

POINCARÉ, MODIFIED LOGARITHMIC SOBOLEV AND ISOPERIMETRIC INEQUALITIES FOR MARKOV CHAINS WITH NON-NEGATIVE RICCI CURVATURE

MATTHIAS ERBAR AND MAX FATHI

ABSTRACT. We study functional inequalities for Markov chains on discrete spaces with entropic Ricci curvature bounded from below. Our main results are that when curvature is non-negative, but not necessarily positive, the spectral gap, the Cheeger isoperimetric constant and the modified logarithmic Sobolev constant of the chain can be bounded from below by a constant that only depends on the diameter of the space, with respect to a suitable metric. These estimates are discrete analogues of classical results of Riemannian geometry obtained by Li and Yau, Buser and Wang.

1. INTRODUCTION

Ricci curvature bounds play an important role in geometric analysis on Riemannian manifolds. For instance, a lower bound on the curvature by a strictly positive constant entails many interesting properties for the manifold, most notably Harnack inequalities, bounds on the eigenvalues of the Laplacian, concentration bounds and isoperimetric inequalities.

In the light of this wide range of implications, considerable effort has been put into developing a notion of Ricci curvature lower bounds for non-smooth spaces. Bakry and Émery [1] proposed a curvature condition for general Markov diffusion operators via the so-called Γ -calculus. Lott-Villani [28] and Sturm [38] presented an approach that applies to (geodesic) metric measure spaces. Such a space has Ricci curvature bounded below by a constant κ provided the entropy is κ -convex along geodesics in the Wasserstein space of probability measures. Subsequently, many of the classical relating curvature bounds to functional inequalities have been generalized to such 'continuous' non-smooth spaces, we refer to [2, 39] for an overview. In recent years, there has been a strong interest in developing an analogous theory for discrete spaces. Unfortunately, the Lott–Sturm–Villani theory does not apply and a number of alternative notions of Ricci bounds have been proposed, see for instance [7, 16, 35]. In this work, we will focus on the notion of *entropic Ricci curvature bounds* put forward in [29, 13] that applies to finite Markov chains and seems to be particularly well suited to study discrete functional inequalities. Here the key point is to replace the L^2 -Wasserstein distance with a new transportation distance \mathcal{W} in the definition of Lott–Sturm–Villani. It has been shown in [13] that a strictly positive entropic Ricci curvature lower bound implies a spectral gap estimate, a modified logarithmic Sobolev inequality and an analogue of Talagrand's transport cost inequality.

2010 *Mathematics Subject Classification.* 60K35, 60J22.

Key words and phrases. discrete Ricci curvature, functional inequalities, spectral gap, zero range process.

In the present work, we are interested in the situation where the curvature is bounded from below but not strictly positive. We show that in this situation relatively weak extra information (for instance a bound on the diameter of the space) still allows one to establish strong functional inequalities.

To state our main results we consider an irreducible and reversible continuous time Markov chain on a finite space \mathcal{X} whose generator is given by

$$L\psi(x) = \sum_{y \in \mathcal{X}} (\psi(y) - \psi(x))Q(x, y) ,$$

where $Q(x, y)$ are the transition rates between x and y and let π be the unique reversible probability measure.

For the purpose of this introduction we state our main results for simplicity under the assumption that the chain has non-negative entropic Ricci curvature. We shall actually derive more general statements allowing for a negative curvature bound in the main text. We refer to Section 2 for a precise definition of entropic Ricci curvature bounds and the functional inequalities we consider.

The first result establishes an isoperimetric inequality using information on the spectral gap (see Theorem 4.1 below).

Theorem 1.1. *If the entropic Ricci curvature of (\mathcal{X}, Q, π) is non-negative, then the Cheeger (or linear isoperimetric) constant h and the spectral gap λ_1 of L satisfy*

$$h \geq \frac{1}{3} \sqrt{Q_* \lambda_1}$$

where $Q_* = \min(Q(x, y) : Q(x, y) > 0)$ is the minimal transition rate. Here the Cheeger constant is defined by

$$h = \max_{A \subset \mathcal{X}} \frac{\pi^+(\partial A)}{\pi(A)(1 - \pi(A))} ,$$

where $\pi^+(\partial A) = \sum_{x \in A, y \in A^c} Q(x, y)\pi(x)$ denotes the perimeter measure of A .

This result is a discrete version of the classical Buser theorem in Riemannian geometry [8]. A simple analytic proof was later obtained by Ledoux [19], and extended to weighted spaces in [21]. A matching upper bound (up to a different universal constant) is valid in any space, without any assumption on the curvature.

The next two results establish estimates on the spectral gap and the logarithmic Sobolev constant in non-negative curvature using information on the diameter of \mathcal{X} . A natural distance $d_{\mathcal{W}}$ on \mathcal{X} is induced by the discrete transport distance \mathcal{W} between probability measures by setting $d_{\mathcal{W}}(x, y) := \mathcal{W}(\delta_x, \delta_y)$. The distance $d_{\mathcal{W}}$ can be compared to more traditional weighted graph distances, see Lemma 2.3, yielding immediate analogues of the results below in terms of weighted graph distance.

Theorem 1.2. *If the entropic Ricci curvature of (\mathcal{X}, Q, π) is non-negative and the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ is bounded by D , then the spectral gap of the generator L satisfies*

$$\lambda_1 \geq \frac{c}{D^2}$$

for some universal constant c .

See Theorem 5.7 below for a more general statement in negative curvature. The continuous version of this statement is a classical result of Li and Yau [26] (extending a previous result

of Li [25] for manifolds without boundary and of [36] for convex sets in Euclidean spaces), and for which the sharp constant was determined in [41]. A version taking into account the dimension has been obtained by Bakry and Qian [3], and recently extended to geodesic metric measure spaces by Cavaletti and Mondino [9, 10] (with a completely different method).

Theorem 1.3. *If the entropic Ricci curvature of the Markov chain is non-negative and the diameter of (X, d_W) is bounded by D , then a modified logarithmic Sobolev inequality holds, with constant $\frac{c'}{D^2}$ for some universal constant c' .*

See Theorem 6.1 below. This last result is a weakened discrete version of a work of Wang [40], where the finite diameter is replaced by a bound on some square-exponential moment of the distance to an arbitrarily fixed point. It immediately implies a discrete version of Talagrand's inequality, via the discrete Otto-Villani theorem of [13], as well as an upper bound on the total variation mixing time, as we shall see in Section 5.2. We shall actually obtain a version of this result only assuming finiteness of a square-exponential moment as in [40] but with a constant that we do not believe to be sharp, see Theorem 6.5 below.

