Lemma 3.6 we infer from finiteness of the left hand side that γ is absolutely continuous w.r.t. the measure $L := \delta_\eta(\mathrm{d}\eta')\Lambda(\eta^1,\eta^2)c_t(\mathrm{d}\eta)\mathrm{d}t$, where $\Lambda(\eta^1,\eta^2) := \Lambda(\frac{\mathrm{d}\eta^1}{\mathrm{d}\sigma},\frac{\mathrm{d}\eta^2}{\mathrm{d}\sigma})\mathrm{d}\sigma$ for any σ such that $\eta^1,\eta^2 \ll \sigma$. Hence there exists a Borel function $U : \mathscr{P}_{2,E}(\mathbb{R}^d)^2 \times (\mathbb{R}^d)^2 \times S^{d-1} \times [0,T] \to \mathbb{R}$ such that $\gamma = UL$ and we can write

$$\mathrm{d}\gamma(\eta,\eta',v,v_*,\omega,t) = \delta_\eta(\mathrm{d}\eta')\mathrm{d}\mathcal{U}_t^\eta(v,v_*,\omega)\mathrm{d}c_t(\eta)\mathrm{d}t\ , \qquad (5.22)$$

where $(\mathcal{U}_t^\eta)_{\eta,t}$ is the Borel family of measures defined by

$$\mathrm{d}\mathcal{U}_t^\eta(v,v_*,\omega) = U(\eta,\eta,v,v_*,\omega,t)\mathrm{d}\Lambda(\eta^1,\eta^2)(v,v_*,\omega)\ .$$

Note further that

$$\mathcal{F}_\alpha(\gamma,\beta^1,\beta^2) = \int_0^T \int \mathcal{A}(\eta,\mathcal{U}_t^\eta)\mathrm{d}c_t(\eta)\mathrm{d}t\ . \qquad (5.23)$$

Step 2: Continuity equation in $\mathbb{R}^\infty$. We now describe the curve (c_t) as an evolution in $\mathscr{P}(R^\infty)$. Fix a countable collection $\{f_i\}_{i\in\mathbb{N}}$ of functions that is dense (w.r.t. uniform

convergence) in the set of 1-Lipschitz functions on $\mathbb{R}^d$ vanishing at 0. Define a map $I : \mathscr{P}_{2,E}(\mathbb{R}^d) \to \mathbb{R}^\infty$ by setting

$$I(\eta) := \big(\langle f_1, \eta \rangle , \langle f_2, \eta \rangle , \dots \big) ,$$

and write $I^m = \pi_m \circ I$. Note that I is injective and continuous w.r.t. the distance W_1 on $\mathscr{P}_{2,E}(\mathbb{R}^d)$ by Kantorovich duality. $I(\mathscr{P}_{2,E}(\mathbb{R}^d))$ is closed in $\mathbb{R}^\infty$, since $(\mathscr{P}_{2,E}(\mathbb{R}^d), W_1)$ is compact, and $I^{-1} : I(X) \to \mathscr{P}_{2,E}(\mathbb{R}^d)$ is continuous w.r.t. W_1.
We define a curve $(\nu_t)_{t\in[0,T]}$ via $\nu_t := I_\# c_t$ and note that it is continuous in duality with smooth cylinder functions by continutity of $t \mapsto c_t$. We define a Borel vector field $\boldsymbol{b} : (0,T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ via

$$\boldsymbol{b}^i_t(x) = \begin{cases} \frac{1}{4} \int \bar{\nabla} f_i \mathrm{d}\mathcal{U}^\eta_t & x = I(\eta) \in I(\mathscr{P}_{2,E}(\mathbb{R}^d)) , \\ 0 , & x \notin I(\mathscr{P}_{2,E}(\mathbb{R}^d)) . \end{cases} \tag{5.24}$$

We claim that $(\nu, \boldsymbol{b})$ satisfies the continuity equation in $\mathbb{R}^\infty$, i.e. (5.16), (5.17). Indeed, (5.16) follows from (5.23) and (5.21) with Corollary 3.9. To show (5.17), fix a smooth cylinder function $F(\boldsymbol{x}) = \psi\big(p_1(x), \dots, p_n(x)\big)$ and $a \in C^\infty_c(0,T)$. From the continuity equation for $(\mu^N_t, \mathcal{V}^N_t)$ we obtain after passing to the empirical measure

$$\int_0^T a'(t) \int F \circ I \mathrm{d}c^N_t \mathrm{d}t = -\frac{1}{2} \int a(t) \big[F\big(I(\eta^{N,v,v_*,\omega})\big) - F\big(I(\eta)\big) \big] \mathrm{d}\gamma^N .$$

Note that $F\big(I(\eta^{N,v,v_*,\omega})\big) - F\big(I(\eta)\big) = \frac{1}{N} \sum_i \partial_i \psi\big(I^m(\eta)\big) \bar{\nabla} f_i(v, v_*, \omega) + o(1)$. We infer from the convergence of c^N_t to c_t and of $\frac{2}{N}\gamma^N$ to γ and (5.22) that

$$\begin{aligned} \int_0^T a'(t) \int F \circ I \mathrm{d}c_t \, \mathrm{d}t &= -\frac{1}{4} \int_0^T a(t) \int \sum_i \partial_i \psi\big(I^m(\eta)\big) \bar{\nabla} f_i \mathrm{d}\mathcal{U}^\eta_t \mathrm{d}c_t(\eta) \mathrm{d}t \\ &= - \int_0^T a(t) \int \langle \boldsymbol{b}_t, \nabla F \rangle \, \mathrm{d}\nu_t \mathrm{d}t , \end{aligned} \tag{5.25}$$

which is (5.17).