In the last two results, the dependence on the diameter is optimal, since it is sharp (up to the values of the constants c and c') for the random walk on the one-dimensional discrete torus. However, it behaves badly in high dimensions. This leads us to formulate conjectures about possible improvements using measure concentration bounds instead of diameter bounds in Section 5. Impressive results in this direction for manifolds have been obtained by Milman [32, 33].

Versions of Theorems 1.1 and 1.2 have been obtained for another notion of curvature, namely a discrete version of the Bakry-Émery Γ_2 condition, in [18] and [11] respectively. The two notions of curvature are known to be *not* equivalent. A Markov chain with non-negative entropic Ricci curvature but negative Bakry-Émery curvature has recently been discovered [12]. However, no analogue of Theorem 1.3 is known using Bakry-Émery curvature. To our knowledge it is not even known whether strictly positive Bakry-Émery curvature is enough to ensure the validity of a modified logarithmic Sobolev inequality as in Theorem 1.3. The proofs of Theorem 1.1 and the main result of [18] are quite close and both based on arguments developed by Ledoux in the continuous setting. For Theorem 1.2, we shall give two proofs. One of them replicates the technique used in [11]. The other one uses an HWI interpolation inequality obtained in [13], for which no analogue is known in the setting of discrete Bakry-Émery curvature. This technique has the advantage that the assumptions can be weakened to a bound on a square exponential moment instead of the diameter. It will also be used to prove Theorem 1.3.

One of the main technical tools in our study is a new equivalent characterization of entropic Ricci curvature lower bounds in terms of gradient estimates for the associated Markov semigroup. In the continuous setting this characterization is one of the cornerstones of the theory initiated by Bakry and Émery [1, 2].

We shall present an application of our results to a particular interacting particle system, namely the zero-range process on the complete graph with constant rates. The best known entropic Ricci bound for this model is 0. Using Theorem 1.3 and easily obtained diameter bound allows us to establish a new bound on the mLSI constant for the zero range process.

Outline. In Section 2, we shall recall the definition and basic results about the discrete transport distance \mathcal{W} and entropic Ricci curvature bounds for Markov chains. In Section 3, we shall give an equivalent characterization of entropic Ricci bounds in terms of gradient

estimates for the Markov semigroup. Section 4 will provide the proof of the discrete Buser theorem, while Sections 5 and 6 will be concerned with the Poincaré and modified log Sobolev inequalities under joint curvature and diameter bounds. Finally, in Section 7, we consider applications to the zero range process.

Acknowledgments. M.F. was supported by NSF FRG grant DMS-1361122. M.E. gratefully acknowledges support by the German Research Foundation through the *Hausdorff Center for Mathematics*. We thank Michel Ledoux, Jan Maas, Emanuel Milman, André Schlichting and Prasad Tetali for discussions on this topic. This work was initiated during a trimester on optimal transport organized at the *Hausdorff Institute for Mathematics* in Bonn, whose support is gratefully acknowledged. We also benefited from the hospitality of the American Institute of Mathematics during the SQUARE meetings *Displacement convexity for interacting Markov chains*.

2. ENTROPIC RICCI CURVATURE BOUNDS FOR MARKOV CHAINS

We briefly recall the definitions of the transport distance \mathcal{W} and of entropic Ricci curvature bounds for finite Markov chains and some of their consequences. For a detailed account we refer to the work of Maas and Mielke [29, 30] where the discrete transport distance and its associated Riemannian structure have been introduced and to [13] where entropic Ricci curvature bounds have been introduced and studied.

2.1. Transport distance and Ricci bounds. Let \mathcal{X} be a finite set let $Q : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ be a collection of transition rates with the convention that $Q(x, x) = 0$ for all x . The operator L acting on functions $\psi : \mathcal{X} \rightarrow \mathbb{R}$ defined by

$$L\psi(x) = \sum_{y \in \mathcal{X}} y (\psi(y) - \psi(x)) Q(x, y) ,$$

is the generator of a continuous time Markov chain on \mathcal{X} . We will assume that Q is irreducible, i.e. for all $x, y \in \mathcal{X}$ there exist $(x_1 = x, x_2, \dots, x_n = y)$ with $Q(x_i, x_{i+1}) > 0$. This implies the existence of a unique stationary probability measure π on \mathcal{X} . We will assume that Q is reversible w.r.t. π in the sense that the detailed balance condition $Q(x, y)\pi(x) = Q(y, x)\pi(y)$ holds for all $x, y \in \mathcal{X}$. We denote by

$$\mathcal{P}(\mathcal{X}) := \left\{ \rho : \mathcal{X} \rightarrow \mathbb{R}_+ \mid \sum_{x \in \mathcal{X}} \pi(x) \rho(x) = 1 \right\}$$

the set of *probability densities* w.r.t. π . Since the measure π is strictly positive and we can identify the set of probability measures on \mathcal{X} with $\mathcal{P}(\mathcal{X})$. The subset consisting strictly positive probability densities is denoted by $\mathcal{P}_*(\mathcal{X})$. We consider the metric \mathcal{W} defined for $\rho_0, \rho_1 \in \mathcal{P}(\mathcal{X})$ by

$$\mathcal{W}(\rho_0, \rho_1)^2 := \inf_{\rho, \psi} \left\{ \frac{1}{2} \int_0^1 \sum_{x, y \in \mathcal{X}} (\psi_t(x) - \psi_t(y))^2 \hat{\rho}_t(x, y) Q(x, y) \pi(x) dt \right\} ,$$

where the infimum runs over all sufficiently regular curves $\rho : [0, 1] \rightarrow \mathcal{P}(\mathcal{X})$ and $\psi : [0, 1] \rightarrow \mathbb{R}^{\mathcal{X}}$ satisfying the continuity equation

$$\begin{cases} \frac{d}{dt} \rho_t(x) + \sum_{y \in \mathcal{X}} (\psi_t(y) - \psi_t(x)) \hat{\rho}_t(x, y) Q(x, y) = 0 & \forall x \in \mathcal{X} , \\ \rho|_{t=0} = \rho_0 , & \rho|_{t=1} = \rho_1 . \end{cases} \quad (2.1)$$

Here, given $\rho \in \mathcal{P}(\mathcal{X})$, we write $\hat{\rho}(x, y) := \theta(\rho(x), \rho(y))$ where $\theta(s, t) := \int_0^1 s^{1-pt} t^p \, dp$ is the logarithmic mean of s and t .

It has been shown in [29] that \mathcal{W} defines a distance on $\mathcal{P}(\mathcal{X})$ and that it is induced by a Riemannian structure on the interior $\mathcal{P}_*(\mathcal{X})$. The logarithmic mean serve the purpose to obtain a discrete chain rule for the logarithm, namely $\hat{\rho}(x, y)(\log \rho(x) - \log \rho(y)) = \rho(x) - \rho(y)$, replacing the usual identity $\rho \nabla \log \rho = \nabla \rho$. The distance \mathcal{W} is constructed in such a way that Markov semigroup $P_t = e^{tL}$ is the gradient flow of the entropy

$$\mathcal{H}(\rho) = \sum_{x \in \mathcal{X}} \pi(x) \rho(x) \log \rho(x), \quad (2.2)$$

w.r.t. the Riemannian structure induced by \mathcal{W} , see [29, 30]. It turns out that every pair of densities $\rho_0, \rho_1 \in \mathcal{P}(\mathcal{X})$ can be joined by a constant speed geodesic, i.e. a curve $(\rho_s)_{s \in [0,1]}$ with $\mathcal{W}(\rho_s, \rho_t) = |s - t| \mathcal{W}(\rho_0, \rho_1)$ for all $s, t \in [0, 1]$. In the spirit of the approach of Lott–Sturm–Villani [28, 38] the following definition was given in [13].

Definition 2.1. *(\mathcal{X}, Q, π) has entropic Ricci curvature bounded from below by $\kappa \in \mathbb{R}$ if for any constant speed geodesic $\{\rho_t\}_{t \in [0,1]}$ in $(\mathcal{P}(\mathcal{X}), \mathcal{W})$ we have*

$$\mathcal{H}(\rho_t) \leq (1 - t)\mathcal{H}(\rho_0) + t\mathcal{H}(\rho_1) - \frac{K}{2}t(1 - t)\mathcal{W}(\rho_0, \rho_1)^2. \quad (2.3)$$

In this case, we write $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$.

We should mention that several other notions of Ricci curvature for Markov chains on discrete spaces have been proposed in the past few years: Ollivier’s coarse Ricci curvature [35], convexity along approximate geodesics [7], discrete Bakry–Émery curvature (defined in [1], and used in the discrete setting for example in [37, 27]), convexity along binomial interpolations [16] or along entropic interpolations [23].

2.2. Riemannian structure and equivalent formulation. We will briefly describe the Riemannian structure induced by \mathcal{W} to give an equivalent formulation of entropic Ricci bounds in terms of a discrete analogue of Bochner’s inequality.

For $\psi \in \mathbb{R}^{\mathcal{X}}$ we denote by $\nabla \psi \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ the discrete gradient $\nabla \psi(x, y) = \psi(y) - \psi(x)$. We denote by $\mathcal{G} = \{\nabla \psi : \psi \in \mathbb{R}^{\mathcal{X}}, \psi(x_0) = 0\}$ the set of discrete gradients modulo constants, for some fixed $x_0 \in \mathcal{X}$. In [29] it has been shown that for each $\rho \in \mathcal{P}_*(\mathcal{X})$ the map

$$\nabla \psi \mapsto \sum_y \nabla \psi(x, y) Q(x, y)$$

is a linear bijection between \mathcal{G} and the tangent space $\mathcal{T} = \{s \in \mathbb{R}^{\mathcal{X}} : \sum_x s(x) \pi(x) = 0\}$ to $\mathcal{P}_*(\mathcal{X})$ at ρ . A Riemannian tensor on $\mathcal{P}_*(\mathcal{X})$ can be defined using this identification by introducing the scalar product $\langle \cdot, \cdot \rangle_\rho$ given by

$$\langle \nabla \varphi, \nabla \varphi \rangle_\rho = \frac{1}{2} \sum_{x, y} \nabla \psi(x, y) \nabla \varphi(x, y) \hat{\rho}(x, y) Q(x, y) \pi(x).$$

Then \mathcal{W} is the associated Riemannian distance. We will set $\mathcal{A}(\rho, \psi) := |\nabla \psi|_\rho^2$. Convexity of the entropy along \mathcal{W} -geodesics is controlled by lower bounds on the Hessian of the entropy \mathcal{H} in the Riemannian structure just defined. An explicit calculation of the Hessian at $\rho \in \mathcal{P}_*(\mathcal{X})$

yields

$$\text{Hess}\mathcal{H}(\rho)[\nabla\psi] = \frac{1}{2} \sum_{x,y} \left[\frac{1}{2} \hat{L}\rho(x,y) |\nabla\psi(x,y)|^2 - \hat{\rho}(x,y) \nabla\psi(x,y) \nabla L\psi(x,y) \right] Q(x,y) \pi(x),$$

where we set $\hat{L}\rho(x,y) = \partial_1\theta(\rho(x),\rho(y))L\rho(x) + \partial_2\theta(\rho(x),\rho(y))L\rho(y)$. Setting for brevity $\mathcal{B}(\rho,\psi) := \text{Hess}\mathcal{H}(\rho)[\nabla\psi]$ we have the following reformulation of entropic Ricci bounds. Note that the statement is non-trivial since the Riemannian structure degenerates at the boundary of $\mathcal{P}(\mathcal{X})$.

Proposition 2.2. *We have $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ if and only if for every $\rho \in \mathcal{P}_*(\mathcal{X})$ and $\psi \in \mathbb{R}^{\mathcal{X}}$ we have*

$$\mathcal{B}(\rho, \psi) \geq \kappa \mathcal{A}(\rho, \psi).$$

2.3. Functional inequalities. Here we introduce the discrete functional inequalities that we shall be interested in. The discrete Dirichlet form \mathcal{E} associated to the Markov triple (\mathcal{X}, Q, π) is given by

$$\mathcal{E}(\varphi, \psi) = \frac{1}{2} \sum_{x,y} \nabla\varphi(x,y) \nabla\psi(x,y) Q(x,y) \pi(x).$$

We say that (\mathcal{X}, Q, π) satisfies for a constant $\lambda > 0$

- a Poincaré inequality $\text{P}(\lambda)$ if

$$\text{Var}_\pi(\psi) \leq \frac{1}{\lambda} \mathcal{E}(\psi, \psi) \quad \forall \psi \in \mathbb{R}^{\mathcal{X}}, \quad (2.4)$$

- a modified logarithmic Sobolev inequality $\text{MLS}(\lambda)$ if

$$\mathcal{H}(\rho) \leq \frac{1}{2\lambda} \mathcal{E}(\rho, \log \rho) \quad \forall \rho \in \mathcal{P}_*(\mathcal{X}), \quad (2.5)$$

- a modified Talagrand inequality $\text{T}_{\mathcal{W}}(\lambda)$ if

$$\mathcal{W}(\rho, \mathbf{1})^2 \leq \frac{2}{\lambda} \mathcal{H}(\rho) \quad \forall \rho \in \mathcal{P}(\mathcal{X}). \quad (2.6)$$

Here $\text{Var}_\pi(\psi) := \pi[\psi^2] - \pi[\psi]^2$. The Poincaré inequality is a spectral estimate: the optimal constant coincides with the smallest nonzero eigenvalue of the generator L of the Markov chain. It is well known that these functional inequalities govern the trend to equilibrium for the Markov semigroup $P_t = e^{tL}$. Indeed, noting that

$$\frac{d}{dt} \mathcal{H}(P_t \rho) = -\mathcal{E}(P_t \rho, \log P_t \rho), \quad \frac{d}{dt} \text{Var}_\pi(P_t \psi) = -\mathcal{E}(P_t \psi, P_t \psi),$$

we see that (2.4) and (2.5) are respectively equivalent to the exponential convergence estimates

$$\mathcal{H}(P_t \rho) \leq e^{-2\lambda t} \mathcal{H}(\rho), \quad \text{Var}_\pi(P_t \psi) \leq e^{-\lambda t} \text{Var}_\pi(\psi). \quad (2.7)$$

We shall often denote the entropy production functional by $\mathcal{I}(\rho) := \mathcal{E}(\rho, \log \rho)$. The modified Talagrand inequality implies concentration properties for the measure π .