Step 3: Probabilistic representation. By Theorem 5.7 there exists a Borel probability measure $\boldsymbol{\lambda}$ on $C\big([0,T], \mathbb{R}^\infty\big)$ concentrated on the solutions $\gamma \in AC_w\big([0,T], \mathbb{R}^\infty\big)$ to the ODE $\dot{\gamma} = \boldsymbol{b}_t(\gamma)$ such that $(e_t)_\# \boldsymbol{\lambda} = \nu_t$ for all t. Since ν_t is concentrated on the closed set $I(\mathscr{P}_{2,E}(\mathbb{R}^d))$ for all t we have that $\boldsymbol{x}_t \in I(\mathscr{P}_{2,E}(\mathbb{R}^d))$ for all $t \in [0,T]$ and $\boldsymbol{\lambda}$ a.e. γ. Thus we can set $\Theta = \iota_\# \boldsymbol{\lambda}$, where ι maps $\gamma \in C\big([0,T], \mathbb{R}^\infty\big)$ to $I^{-1} \circ \gamma \in C\big([0,T], \mathscr{P}_{2,E}(\mathbb{R}^d)\big)$. It remains to check that Θ has the desired properties.
Since $\nu_t = I_\# c_t$ we immediately get $(e_t)_\# \Theta = c_t$ for all t. Further, since for fixed i we have $\langle f_i, \iota(\gamma) \rangle = \pi_i(\gamma)$, we have by (5.24) that $t \mapsto \langle f_i, \eta_t \rangle$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f_i, \eta_t \rangle = +\frac{1}{4} \int \bar{\nabla} f_i \mathrm{d}\mathcal{U}^{\eta_t}_t \quad \text{for a.e. } t \in (0,T), \text{ for } \Theta\text{-a.e. } \eta . \tag{5.26}$$

From (5.23) and (5.21) we obtain with Corollary 3.9 that the integrability condition (3.3) holds. This allows to extend (5.26) to all Lipschitz f. Hence for Θ-a.e. curve η we have that $t \mapsto (\eta_t, \mathcal{U}^{\eta_t}_t)$ belongs to $\mathcal{CRE}^E_T$. □

Proof of Proposition 5.6: We recall from (5.21) and (5.23) that

$$\int_0^T \int \mathcal{A}(\eta, \mathcal{U}^\eta_t) \mathrm{d}c_t(\eta) \mathrm{d}t \le \liminf_N \frac{1}{N} \mathcal{A}^N_T(\mu^N) . \tag{5.27}$$

We obtain a lim inf estimate for the dissipation in a similar fashion. We note that $D^N(\mu^N_t) = 2\mathcal{G}(\mu^{N,1}, \mu^{N,2})$, where $\mathcal{G}$ is the integral functional defined in the proof of Lemma

2.5. From Lemma 2.4 we obtain

$$\liminf_N \int_0^T \frac{1}{N} D^N(\mu_t^N)\mathrm{d}t \geq \liminf_N \mathcal{G}\left(\frac{2}{N}\beta^{N,1}, \frac{2}{N}\beta^{N,2}\right) \geq \mathcal{G}\left(\beta^1, \beta^2\right)$$
$$= \int_0^T \int D(\eta)\mathrm{d}c_t(\eta)\mathrm{d}t \ , \tag{5.28}$$

where we recall the definition of $\beta^{N,k}$ and β^k from (5.18), (5.19), (5.20). By Proposition 5.5 we can then rewrite (5.27) and (5.28) as (5.14) and (5.15), noting that Θ-a.e. curve $(\eta_t)_t$ satisfies $\mathcal{A}_T(\eta) \leq \int_0^T \mathcal{A}(\eta_t, \mathcal{U}_t^{\eta_t})\mathrm{d}t$. □

5.2.3. *Limit for the relative entropy.*

Proposition 5.8 (lim inf-inequality for the entropy)**.** *Let $(\mu^N)_N$ be a sequence of measures in $\mathscr{P}(\mathcal{X}_N)$ such that $c^N = (L_N)_\# \mu^N$ converges weakly to $c \in \mathscr{P}(\mathscr{P}_{2,E}(\mathbb{R}^d))$. Then we have that*

$$\liminf_N \frac{1}{N}\mathcal{H}(\mu^N|\pi_N) \geq \int \mathcal{H}(\eta|M)\ \mathrm{d}c(\eta) \ . \tag{5.29}$$

To prove this result, we will rely on ideas from large deviation theory. Namely, we will exploit the fact that the empirical measure of independent Gaussian distributed points in $\mathbb{R}^d$ satisfies a large deviation principle and that this implies a $\Gamma - \liminf$ inequality for the relative entropy w.r.t. the law of this empirical measure. Then we will conclude by relating the entropy w.r.t. π_N to the entropy w.r.t. the product Gaussian distribution. Let us briefly explain the concepts we will be using. For background on large devition theory we refer to [7].

Let X be a Polish space and equip the set of Borel probability measures $\mathscr{P}(X)$ with the weak topology. Let $I : X \to [0, \infty]$ be a lower semiconinuous function. A sequence of measures $(m_N)_N$ in $\mathscr{P}(X)$ is said to satisfy a *large deviation principle* with *rate function* I (and speed N) if for any open set O and any closed set C in X their probabilities are asymptotically controled as:

$$\liminf_N \frac{1}{N}\log m_N(O) \geq -\inf_{x\in O} I(x) \ , \quad \limsup_N \frac{1}{N}\log m_N(C) \leq -\inf_{x\in C} I(x) \ .$$

If the second ineqality holds only for all compact sets C, we speak of a *weak large diviation upper bound*. This weak upper bound is equivalent to a $\Gamma-\liminf$ inequality for the relative entropy:

Lemma 5.9 ([13, Thm. 3.5] (P1)⇔(H2))**.** *(m_N) satisfies a weak large deviation upper bound with rate function I and speed N if and only if for any sequence (μ_N) in $\mathscr{P}(X)$ converging to μ we have*

$$\liminf_N \frac{1}{N}\mathcal{H}(\mu_N|m_N) \geq \int_X I\mathrm{d}\mu \ .$$

We will also use the following desintegration principle for the relative entropy, which can be verified by a direct computation.