It has been shown in [13] that entropic Ricci bounds are intimately related with the functional inequalities above. The bound $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ for $\kappa \in \mathbb{R}$ implies the $\mathcal{HWI}(\kappa)$ inequality

$$\mathcal{H}(\rho_1) - \mathcal{H}(\rho_0) \leq \mathcal{W}(\rho_0, \rho_1) \sqrt{\mathcal{I}(\rho_1)} - \frac{\kappa}{2} \mathcal{W}(\rho_0, \rho_1)^2. \quad (2.8)$$

If $\kappa > 0$, this inequality readily implies $\text{MLSI}(\kappa)$. The latter in turn was shown to imply the modified Talagrand inequality $\text{T}_{\mathcal{W}}(\kappa)$ in analogy with the result of Otto–Villani in the continuous case. By a linearization argument, it can be shown that $\text{MLSI}(\kappa)$ and $\text{T}_{\mathcal{W}}(\kappa)$ both imply the Poincaré inequality $\text{P}(\kappa)$. The converse is not true in general, see for example [2].

2.4. Distances on \mathcal{X} . The transport distance \mathcal{W} on $\mathcal{P}(\mathcal{X})$ gives rise to a distance $d_{\mathcal{W}}$ on \mathcal{X} by restriction to Dirac masses. More precisely, we set for $x, y \in \mathcal{X}$

$$d_{\mathcal{W}}(x, y) = \mathcal{W}(\delta_x, \delta_y).$$

We can give an upper bound on $d_{\mathcal{W}}$ in terms of a weighted graph distance. Let us define a distance d_Q on \mathcal{W} by setting

$$d_Q(x, y) = \inf \left\{ \sum_{i=0}^{n-1} \frac{1}{\sqrt{\min(Q(z_i, z_{i+1}), Q(z_{i+1}, z_i))}} \right\},$$

where the infimum runs over all sequences $z_0 = x, z_1, \dots, z_n = y$ such that $Q(z_i, z_{i+1}) > 0$ (and hence also $Q(z_{i+1}, z_i) > 0$ by detailed balance).

Lemma 2.3. *For any $x, y \in \mathcal{X}$ we have*

$$d_{\mathcal{W}}(x, y) \leq c d_Q(x, y), \quad c = \int_0^1 \frac{dr}{\sqrt{2\theta(1-r, 1+r)}} \approx 1.56.$$

This is a reinforcement of [13, Lem. 2.13], where $d_{\mathcal{W}}$ has been compared to $d_g/\sqrt{Q_*}$, where d_g is the unweighted graph distance obtained by replacing the summands in the definition of d_Q by 1 and $Q_* = \min\{Q(x, y) : Q(x, y) > 0\}$ is the minimal transition rate.

Proof. We argue similarly as in [13, Lem. 2.13]. We shall use that for $x, y \in \mathcal{X}$ with $Q(x, y) > 0$ the distance $\mathcal{W}(\delta_x, \delta_y)$ can be estimated by the distance on a two point space. More precisely, [29, Thm. 2.4, Lem. 3.14] and their proofs yield the estimate

$$\mathcal{W} \left(\frac{\mathbf{1}_{\{x\}}}{\pi(x)}, \frac{\mathbf{1}_{\{y\}}}{\pi(y)} \right) \leq c \sqrt{\frac{\max(\pi(x), \pi(y))}{Q(x, y)\pi(x)}} = c \sqrt{\frac{1}{\min(Q(x, y), Q(y, x))}},$$

where we have used detailed balance in the last step. The result then follows by the triangle inequality for \mathcal{W} and by taking the infimum over all sequences connecting x to y . \square

2.5. Notation. In order to alleviate notation in the sequel we introduce the following concepts. Given two functions $\varphi, \psi \in \mathbb{R}^{\mathcal{X}}$ we denote their scalar product in $L^2(\pi)$ by

$$\langle \varphi, \psi \rangle_{\pi} = \sum_{x \in \mathcal{X}} \varphi(x) \psi(x) \pi(x).$$

For two functions $\Phi, \Psi \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ defined on edges we define

$$\langle \Phi, \Psi \rangle_{\pi} = \frac{1}{2} \sum_{x, y \in \mathcal{X}} \Phi(x, y) \Psi(x, y) Q(x, y) \pi(x).$$

As a consequence of the detailed balance assumption, the generator L is selfadjoint and we have an integration by parts formula

$$\langle \psi, L\varphi \rangle_{\pi} = -\langle \nabla \varphi, \nabla \psi \rangle_{\pi} = \langle L\psi, \varphi \rangle_{\pi}.$$

We note moreover, that that the Riemannian metric tensor associate to \mathcal{W} can be rewritten as $\mathcal{A}(\rho, \psi) = \langle \nabla \psi, \nabla \psi \rangle_\rho = \langle \hat{\rho} \cdot \nabla \psi, \nabla \psi \rangle_\pi$, where the product $\hat{\rho} \cdot \nabla \psi$ is defined component-wise. Similarly, the Hessian of the entropy can be rewritten in the compact form

$$\mathcal{B}(\rho, \psi) = \frac{1}{2} \langle \hat{L}\rho \cdot \nabla \psi, \nabla \psi \rangle_\pi - \langle \nabla \psi, \hat{\rho}, \nabla L\psi \rangle_\pi .$$

We also define the Γ -operator $\Gamma : \mathbb{R}^{\mathcal{X}} \times \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}}$ by setting

$$\Gamma(\varphi, \psi)(x) = \sum_y \nabla \varphi(x, y) \nabla \psi(x, y) Q(x, y) ,$$

and set $\Gamma(\varphi) = \Gamma(\varphi, \varphi)$.

3. GRADIENT ESTIMATES

In this section we show that entropic Ricci curvature lower bounds are equivalent to certain gradient estimates for the Markov semigroup $P_t = e^{tL}$. These will be crucial in establishing functional inequalities in the sequel.