Let Y be a further Polish space, μ, m two probability measures on X, and $T : X \to Y$ be a Borel map. Let $\mu(\cdot|T = y)$ and $m(\cdot|T = y)$ denote the desintegration of μ and m w.r.t. T. I.e. $\mu(\cdot|T = y)$ are probability measures concentrated on $T^{-1}(y)$ such that for any measurable set $A \subset \mathcal{X}$, $y \mapsto \mu(A|T = y)$ is measurable, and

$$\mu(A) = \int_Y \mu(A|T = y)\mathrm{d}T_\#\mu(y) \ ,$$

and similarly for m. Then we have that

$$\mathcal{H}(\mu|m) = \mathcal{H}(T_\#\mu|T_\#m) + \int_Y \mathcal{H}\big(\mu(\cdot|T = y)|m(\cdot|T = y)\big)\mathrm{d}T_\#\mu(y) \ . \tag{5.30}$$

Since the relative entropy is non-negative, we have in particular

$$\mathcal{H}(\mu|m) \geq \mathcal{H}(T_\#\mu|T_\#m) \ . \tag{5.31}$$

Proof of Proposition 5.8: (i) Let $\gamma_N \in \mathscr{P}(\mathbb{R}^{Nd})$ denote the distribuion of N independent standard d-dimensional Gaussian vectors, i.e. γ_N has density

$$g_N(v_1, \dots, v_N) = (2\pi)^{-Nd/2} \exp\left(-\sum_{i=1}^N \frac{|v_i|^2}{2}\right)$$

w.r.t. Lebesgue measure on $\mathbb{R}^{Nd}$. Note that π_N is obtained by conditioning γ_N to $\mathcal{X}_N \subset \mathbb{R}^{dN}$, i.e.

$$\pi_N = \gamma_N\left(\cdot \mid \mathcal{M}^N = 0, \mathcal{E}^N = d\right) = \frac{g_N}{\int_{\mathcal{X}_N} g_N \mathrm{d}\pi_N}\pi_N \ ,$$

with $\mathcal{M}^N(\boldsymbol{v}) = 1/N \sum_i v_i$ and $\mathcal{E}^N(\boldsymbol{v}) = 1/N \sum_i |v_i|^2$. This follows immediately from g_N beeing constant on $\mathcal{X}_N$.
(ii) We now claim that the analog of (5.29) holds for γ_N: if $\widetilde{\mu}^N$ is a sequence in $\mathscr{P}(\mathbb{R}^{Nd})$ such that $c^N = (L_N)_\#\widetilde{\mu}^N$ converges weakly to c, then

$$\liminf_N \frac{1}{N}\mathcal{H}(\widetilde{\mu}^N|\gamma_N) \geq \int \mathcal{H}(\eta|M) \ \mathrm{d}c(\eta) \ . \tag{5.32}$$

Setting $m_N := (L_N)_\#\gamma_N$ we obtain from (5.31) that $\mathcal{H}(\widetilde{\mu}^N|\gamma_N) \geq \mathcal{H}(c^N|m_N)$. Thus, it suffices to show that

$$\liminf_N \frac{1}{N}\mathcal{H}(c^N|m_N) \geq \int \mathcal{H}(\eta|M)c(\mathrm{d}\eta) \ . \tag{5.33}$$

By Sanov's theorem on large deviations for empirical measures [7, Thm. 6.2.10], m_N satisfies a large deviation principle with rate function $\mathcal{H}(\cdot|M)$ on $\mathscr{P}(\mathbb{R}^d)$ equipped with the weak topology. Thus, (5.33) follows from Lemma 5.9.
(iii) Finally, we will conclude by relating $\mathcal{H}(\cdot|\gamma_N)$ and $\mathcal{H}(\cdot|\pi_N)$. For $m \in \mathbb{R}^d$, $E > 0$, define $\Psi_{m,E} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ by $\Psi_{m,E}(\boldsymbol{v}) = (\sqrt{E}v_1 + m, \dots, \sqrt{E}v_n + m)$. Let $Q_N = (\mathcal{M}^N, \mathcal{E}^N)_\#\gamma_N$ in $\mathscr{P}\big(\mathbb{R}^d \times [0,\infty)\big)$ be the distribution of momentum and energy under γ_N. We have $\gamma_N(\cdot \mid \mathcal{M}^N = m, \mathcal{E}^N = E) = (\Psi_{m,E/d})_\#\pi_N$ as in (i). Hence γ_N desintegrates as $\gamma_N = \int (\Psi_{m,E/d})_\#\pi_N \mathrm{d}Q_N(m,E)$. Define a map $\boldsymbol{\Psi} : \mathscr{P}(\mathcal{X}_N) \to \mathscr{P}(\mathbb{R}^{Nd})$ via

$$\boldsymbol{\Psi}(\mu) = \int (\Psi_{m,E/d})_\#\mu \ \mathrm{d}Q_N(m,E) \ .$$

Note that $(\mathcal{M}^N, \mathcal{E}^N)_\#\boldsymbol{\Psi}(\mu) = Q_N$. Thus, the desintegration formula (5.30) with $T = (\mathcal{M}^N, \mathcal{E}^N)$ gives

$$\mathcal{H}\big(\boldsymbol{\Psi}(\mu)|\gamma_N\big) = \int \mathcal{H}\big((\Psi_{m,E/d})_\#\mu|(\Psi_{m,E/d})_\#\pi_N\big)\mathrm{d}Q_N(m,E) = \mathcal{H}(\mu|\pi_N) \ ,$$

where the last equality follows from (5.31) and $\Psi_{m,E}$ being bijective. Since $\mu^N \rightharpoonup \mu$ implies $\boldsymbol{\Psi}(\mu^N) \rightharpoonup \boldsymbol{\Psi}(\mu)$, we can now deduce (5.29) from (5.32). □

Appendix A. The collision distance

In this section, we present a new type of distance between probability measures on $\mathbb{R}^d$ which is formally the Riemannian distance associated to the Onsager operator $\mathcal{K}^B$, see (1.5). The Riemannian distance $\mathcal{W}_B$ between to probability densities f_0, f_1 is formally given as

$$\mathcal{W}_B(f_0, f_1)^2 = \inf\left\{\frac{1}{4}\int_0^1 \int |\bar{\nabla}\psi_t|^2 \Lambda(f_t) B(v - v_*, \omega)\mathrm{d}\omega \mathrm{d}v_* \mathrm{d}v \mathrm{d}t\right\} \ , \tag{A.1}$$

where the infimum runs over all curves of densities $t \mapsto f_t$ connecting f_0 to f_1 and all functions $\psi : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ related via

$$\partial_t f_t(v) + \int \bar{\nabla}\psi_t \Lambda(f_t) B(v - v_*, \omega) \mathrm{d}\omega \mathrm{d}v_* = 0 \ . \tag{A.2}$$

Note that the definition of $\mathcal{W}_B$ resembles the dynamic formulation of the L^2-Wasserstein distance, known as the Benamou–Brenier formula [4]. Here, the *collision rate equation* (A.2) takes over the role of the usual continuity equation.
The distance $\mathcal{W}_B$ will be constructed by relaxing the minization problem above to a measure valued framework and by minimizing the action as defined in Section 3 over curves connecting two given probability measures via the collision rate equation.
In this section, we will relax the assumptions on the collision kernel and require:

Assumption A.1. *$B : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}_+$ is measurable, invariant under the transformation (1.3), and satisfies*

(i) *for any function $\xi \in C(S^{d-1})$ the map*

$$k \mapsto \int_{S^{d-1}} \xi(\omega) B(k, \omega) \mathrm{d}\omega$$

is continuous;

(ii) *there exists a constant C_B such that*

$$\int_{S^{d-1}} B(k, \omega) \mathrm{d}\omega \leq C_B \quad \forall k \in \mathbb{R}^d \ . \tag{A.3}$$

Note that Assumption A.1 is satisfied if (A.3) holds and B is a Carathéodory integrand, i.e. the map $k \mapsto B(k, \omega)$ is continuous for a.e. ω (w.r.t. the Hausdorff measure).