Theorem 3.1 (Gradient estimate). *A Markov triple (\mathcal{X}, Q, π) satisfies the entropic Ricci bound $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ if and only if for every $\psi \in \mathbb{R}^{\mathcal{X}}$ and $\rho \in \mathcal{P}(\mathcal{X})$ we have*

$$|\nabla P_t \psi|_\rho^2 \leq e^{-2\kappa t} |\nabla \psi|_{P_t \rho}^2 . \quad (3.1)$$

Remark 3.2. More explicitly, the gradient estimate reads as follows:

$$\frac{1}{2} \sum_{x,y} |\nabla P_t \psi|^2(x, y) \hat{\rho}(x, y) Q(x, y) \pi(x) \leq e^{-2\kappa t} \frac{1}{2} \sum_{x,y} |\nabla \psi|^2(x, y) \widehat{P_t \rho}(x, y) Q(x, y) \pi(x) .$$

The proof follows the standard semigroup interpolation argument, slightly adapted to our setting.

Proof. First we note that if (3.1) holds for all ψ and $\rho \in \mathcal{P}_*(\mathcal{X})$ then it also holds for all $\rho \in \mathcal{P}(\mathcal{X})$. Now fix $\psi \in \mathbb{R}^{\mathcal{X}}$, $\rho \in \mathcal{P}_*(\mathcal{X})$ and $t > 0$. We define for $s \in [0, t]$:

$$\Phi(s) = e^{-2\kappa s} |\nabla P_{t-s} \psi|_{P_s \rho}^2 = e^{-2\kappa s} \langle \nabla P_{t-s} \psi, \widehat{P_s \rho} \cdot \nabla P_{t-s} \psi \rangle_\pi .$$

Note that $\Phi(0) = |\nabla P_t \psi|_\rho^2$ and $\Phi(t) = e^{-2\kappa t} |\nabla \psi|_{P_t \rho}^2$. We immediately compute the derivative of Φ . Putting $\psi_s = P_s \psi$ and $\rho_s = P_s \rho$ we get:

$$\begin{aligned} \Phi'(s) &= e^{-2\kappa s} \left[\langle \nabla \psi_{t-s}, \hat{L}\rho_s \cdot \nabla \psi_{t-s} \rangle_\pi - 2 \langle \nabla \psi_{t-s}, \hat{\rho}_s \cdot \nabla L\psi_{t-s} \rangle_\pi - 2\kappa \langle \nabla \psi_{t-s}, \hat{\rho}_s \nabla \psi_{t-s} \rangle_\pi \right] \\ &= 2e^{-2\kappa s} \left[\mathcal{B}(\rho_s, \psi_{t-s}) - \kappa \mathcal{A}(\rho_s, \psi_{t-s}) \right] . \end{aligned}$$

If $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ holds, we conclude from Proposition 2.2 that $\Phi'(s) \geq 0$ and obtain (3.1). For the converse implication we derivate (3.1) at $t = 0$. More precisely, assume that (3.1) holds. Then we have that

$$\begin{aligned} 0 &\leq e^{-2\kappa t} |\psi|_{P_t \rho}^2 - |P_t \psi|_\rho^2 \\ &= (e^{-2\kappa t} - 1) |\psi|_\rho^2 + e^{-2\kappa t} (|\psi|_{P_t \rho}^2 - |\psi|_\rho^2) - |P_t \psi|_\rho^2 + |\psi|_\rho^2 . \end{aligned}$$

Dividing by t and letting $t \rightarrow 0$ we obtain that $\mathcal{B}(\rho, \psi) - \kappa \mathcal{A}(\rho, \psi) \geq 0$ which yields the claim again by Proposition 2.2. \square

Remark 3.3. The previous result is in close analogy with the classical Bakry–Émery gradient estimate for the heat semigroup on a Riemannian manifold M with $\text{Ric} \geq \kappa$ which states that for any smooth function ψ it holds $|\nabla P_t \psi|^2 \leq e^{-2\kappa t} P_t |\nabla \psi|^2$. Integrating this estimate against a function ρ yields

$$\int_M |\nabla P_t \psi|^2 \rho \leq e^{-2\kappa t} \int_M |\nabla \psi|^2 P_t \rho,$$

which closely resembles (3.1) except for the appearance of the logarithmic mean.

Using the freedom in the choice of ρ in the gradient estimate (3.1), we can deduce a more explicit estimate, that does not involve logarithmic means. To this end we introduce the heat kernel associated to the continuous-time Markov chain. We put $p_t(x, y) = \pi(y)^{-1} P_t \mathbf{1}_y(x)$. From the symmetry and linearity of P_t it is immediate to check that $P_t f(x) = \sum_y p_t(x, y) f(y) \pi(y)$ and $p_t(x, y) = p_t(y, x)$.

Corollary 3.4. *If $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ holds, we have for any $\psi \in \mathbb{R}^{\mathcal{X}}$ and all $x, y \in \mathcal{X}$:*

$$\frac{1}{2} |\nabla P_t \psi|^2(x, y) Q(x, y) \pi(x) \leq e^{-2\kappa t} [P_t \Gamma(\psi)(x) \pi(x) + P_t \Gamma(\psi)(y) \pi(y)]. \quad (3.2)$$

Proof. Starting from (3.1) we choose $\rho = \mathbf{1}_x + \mathbf{1}_y$. Note that it is not a probability density, but this does no harm, since both sides of (3.1) are homogeneous in ρ . Note that $P_t \rho(u) = p_t(x, u) \pi(x) + p_t(y, u) \pi(y)$. Using the estimate $\theta(s, t) \leq (s + t)/2$ and using symmetry we obtain

$$\begin{aligned} \frac{1}{2} |\nabla P_t \psi|^2(x, y) Q(x, y) \pi(x) &\leq e^{-2\kappa t} \frac{1}{2} \sum_{u, v} |\nabla \psi|^2(u, v) [p_t(x, u) \pi(x) + p_t(y, u) \pi(y)] Q(u, v) \pi(u) \\ &= e^{-2\kappa t} [P_t \Gamma(\psi)(x) \pi(x) + P_t \Gamma(\psi)(y) \pi(y)]. \end{aligned}$$

□

Next, we derive a reverse Poincaré inequality along the Markov semigroup.

Theorem 3.5 (Reverse Poincaré inequality). *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$. Then we have for any $\rho \in \mathcal{P}(\mathcal{X})$ and $\psi \in \mathbb{R}^{\mathcal{X}}$:*

$$\langle \psi^2, P_t \rho \rangle_\pi - \langle (P_t \psi)^2, \rho \rangle_\pi \geq \frac{e^{2\kappa t} - 1}{\kappa} |\nabla P_t \psi|_\rho^2. \quad (3.3)$$

Proof. We put $\Phi(s) = \langle (P_{t-s} \psi)^2, P_s \rho \rangle_\pi$ and calculate, putting $g = P_{t-s} \psi$ and $h = P_s \rho$,

$$\begin{aligned} \Phi'(s) &= \langle \Delta(P_{t-s} \psi)^2 - 2P_{t-s} \psi \Delta P_{t-s} \psi, P_s \rho \rangle_\pi \\ &= \sum_{x, y} h(x) [g(y)^2 - g(x)^2] Q(x, y) \pi(x) - 2 \sum_{x, y} h(x) g(x) [g(y) - g(x)] Q(x, y) \pi(x) \\ &= \sum_{x, y} h(x) [g(y) - g(x)]^2 Q(x, y) \pi(x) \\ &= \sum_{x, y} \frac{h(x) + h(y)}{2} [g(y) - g(x)]^2 Q(x, y) \pi(x) \\ &\geq \sum_{x, y} \theta(h(x), h(y)) [g(y) - g(x)]^2 Q(x, y) \pi(x) \\ &= 2 |\nabla P_{t-s} \psi|_{P_s \rho}^2. \end{aligned}$$

Using the gradient estimate (3.1) we conclude $\Phi'(s) \geq e^{2\kappa s} |\nabla P_t \psi|_\rho^2$ which yields the claim. \square

From the previous theorem we can derive a uniform bound on the gradient. Fix $x, y \in \mathcal{X}$ and put $\rho = \frac{1}{2\pi(x)}\delta_x + \frac{1}{2\pi(y)}\delta_y$. Then (3.3) yields

$$\frac{e^{2\kappa t} - 1}{\kappa} |\nabla P_t \psi|^2(x, y) \theta(Q(x, y), Q(y, x)) \leq \|\psi\|_\infty^2.$$

Putting $Q_* = \min\{Q(x, y) : x, y \text{ s.t. } Q(x, y) > 0\}$ we obtain

$$\frac{e^{2\kappa t} - 1}{\kappa} \max_{x, y: Q(x, y) > 0} |\nabla P_t \psi|(x, y) \leq \frac{1}{\sqrt{Q_*}} \|\psi\|_\infty. \quad (3.4)$$

As a consequence we obtain the following L^1 bound for the semigroup.

Lemma 3.6. *If $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ we have for any $\psi \in \mathbb{R}^{\mathcal{X}}$ and $t \leq 1/2|\kappa|$:*

$$\|\psi - P_t \psi\|_{L^1(\pi)} \leq \frac{2\sqrt{t}}{\sqrt{Q_*}} \|\nabla \psi\|_{L^1}, \quad (3.5)$$

or more explicitly

$$\sum_x |\psi(x) - P_t \psi(x)| \pi(x) \leq \frac{2\sqrt{t}}{\sqrt{Q_*}} \sum_{x, y} |\nabla \psi|(x, y) Q(x, y) \pi(x).$$

Proof. Fix a function g with $|g| \leq 1$. Then we estimate

$$\begin{aligned} \langle g, \psi - P_t \psi \rangle_\pi &= - \int_0^t \langle g, \Delta P_s \psi \rangle_\pi \, ds \\ &= \int_0^t \langle \nabla P_s g, \nabla \psi \rangle_\pi \, ds \\ &\leq \|\nabla \psi\|_{L^1} \frac{1}{\sqrt{Q_*}} \int_0^t \frac{1}{\sqrt{s}} \, ds \\ &= \|\nabla \psi\|_{L^1} \frac{2\sqrt{t}}{\sqrt{Q_*}}. \end{aligned}$$

Here we have used (3.4) and the fact that $(e^{2\kappa t} - 1)/\kappa \geq t$ for $0 < t \leq \frac{1}{2|\kappa|}$. Taking the supremum over g yields the claim. \square

Finally, we derive an exponential decay estimate for the Γ operator.

Proposition 3.7. *Assuming that $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa \in \mathbb{R}$, we have for any $f \in \mathbb{R}^{\mathcal{X}}$*

$$\pi[\Gamma(P_t f)] \leq e^{-2\kappa t} \pi[\Gamma(f)].$$

Moreover, $P_t f$ is Lipschitz with constant $\|f\|_\infty \sqrt{\kappa/(e^{2\kappa t} - 1)}$ for the distance d_W .

Proof. The first part is a direct consequence of the gradient estimate (3.1) when taking $\rho \equiv 1$. For the second part, fix $\varepsilon > 0$ and consider a pair $(\rho_s, \psi_s)_{s \in [0,1]}$ satisfying (2.1) such that $\mathcal{W}(\delta_x, \delta_y)^2 \leq \int_0^1 |\psi_s|_{\rho_s}^2 \, ds + \varepsilon$. Then we can estimate using Jensen's inequality and the reverse

Poincaré inequality 3.5:

$$\begin{aligned}
 P_t f(x) - P_t f(y) &= \int_0^1 \langle \nabla P_t f, \nabla \psi_s \rangle_{\rho_s} ds \leq \int_0^1 |\nabla P_t f|_{\rho_s} |\nabla \psi_s|_{\rho_s} ds \\
 &\leq \|f\|_\infty \sqrt{\kappa/(e^{2\kappa t} - 1)} \int_0^1 |\nabla \psi_s|_{\rho_s} ds \\
 &\leq \|f\|_\infty \sqrt{\kappa/(e^{2\kappa t} - 1)} \sqrt{\mathcal{W}(\delta_x, \delta_y)^2 + \varepsilon}.
 \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ yields the claim. \square

4. ISOPERIMETRIC ESTIMATE AND BUSER INEQUALITY

Now we can formulate an isoperimetric estimate which will immediately yield the discrete Buser inequality, see Theorem 1.1. To this end we recall that the perimeter measure π^+ of a subset $A \subset \mathcal{X}$ is defined by

$$\pi^+(\partial A) = \sum_{x \in A, y \in A^c} Q(x, y) \pi(x).$$

Theorem 4.1. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ holds and let λ_1 be the spectral gap of L . Then for any subset $A \subset \mathcal{X}$ we have*

$$\pi^+(\partial A) \geq \frac{1}{3} \sqrt{Q_*} \min\left(\frac{\lambda_1}{\sqrt{\kappa}}, \sqrt{\lambda_1}\right) \pi(A)(1 - \pi(A)). \quad (4.1)$$