The following result will allow us to extract subsequential limits from sequences of solutions to the collision rate equation with uniformly bounded action and energy. Reccall that $\mathcal{CRE}_T^E$ is the set of $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ such that $\mathcal{E}(\mu_t) \leq E$ for all $t \in [0,T]$.

Proposition A.2 (Compactness of solutions with bounded action and energy)**.** *Let $(\mu^n, \mathcal{U}^n)$ be a sequence in $\mathcal{CRE}_T^E$ such that*

$$\sup_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n) \mathrm{d}t \ < \ \infty \ . \tag{A.4}$$

Then there exists a couple $(\mu, \mathcal{U}) \in \mathcal{CRE}_T^E$ such that up to extraction of a subsequence

$$\mu_t^n \rightharpoonup \mu_t \quad \textit{weakly in } \mathscr{P}(\mathbb{R}^d) \textit{ for all } t \in [0,T] \ ,$$
$$\mathcal{U}^n \rightharpoonup^* \mathcal{U} \quad \textit{weakly* in } \mathcal{M}(G \times [0,T]) \ .$$

Moreover, along this subsequence we have :

$$\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t) \mathrm{d}t \ \leq \ \liminf_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n) \mathrm{d}t \ .$$

Proof. Thank to the uniform bounds on action and energy, we can proceed verbatim as in the proof of Proposition 3.11 to obtain existence of a Borel family $(\mathcal{U}_t)_{t \in [0,T]}$ satisfying (iii) of Definition 3.1 such that $\mathcal{U}_t^n \mathrm{d}t$ converges weakly* to $\mathcal{U}_t \mathrm{d}t$ and the convergence (3.15) holds. By a further argument based on (3.14), we can aproximate the indicator function $\mathbf{1}_{(t_0, t_1)}$ for any $0 \leq t_0 < t_1 \leq T$ by functions $a \in C\big([0,T]\big)$ and obtain for any $\xi \in C_b(\mathbb{R}^d)$:

$$\int_{t_0}^{t_1} \int \bar{\nabla}\xi \mathrm{d}\mathcal{U}_t^n \mathrm{d}t \ \overset{n \to \infty}{\longrightarrow} \ \int_{t_0}^{t_1} \int \bar{\nabla}\xi \mathrm{d}\mathcal{U}_t \mathrm{d}t \ . \tag{A.5}$$

Finally, we show existence of a limiting curve $(\mu_t)_{t \in [0,T]}$. Since $\mathscr{P}_{2,E}(\mathbb{R}^d)$ is compact w.r.t. weak convergence, after extraction of another subsequence we can assume that

$\mu_0^n \rightharpoonup \mu_0$ weakly for some $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$. Using this, the convergence (A.5) and the collision rate equation in the form (3.2) infer that μ_t^n converges weakly to some probability measure μ_t for every $t \in [0,T]$ and that $(\mu,\mathcal{U})$ satisfies (3.2). Inparticular, $t \mapsto \mu_t$ is weakly continuous and hence $(\mu,\mathcal{U}) \in \mathcal{CRE}_T$. By lower semicontinuity of moments, we infer $\mathcal{E}(\mu_t) \leq E$ for all t. The lower semicontinuity statement follows from Lemma 2.4 by noting that $\int_0^T \mathcal{A}(\mu_t^n,\mathcal{U}_t^n)\mathrm{d}t = \mathcal{F}_\alpha(\mu^{n,1},\mu^{n,2},\mathcal{U}^n)$ with $\mu^{n,k} = \mu_t^{n,k}\mathrm{d}t$. □

We can now define the distance.

Definition A.3 (Distance)**.** *For $\mu_0,\mu_1 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ we define*

$$\mathcal{W}_B(\mu_0,\mu_1)^2 \;:=\; \inf\left\{\int_0^1 \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t \;:\; (\mu,\mathcal{U}) \in \mathcal{CRE}_1^E(\mu_0,\mu_1)\right\}, \tag{A.6}$$

with the convention that $\mathcal{W}_B(\mu_0,\mu_1) = +\infty$ if $\mathcal{CRE}_1^E(\mu_0,\mu_1)$ is empty.

Remark A.4. In the same way one could construct a (a priori smaller) extended distance on the full space $\mathscr{P}(\mathbb{R}^d)$ by dropping the moment condition and minimizing over $(\mu,\mathcal{U}) \in \mathcal{CRE}_1$ instead of $\mathcal{CRE}_1^E$. We will not consider this possibility here.

Let us give an equivalent characterization of the infimum in (A.6).

Lemma A.5. *For any $T > 0$ and $\mu_0,\mu_1 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ we have :*

$$\mathcal{W}_B(\mu_0,\mu_1) \;=\; \inf\left\{\int_0^T \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t \;:\; (\mu,\mathcal{U}) \in \mathcal{CRE}_T^E(\mu_0,\mu_1)\right\}.$$

Proof. This follows from a standard reparametrization argument. See [1, Lem. 1.1.4] or [9, Thm. 5.4] for details in similar situations. □

The next result shows that the infimum in the definition above is in fact a minimum.