Proof. We apply (3.5) to the indicator function χ_A and obtain

$$\begin{aligned}
 \frac{2\sqrt{t}}{\sqrt{Q_*}} \cdot \pi^+(\partial A) &= \frac{2\sqrt{t}}{\sqrt{Q_*}} \|\nabla \chi_A\|_{L^1} \geq \|\chi_A - P_t \chi_A\|_{L^1(\pi)} \\
 &= 2\pi(A) - 2\|P_{\frac{t}{2}} \chi_A\|_{L^2(\pi)}^2 \\
 &= 2\left[\pi(A) - \pi(A)^2 - \|P_{\frac{t}{2}}(\chi_A - \pi(A))\|_{L^2(\pi)}^2\right] \\
 &\geq 2\left[\pi(A) - \pi(A)^2 - e^{-\lambda_1 t} \|\chi_A - \pi(A)\|_{L^2(\pi)}^2\right] \\
 &= 2\pi(A)(1 - \pi(A))(1 - e^{-\lambda_1 t}).
 \end{aligned}$$

Now we conclude by optimizing in t . That is, if $\lambda_1 \geq 2|\kappa|$ we can take $t = \frac{1}{\lambda_1}$ and in the opposite case we take $t = \frac{1}{2\kappa}$. \square

We recall that the Cheeger constant h of the Markov chain (\mathcal{X}, Q, π) is defined as the optimal constant in the previous isoperimetric estimate, i.e.

$$h = \max_{A \subset \mathcal{X}} \frac{\pi^+(\partial A)}{\pi(A)(1 - \pi(A))}.$$

Corollary 4.2 (Buser inequality). *If $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$, we have the following Buser inequality:*

$$h \geq \frac{1}{3} \sqrt{Q_*} \min\left(\frac{\lambda_1}{\sqrt{\kappa}}, \sqrt{\lambda_1}\right). \quad (4.2)$$

In particular, if $\kappa \geq 0$, we have $h \geq \frac{\sqrt{\lambda_1}}{3} \sqrt{Q_}$.*

Proof. This follows immediately from the previous proposition and the definition of the Cheeger constant h . The second statement follows by recalling that if $\kappa > 0$, $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ implies that $\lambda_1 \geq \kappa$. \square

Corollary 4.3. *When $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$, the Poincaré inequality $P(\lambda_1)$ implies an L^1 Poincaré inequality*

$$\sum_x |\psi(x) - \pi[\psi]| \pi(x) \leq \frac{4}{c(\kappa, \lambda_1)} \sum_{x,y} |\nabla \psi| Q(x, y) \pi(x) \quad \forall \psi \in \mathbb{R}^{\mathcal{X}}.$$

where $c(\kappa, \lambda_1)$ is the constant in the right hand side of (4.2).

The equivalence of the Cheeger isoperimetric inequality and the L^1 Poincaré inequality is a well-established fact. We briefly prove that the Cheeger inequality (4.1) implies the L^1 Poincaré inequality for the sake of completeness. The reverse implication is immediately established by applying the L^1 Poincaré inequality to indicators of sets.

Proof. First, we shall establish the inequality when $\pi[\psi]$ is replaced by a median of ψ . Let ψ be a function with median 0. Writing χ_A for the indicator of a set A , we have

$$\begin{aligned} \sum_{x,y} |\psi(x) - \psi(y)| Q(x, y) \pi(x) &= \int_{-\infty}^{+\infty} \sum_{x,y} \chi_{[\psi(y), \psi(x)]}(t) Q(x, y) dt \\ &= \int_{-\infty}^{+\infty} \pi^+(\psi > t) dt \\ &\geq \int_{-\infty}^{+\infty} c(\kappa, \lambda_1) \pi(\psi > t) \pi(\psi < t) dt \\ &\geq c(\kappa, \lambda_1) \int_{-\infty}^0 \frac{\pi(\psi < t)}{2} dt + c(\kappa, \lambda_1) \int_0^{+\infty} \frac{\pi(\psi > t)}{2} dt \\ &= \frac{c(\kappa, \lambda_1)}{2} \sum_x |\psi(x)| \pi(x). \end{aligned}$$

We can then replace the median by the mean since $\pi[|\psi - \pi[\psi]|] \leq 2\pi[|\psi|]$. \square

5. POINCARÉ INEQUALITY

The aim in this section is to prove that if the curvature of a Markov is non-negative, then the spectral gap is controlled by the diameter. This is a discrete analog of the following classical result of Li and Yau [26]:

Theorem 5.1. *Let M be a manifold with non-negative curvature and finite diameter D . Then the spectral gap of the manifold satisfies*

$$\lambda_1 \geq \frac{\pi^2}{D^2}.$$

In the continuous setting, the constant π^2 is known to be optimal. Moreover, this result is rigid, in the following sense: if equality holds, then the manifold is isometric to the one-dimensional torus of diameter D [17]. As we shall discuss in Section 6.3, results of Milman [32, 31, 33] show that on geodesic spaces the spectral gap can be controlled even by measure concentration properties.

The proof is inspired by [15], although the proof of the weak Poincaré inequality follows a different line of arguments, inspired by [22]. First we establish a weak Poincaré inequality, under the assumption that curvature is non-negative and that the invariant measures has the exponential concentration property. Then we establish a tight Poincaré inequality under the stronger assumption of bounded diameter.

5.1. Weak Poincaré inequality under exponential concentration.

Definition 5.2. *A probability measure on a metric space is said to satisfy a concentration property with profile $\beta : \mathbb{R}_+ \rightarrow [0, 1]$ if, for any set A such that $\mu(A) \geq 1/2$, we have*

$$\mu(A_r^c) \leq \beta(r); \quad A_r^c := \{x; d(x, A) > r\}. \quad (5.1)$$

In particular, we shall say that a probability measure satisfies the exponential concentration property with constants M and α if (5.1) holds with $\beta(r) := Me^{-\alpha r}$. Similarly, it is said to satisfy a Gaussian concentration property with constants M and ρ if it admits $\beta(r) = Me^{-\rho r^2}$ as a concentration profile.

A concentration profile governs the tail behavior of the measure. Heuristically, the exponential (resp. Gaussian) concentration property compares the behavior at infinity with that of the exponential (resp. Gaussian) measure. When the diameter is bounded by D , it is easy to see that an exponential (resp. Gaussian) concentration property holds with constant $\alpha = 1/D$ and $M = e$ (resp. $\rho = 1/D^2$ and $M = e$). However, this estimate is often much worse than the optimal concentration estimate, and concentration can hold for unbounded spaces (for example, \mathbb{R}^d equipped with a Gaussian measure).

A classical result in the study of concentration of measure is that one can use functional inequalities to establish concentration with some profile. In particular, the Poincaré inequality implies an exponential concentration property, while a (modified) logarithmic Sobolev inequality implies Gaussian concentration [20]. Moreover, one can show that Gaussian concentration is equivalent to a transport-entropy inequality for the Wasserstein distance W_1 [6]. In the Riemannian setting, Milman [32, 31, 33] showed that the converse is true when curvature is non-negative: functional inequalities and concentration properties are then actually *equivalent*. We shall discuss this aspect further in Section 6.3.