Proposition A.6. *Let $\mu_0,\mu_1 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ be such that $W := \mathcal{W}_B(\mu_0,\mu_1)$ is finite. Then the infimum in (A.6) is attained by a curve $(\mu,\mathcal{U}) \in \mathcal{CRE}_1^E(\mu_0,\mu_1)$ satisfying $\mathcal{A}(\mu_t,\mathcal{U}_t) = W^2$ for a.e. $t \in [0,1]$.*

Proof. Existence of a minimizing curve $(\mu,\mathcal{U}) \in \mathcal{CRE}_1^E(\mu_0,\mu_1)$ follows immediately by the direct method taking into account Proposition A.2. Invoking Lemma A.5 and Jensen's inequality we see that this curve satisfies

$$\int_0^1 \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t \;\geq\; W \;=\; \left(\int_0^1 \mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t\right)^{\frac{1}{2}} \;\geq\; \int_0^1 \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t\,.$$

Hence we must have $\mathcal{A}(\mu_t,\mathcal{U}_t) = W^2$ for a.e. $t \in [0,1]$. □

We have the following properties of the function $\mathcal{W}_B$.

Theorem A.7. *$\mathcal{W}_B$ defines an (extended) distance on $\mathscr{P}_{2,E}(\mathbb{R}^d)$. The topology it induces is stronger than the weak topology and bounded sets w.r.t. $\mathcal{W}_B$ are weakly compact. Moreover, the map $(\mu_0,\mu_1) \mapsto \mathcal{W}_B(\mu_0,\mu_1)$ is lower semicontinuous w.r.t. weak convergence. For each $\tau \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ the set $\mathscr{P}_\tau := \{\mu \in \mathscr{P}_{2,E}(\mathbb{R}^d) \;:\; \mathcal{W}_B(\mu,\tau) < \infty\}$ equipped with the distance $\mathcal{W}_B$ is a complete geodesic space.*

Here, we call a function $d : X \times X \to [0,\infty]$ an *extended distance* on the set X, if it is symmetric, satisfies the triangle inequality and vanishes precisely on the diagonal.

Proof. Symmetry of $\mathcal{W}_B$ is obvious from the fact that $\alpha(w,\cdot,\cdot) = \alpha(-w,\cdot,\cdot)$. Equation (3.2) shows that two curves in $\mathcal{CRE}_1^E$ can be concatenated to obtain a curve in $\mathcal{CRE}_2^E$. Hence the triangle inequality follows easily using Lemma A.5. To see that $\mathcal{W}_B(\mu_0,\mu_1) > 0$ whenever $\mu_0 \neq \mu_1$ assume that $\mathcal{W}_B(\mu_0,\mu_1) = 0$ and choose a minimizing curve $(\mu,\mathcal{U}) \in$

$\mathcal{CRE}_1^E(\mu_0,\mu_1)$. Then we must have $\mathcal{A}(\mu_t,\mathcal{U}_t)=0$ and hence $\mathcal{U}_t=0$ for a.e. $t\in(0,1)$. From the continuity equation in the form (3.2) we infer $\mu_0=\mu_1$.
The compactness assertion and lower semicontinuity of $\mathcal{W}_B$ follow immediately from Proposition A.2. These in turn imply that the topology induced by $\mathcal{W}_B$ is stronger than the weak one.
Let us now fix $\tau\in\mathscr{P}_{2,E}(\mathbb{R}^d)$ and let $\mu_0,\mu_1\in\mathscr{P}_\tau$. By the triangle inequality we have $\mathcal{W}_B(\mu_0,\mu_1)<\infty$ and hence Proposition A.6 yields existence of a minimizing curve $(\mu,\mathcal{U})\in\mathcal{CRE}_1^E(\mu_0,\mu_1)$. The curve $t\mapsto\mu_t$ is then a constant speed geodesic in $\mathscr{P}_\tau$ since it satisfies

$$\mathcal{W}_B(\mu_s,\mu_t) \;=\; \int_s^t \sqrt{\mathcal{A}(\mu_r,\mathcal{U}_r)}\mathrm{d}r \;=\; (t-s)\mathcal{W}_B(\mu_0,\mu_1) \quad \forall 0\le s\le t\le 1\ .$$

To show completeness, let $(\mu^n)_n$ be a Cauchy sequence in $\mathscr{P}_\tau$. In particular the sequence is bounded w.r.t. $\mathcal{W}_B$ and we can find a subsequence (still indexed by n) and $\mu^\infty\in\mathscr{P}_{2,E}(\mathbb{R}^d)$ such that $\mu^n\rightharpoonup\mu^\infty$ weakly. Invoking lower semicontinuity of $\mathcal{W}_B$ and the Cauchy condition we infer that $\mathcal{W}_B(\mu^n,\mu^\infty)\to 0$ as $n\to\infty$ and that $\mu^\infty\in\mathscr{P}_\tau$. □

It is not yet clear when precisely the distance $\mathcal{W}_B$ is finite. However, it is easily seen to be finite along solutions to the Boltzmann equation: if f_t is a solution according to Theorem 2.2 and we set $\mu_t=f_t\mathcal{L}$ and

$$\mathcal{U}_t=\bar\nabla\log f_t\Lambda(f_t)\mathcal{B}=\big[(f')_t(f'_*)_t-f_t(f_*)_t\big]\mathcal{B}\ ,$$

then $(\mu,\mathcal{U})\in\mathcal{CRE}^E$ and we have $\mathcal{A}(\mu_t,\mathcal{U}_t)=D(\mu_t)$. Thus,

$$\mathcal{W}_B(\mu_0,\mu_T)\le\int_0^T\sqrt{D(\mu_t)}\mathrm{d}t\le\sqrt{T}\left(\int_0^T D(\mu_t)\mathrm{d}t\right)^{\frac12}=\sqrt{T}\sqrt{\mathcal{H}(\mu_0)-\mathcal{H}(\mu_T)}\ .$$

The following result shows that the distance $\mathcal{W}_B$ can be bounded from below by the L^1-Wasserstein distance. Recall that the L^1-Wasserstein distance is defined for $\mu_0,\mu_1\in\mathscr{P}(\mathbb{R}^d)$ by

$$W_1(\mu_0,\mu_1)\;:=\;\inf_\pi\int|x-y|\pi(\mathrm{d}x,\mathrm{d}y)\ ,$$

where the infimum is taken over all probability measures $\pi\in\mathscr{P}(\mathbb{R}^d\times\mathbb{R}^d)$ whose first and second marginal are μ_0 and μ_1 respectively.

Proposition A.8. *For any $\mu_0,\mu_1\in\mathscr{P}_{2,E}(\mathbb{R}^d)$ we have the bound*

$$W_1(\mu_0,\mu_1)\;\le\;\sqrt{2C_BE}\mathcal{W}_B(\mu_0,\mu_1)\ .$$

Proof. We can assume that $\mathcal{W}_B(\mu_0,\mu_1)<\infty$. Take a minimizing curve $(\mu,\mathcal{U})\in\mathcal{CRE}_1^E(\mu_0,\mu_1)$ and let $\varphi:\mathbb{R}^d\to\mathbb{R}$ be a bounded 1-Lipschitz function. This implies that $|\bar\nabla\varphi|\le 2|v-v_*|$. Taking into account Remark 3.3 and using Lemma 3.8, we estimate

$$\begin{aligned}
\left|\int\varphi\mathrm{d}\mu_1-\int\varphi\mathrm{d}\mu_0\right| &= \frac14\left|\int_0^1\int\bar\nabla\varphi\mathrm{d}\mathcal{U}_t\mathrm{d}t\right|\\
&\le \frac12\int_0^1\int|v-v_*|\mathrm{d}\,|\mathcal{U}_t|\,(v,v_*,\omega)\mathrm{d}t\\
&\le \left(\int_0^1\mathcal{A}(\mu_t,\mathcal{U}_t)\mathrm{d}t\right)^{\frac12}\left(\int_0^1\int|v|^2+|v_*|^2B(v-v_*,\omega)\mu_t(\mathrm{d}v)\mu_t(\mathrm{d}v_*)\mathrm{d}t\right)^{\frac12}\\
&\le \sqrt{2C_BE}\mathcal{W}_B(\mu_0,\mu_1)\ .
\end{aligned}$$

Here we have also used (A.3) and the fact that μ_t has energy E in the last inequality. Taking the supremum over all bounded 1-Lipschitz functions φ yields the claim by Kantorovich–Rubinstein duality (see [21, Thm. 5.10, 5.16]). □

We now give a characterization of absolutely continuous curves with respect to $\mathcal{W}_B$. See (B.1) and (B.2) for the definition of absolutely continuous curves and their metric derivative.

Proposition A.9 (Metric velocity)**.** *A curve $(\mu_t)_{t\in[0,T]}$ in $\mathscr{P}_{2,E}(\mathbb{R}^d)$ is absolutely continuous with respect to $\mathcal{W}_B$ if and only if there exists a Borel family $(\mathcal{U}_t)_{t\in[0,T]}$ such that $(\mu,\mathcal{U}) \in \mathcal{CRE}_T^E$ and*

$$\int_0^T \sqrt{\mathcal{A}(\mu_t,\mathcal{U}_t)}\mathrm{d}t \ < \ \infty \ .$$

In this case, the metric derivative is bounded as $|\dot{\mu}|^2(t) \leq \mathcal{A}(\mu_t,\mathcal{U}_t)$ for a.e. $t \in [0,T]$. Moreover, there exists a unique Borel family $\widetilde{\mathcal{U}}_t$ with $(\mu,\widetilde{\mathcal{U}}) \in \mathcal{CRE}_T$ such that

$$|\dot{\mu}|^2(t) = \mathcal{A}(\mu_t,\widetilde{\mathcal{U}}_t) \qquad \text{for a.e. } t \in [0,T] \ . \tag{A.7}$$

Proof. The proof follows from the very same arguments as in [9, Thm. 5.17]. □

We can describe the optimal velocity measures $\widetilde{\mathcal{U}}_t$ appearing in the preceding proposition in more detail. We define T_μ to be the set of all $\mathcal{U} \in \mathcal{M}(G)$ such that $\mathcal{A}(\mu,\mathcal{U}) < \infty$ and $\mathcal{A}(\mu,\mathcal{U}) \ \leq \ \mathcal{A}(\mu,\mathcal{U}+\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in \mathcal{M}(G)$ satisfying

$$\frac{1}{4}\int_G \bar{\nabla}\xi \mathrm{d}\boldsymbol{\eta} \ = \ 0 \qquad \forall \xi \in C_c^\infty(G) \ .$$

Corollary A.10. *Let $(\mu,\mathcal{U}) \in \mathcal{CRE}_T^E$ such that the curve $t \mapsto \mu_t$ is absolutely continuous w.r.t. $\mathcal{W}_B$. Then $\mathcal{U}$ satisfies (A.7) if and only if $\mathcal{U}_t \in T_{\mu_t}$ for a.e. $t \in [0,T]$.*

If μ is absolutely continuous with respect to Lebesgue measure $\mathcal{L}$ we can give an explicit description of T_μ. Recall that $\mathcal{B} \in \mathcal{M}(G)$ is the measure given by $\mathrm{d}\mathcal{B}(v,v_*,\omega) = B(v-v_*,\omega)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\omega$.

Proposition A.11. *Let $\mu = fm \in \mathscr{P}_{2,E}(\mathbb{R}^d)$. Then we have $\mathcal{U} \in T_\mu$ if and only if $\mathcal{U} = U\Lambda(f)\mathcal{B}$ is absolutely continuous w.r.t. the measure $\Lambda(f)\mathcal{B}$ and*

$$U \ \in \ \overline{\{\bar{\nabla}\varphi \mid \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\Lambda(f)\mathcal{B})} \ =: \ T_f \ .$$

Proof. If $\mathcal{A}(\mu,\mathcal{U})$ is finite we infer from Lemma 3.6 that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ for some density $U : G \to \mathbb{R}$ and that $\mathcal{A}(\mu,\mathcal{U}) = \|U\|^2_{L^2(\Lambda(f)\mathcal{B})}$. Now the optimality condition in the definition of T_μ is equivalent to

$$\|U\|_{L^2(\Lambda(f)\mathcal{B})} \ \leq \ \|U+V\|_{L^2(\Lambda(f)\mathcal{B})} \qquad \forall V \in N_f \ ,$$

where $N_f := \{V \in L^2(\Lambda(f)\mathcal{B}) \ : \ \int \bar{\nabla}\xi V\Lambda(f)\mathcal{B} = 0 \ \forall \xi \in C_c^\infty(\mathbb{R}^d)\}$. This implies the assertion of the proposition after noting that N_f is the orthogonal complement in L^2 of T_f. □

In the light of the formal Riemannian interpretation of the distance $\mathcal{W}_B$ one should view T_μ as the tangent space at the measure μ. This is reminiscent of Otto's Riemannian interpretation of the L^2-Wasserstein space [17].

Appendix B. Metric gradient flow

In this section, we recast the variational characterization of Section 4 in the language of the theory of gradient flows in metric spaces. Let us briefly recall the basic theory of gradient flow in metric spaces. For a detailed account we refer the reader to [1].
Let (X,d) be a complete metric space and let $E : X \to (-\infty,\infty]$ be a function with proper domain, i.e. the set $D(E) := \{x : E(x) < \infty\}$ is non-empty.

A curve $(x_t)_{t\in(a,b)}$ in (X,d) is called p-absolutely continuous for $p \geq 1$ if there exists $m \in L^p((a,b))$ such that

$$d(x_s, x_t) \;\leq\; \int_s^t m(r)\mathrm{d}r \quad \forall\, a \leq s \leq t \leq b\;. \tag{B.1}$$

In this case we write $x \in AC^p\big((a,b);(X,d)\big)$. For $p=1$ we simply drop p in the notation. Similarly, one defines locally p-absolutely continuous curves. For a locally absolutely continuous curve the metric derivative defined by

$$|\dot{x}|\,(t) \;:=\; \lim_{h\to 0} \frac{d(x_{t+h}, x_t)}{|h|} \tag{B.2}$$

exists for a.e. t and is the minimal m in (B.1), see [1, Thm.1.1.2].
The following notion plays the role of the modulus of the gradient in a metric setting.

Definition B.1 (Strong upper gradient)**.** *A function $g : X \to [0,\infty]$ is called a* strong upper gradient *of E if for any $x \in AC\big((a,b);(X,d)\big)$ the function $g \circ x$ is Borel and*

$$|E(x_s) - E(x_t)| \;\leq\; \int_s^t g(x_r)|\dot{x}|(r)\mathrm{d}r \quad \forall\, a \leq s \leq t \leq b\;.$$

Note that by the definition of strong upper gradient, and Young's inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we have that for all $s \leq t$:

$$E(x_t) - E(x_s) + \frac{1}{2}\int_s^t g(x_r)^2 + |\dot{x}|^2(r)\mathrm{d}r \geq 0\;.$$

Definition B.2 (Curve of maximal slope)**.** *A locally 2-absolutely continuous curve $(x_t)_{t\in(0,\infty)}$ is called a curve of maximal slope of E w.r.t. its strong upper gradient g if $t \mapsto E(x_t)$ is non-increasing and*

$$E(x_t) - E(x_s) + \frac{1}{2}\int_s^t g(x_r)^2 + |\dot{x}|^2(r)\mathrm{d}r \leq 0 \quad \forall\, 0 < s \leq t\;. \tag{B.3}$$

We say that a curve of maximal slope starts from $x_0 \in X$ if $\lim_{t\searrow 0} x_t = x_0$.

Equivalently, we can require equality in (B.3). If a strong upper gradient g of E is fixed we also call a curve of maximal slope of E (relative to g) a *gradient flow curve*.
Finally, we define the (descending) metric slope of E as the function $|\partial E| : D(E) \to [0,\infty]$ given by

$$|\partial E|(x) = \limsup_{y\to x} \frac{\max\{E(x) - E(y), 0\}}{d(x,y)}\;. \tag{B.4}$$

The metric slope is in general only a weak upper gradient E, see [1, Thm. 1.2.5]. In our application to the homogeneous Boltzmann equation, we will show that the square root of the dissipation D provides a strong upper gradient for the entropy $\mathcal{H}$.
Let us assume that the collision kernel B satisfies Assumption 2.1. Then we have the following

Corollary B.3 (Boltzmann equation as curve of maximal slope)**.** *$\sqrt{D}$ is a strong upper gradient for $\mathcal{H}$ on $(\mathscr{P}_{2,E}(\mathbb{R}^d), \mathcal{W}_B)$. Moreover, for any $\mu_0 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$, the curves of maximal slope of $\mathcal{H}$ w.r.t. the strong upper gradient $\sqrt{D}$ starting from μ_0 are precisely the weak solutions to the Boltzmann equation satisfying* (4.18).

Proof. Let $(\mu_r)_r$ be an absolutely continuous curve such that $\int_s^t \sqrt{D(\mu_r)}|\dot{\mu}|(r)\mathrm{d}r < \infty$. This implies that μ_r has a density f_r (and hence by Lemma 3.6 $\mathcal{U}_r$ has a density U_r) for a.e. r. We can also assume that one of the measures μ_s, μ_t has finite entropy, say μ_s. Then, Proposition 4.1 together with the estimate (4.15) yield immediately that $\sqrt{D}$ is a strong upper gradient. Theorem 4.3 gives the identification of curves of maximal slope. □

Appendix C. Variational approximation scheme

In this section, we consider a time-discrete variational approximation scheme for the homogeneous Boltzmann equation. Recall that we make Assumption 2.1 on the collision kernel B. The scheme can be interpreted as the implicit Euler scheme for the gradient flow equation. Given a time step $\tau > 0$ and an initial datum $\mu_0 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$, we consider a sequence $(\mu_n^\tau)_n$ in $\mathscr{P}_{2,E}(\mathbb{R}^d)$ defined recursively via

$$\mu_0^\tau = \mu_0 \ , \quad \mu_n^\tau \in \operatorname*{argmin}_\nu \Big[\mathcal{H}(\nu) + \frac{1}{2\tau}\mathcal{W}_B(\nu, \mu_{n-1}^\tau)^2\Big] \ . \tag{C.1}$$

Then we build a discrete gradient flow trajectory as the piece-wise constant interpolation $(\bar\mu_t^\tau)_{t\geq 0}$ given by

$$\bar\mu_0^\tau = \mu_0 \ , \quad \bar\mu_t^\tau = \mu_n^\tau \text{ if } t \in \big((n-1)\tau, n\tau\big] \ . \tag{C.2}$$

Then we have the following result.

Theorem C.1. *For any $\tau > 0$ and $\mu_0 \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$ the variational scheme (C.1) admits a solution $(\mu_n^\tau)_n$. As $\tau \to 0$, for any family of discrete solutions there exists a sequence $\tau_k \to 0$ and a locally 2-absolutely continuous curve $(\mu_t)_{t\geq 0}$ such that*

$$\bar\mu_t^{\tau_k} \rightharpoonup \mu_t \quad \forall t \in [0, \infty) \ . \tag{C.3}$$

Moreover, any such limit curve is a gradient flow of the entropy, i.e. a weak solution to the Boltzmann equation satisfying (4.18).

With the knowledge that the Boltzmann equation in our setting has a unique solution, we obtain convergence of $\bar\mu_t^\tau$ to the solution to the Boltzmann equation for any sequence of time steps $\tau \to 0$.

With the work we have done so far, Theorem C.1 follows basically from standard general results for metric gradient flows where (C.1) is known as the minimizing movement scheme, see [1, Sec. 2.3]. We need one small additional ingredient relatiing the dissipation D to the metric slope $|\partial\mathcal{H}|$ of the entropy in the metric space $(\mathscr{P}_{2,E}(\mathbb{R}^d), \mathcal{W}_B)$. Recall (B.4) for the definition of the metric slope. We consider its sequentially lower semicontinuous envelope, or *relaxed slope* $|\partial^-\mathcal{H}|$ given by

$$|\partial^-\mathcal{H}|(\mu) = \inf\Big\{\liminf_{n\to\infty} |\partial\mathcal{H}(\mu_n) \ : \ \mu_n \rightharpoonup \mu, \ \sup_n\{\mathcal{W}_B(\mu_n, \mu), \mathcal{H}(\mu_n)\} < \infty\Big\} \ .$$

Lemma C.2. *For any $\mu \in \mathscr{P}_{2,E}(\mathbb{R}^d)$ with $\mathcal{H}(\mu) < \infty$ we have that*

$$\sqrt{D(\mu)} \leq |\partial^-\mathcal{H}(\mu)| \ .$$

Proof. Let f be the density of μ and consider the solution (f_t) to the homogeneous Boltzmann equation with initial datum f. Set $\mu_t = f_t\mathcal{L}$ and observe that

$$\begin{aligned} D(f) &\leq \lim_{t\searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{t} = \lim_{t\searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{\mathcal{W}_B(\mu_t, \mu)} \frac{\mathcal{W}_B(\mu_t, \mu)}{t} \\ &\leq |\partial\mathcal{H}(\mu)||\dot\mu|(0) \leq |\partial\mathcal{H}(\mu)|\sqrt{D(\mu)} \ . \end{aligned}$$

Thus, we have $\sqrt{D(\mu)} \leq |\partial\mathcal{H}(\mu)|$ for any such μ. The claim follows immediately from the lower semicontinuity of D, Lemma 2.5. □

Proof of Theorem C.1. We verify that the present situation is consistent with the abstract setting considered in [1, Sec. 2].

We consider the metric space $(\mathscr{P}_{\mu_0}(\mathbb{R}^d), \mathcal{W}_B)$ and endow it with the weak topology σ. By Theorem A.7, $(\mathscr{P}_{\mu_0}(\mathbb{R}^d), \mathcal{W}_B)$ is complete, $\mathcal{W}_B$ is lower semicontinuous w.r.t. σ and induces a stronger topology. Recall from Section 2 that the entropy $\mathcal{H}$ is bounded below

on $\mathscr{P}_{2,E}(\mathbb{R}^d)$ and lower semicontinuous w.r.t. weak convergence. Moreover, $\mathscr{P}_{2,E}(\mathbb{R}^d)$ is compact w.r.t. weak convergence. Thus, [1, Assumption 2.1 a,b,c] are satisfied.
Existence of a solution to the variational scheme (C.1) and of a subsequential limit curve $(\mu_t)_t$ now follows from [1, Cor. 2.2.2, Prop. 2.2.3]. Moreover, [1, Thm. 2.3.2] gives that the limit curve is a curve of maximal slope for the strong upper gradient $|\partial^-\mathcal{H}|$, i.e.

$$\frac{1}{2}\int_0^t |\dot{\mu}|^2(r) + |\partial^-\mathcal{H}(\mu_r)|^2 \mathrm{d}r + \mathcal{H}(\mu_t) \leq \mathcal{H}(\mu_0) \ .$$

Thus, by Lemma C.2, it is also a curve of maximal slope for the strong upper gradient $\sqrt{D}$. □

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH. Birkhäuser, Zürich, 2008.

[2] Luigi Ambrosio and Dario Trevisan. Well-posedness of Lagrangian flows and continuity equations in metric measure spaces. *Anal. PDE*, 7(5):1179–1234, 2014.

[3] L. Arkeryd. On the Boltzmann equation. I. Existence. *Arch. Rational Mech. Anal.*, 45:1–16, 1972.

[4] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3):375–393, 2000.

[5] G. Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations*. Pitman Research Notes in Mathematics Series. Longman Scientific and Technical, Harlow, 1989.

[6] E. A. Carlen and M. C. Carvalho. Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. *J. Statist. Phys.*, 67(3-4):575–608, 1992.

[7] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[8] L. Desvillettes. Boltzmann's kernel and the spatially homogeneous Boltzmann equation. *Riv. Mat. Univ. Parma (6)*, 4*:1–22, 2001. Fluid dynamic processes with inelastic interactions at the molecular scale (Torino, 2000).

[9] J. Dolbeault, B. Nazaret, and G. Savaré. A new class of transport distances between measures. *Calc. Var. Partial Differential Equations*, 34(2):193–231, 2009.

[10] M. Erbar. Gradient flows of the entropy for jump processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(3):920–945, 2014.

[11] M. Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 171–197. University of California Press, Berkeley and Los Angeles, 1956.

[12] J. Maas. Gradient flows of the entropy for finite Markov chains. *J. Funct. Anal.*, 261(8):2250 – 2292, 2011.

[13] Mauro Mariani. A Γ-convergence approach to large deviations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(3):951–976, 2018.

[14] Alexander Mielke. A gradient structure for reaction-diffusion systems and for energy-drift-diffusion systems. *Nonlinearity*, 24(4):1329–1346, 2011.

[15] Stéphane Mischler and Clément Mouhot. Kac's program in kinetic theory. *Invent. Math.*, 193(1):1–147, 2013.

[16] James Norris. A consistency estimate for Kac's model of elastic collisions in a dilute gas. *Ann. Appl. Probab.*, 26(2):1029–1081, 2016.

[17] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[18] E. Sandier and S. Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, 57(12):1627–1672, 2004.

[19] Alain-Sol Sznitman. Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrsch. Verw. Gebiete*, 66(4):559–592, 1984.

[20] G. Toscani and C. Villani. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. *Comm. Math. Phys.*, 203(3):667–706, 1999.

[21] C. Villani. *Optimal transport, Old and new*, volume 338 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2009.

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