Proposition 5.3. *Let (\mathcal{X}, Q, π) be a Markov chain with $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa \in \mathbb{R}$ and assume that π has exponential concentration with respect to the distance $d_{\mathcal{W}}$ with constants α and M . Then for any $t > 0$ and $f \in \mathbb{R}^{\mathcal{X}}$ we have*

$$\text{Var}_{\pi}(f) \leq \frac{1 - e^{-2\kappa t}}{\kappa} \pi[\Gamma(f)] + \frac{\kappa}{e^{2\kappa t} - 1} \frac{2\|f\|_{\infty}^2}{\alpha^2}.$$

In particular, if $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$, then

$$\text{Var}_{\pi}(f) \leq 2t\pi[\Gamma(f)] + \frac{M\|f\|_{\infty}^2}{\alpha^2 t}.$$

If moreover the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ is bounded by D , we have

$$\text{Var}_{\pi}(f) \leq 2t\pi[\Gamma(f)] + \frac{D^2\|f\|_{\infty}^2}{4t}.$$

Note that by comparing $d_{\mathcal{W}}$ with the weighted graph distance, we can use concentration with respect to the graph distance instead. This will worsen the constant by a universal factor, see Lemma 2.3.

Proof. First, we have by Proposition 3.7

$$\begin{aligned} \mathrm{Var}_\pi(f) &= \mathrm{Var}_\pi(P_t f) + 2 \int_0^t \pi [\Gamma(P_t f)] dt \\ &\leq \mathrm{Var}_\pi(P_t f) + \frac{1 - e^{-2\kappa t}}{\kappa} \pi [\Gamma(f)] . \end{aligned}$$

Since π satisfies the exponential concentration property with constant α and M , for any Lipschitz function f , we have

$$\mathrm{Var}_\pi(f) \leq \frac{2M}{\alpha^2} \|f\|_{lip}^2 \quad (5.2)$$

and applying this to $P_t f$ using Proposition 3.7 yields the result. The variance bound (5.2) is obtained just by integrating the concentration bound, and using the fact that $\mathrm{Var}_\pi(f) \leq \pi[(f - m)^2]$, where m is a median of f . The estimate for the case when the diameter is bounded by D is obtained by using the estimate $\mathrm{Var}_\pi(f) \leq \frac{D^2}{2} \|f\|_{lip}^2$. \square

5.2. Tight Poincaré inequality under a diameter bound. The proof of Theorem 1.2 follows the ideas of [15] and relies on a discrete analogue of the HWI inequality that is given by $\mathcal{HWI}(\kappa)$ in (2.8).

One of the obstacles to applying the strategy of [15] is that the quantity $\mathcal{I}(\rho)$ that appears in the $\mathcal{HWI}(\kappa)$ inequality is given by

$$\mathcal{I}(\rho) = \mathcal{E}(\rho, \log \rho) = \pi [\Gamma(\rho, \log \rho)]$$

and that in the discrete setting this is the latter expression is different from the term $\pi [\Gamma(\sqrt{\rho})]$ which naturally appears in the Poincaré inequality. To this end we need the following comparison result.

Lemma 5.4. *For any $\alpha > 0$ the following are equivalent:*

$$\begin{aligned} (i) \quad \mathrm{Var}_\pi(f) &\leq \frac{1}{\alpha} \pi [\Gamma(f)] \quad \forall f \in \mathbb{R}^{\mathcal{X}} , \\ (ii) \quad \mathrm{Var}_\pi(f) &\leq \frac{1}{4\alpha} \pi [\Gamma(f^2, \log f^2)] \quad \forall f \in \mathbb{R}^{\mathcal{X}} . \end{aligned}$$

Proof. (i) \Rightarrow (ii) just follows from the inequality $\Gamma(f) \leq \frac{\Gamma(f^2, \log f^2)}{4}$.

To prove (ii) \Rightarrow (i), we linearize (ii) taking $f = 1 + \varepsilon h$ for some $h \in \mathbb{R}^{\mathcal{X}}$ with $\pi[h] = 0$ and let $\varepsilon \rightarrow 0$. \square

This equivalence is not true for non-tight versions of the Poincaré inequality, for which we only have (i) \Rightarrow (ii). So we shall prove non-tight inequalities with the modified Dirichlet form, deduce a tight inequality with the modified Dirichlet form, and finally obtain the usual Poincaré inequality in the end.

Lemma 5.5. *Assume that $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq -\kappa$ for some $\kappa \geq 0$ and the diameter of $(\mathcal{X}, \delta_{\mathcal{W}})$ is bounded by D . Then for any $\delta > 0$ and any $f \in \mathbb{R}^{\mathcal{X}}$, we have*

$$\pi[f^2] \leq \frac{1}{4\delta} \pi [\Gamma(f^2, \log f^2)] + e^{D^2(\delta+\kappa/2)} \pi[|f|]^2 . \quad (5.3)$$

Proof. Since the distance \mathcal{W} can be bounded by the L^2 Wasserstein distance built from $d_{\mathcal{W}}$, see [13, Prop. 3.12], we have that $\mathcal{W}(\rho, \rho') \leq D$ for all $\rho, \rho' \in \mathcal{P}(\mathcal{X})$. After multiplying f with

a constant we can assume that $\pi[f^2] = 1$. The $\mathcal{HWI}(-\kappa)$ inequality applied to f^2 together with Young's inequality and the diameter bound yields

$$\begin{aligned} \pi[f^2 \log f^2] &= \mathcal{H}(f^2) \leq \mathcal{W}(f^2, 1) \sqrt{\pi[\Gamma(f^2, \log f^2)]} + \frac{\kappa}{2} \mathcal{W}(f^2, 1)^2 \\ &\leq \frac{1}{4\delta} \pi[\Gamma(f^2, \log f^2)] + D^2 \left(\delta + \frac{\kappa}{2} \right). \end{aligned}$$

To obtain the result from this inequality, we can then just follow the proof of [15, Lemma 3.5] (The proof uses a different Dirichlet form, but in this case it makes no difference). \square

We will use the following tightening result.

Proposition 5.6. *Assume that for any function $f \in \mathbb{R}^{\mathcal{X}}$ we have*

$$\mathrm{Var}_\pi(f) \leq \alpha_1 \pi[\Gamma(f^2, \log f^2)] + \beta_1 \|f\|_\infty^2 \quad (5.4)$$

and

$$\mathrm{Var}_\pi(f) \leq \alpha_2 \pi[\Gamma(f^2, \log f^2)] + \beta_2 \pi[|f|]^2 \quad (5.5)$$

with constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfying $\frac{\beta_2}{2} + \frac{1}{2} \sqrt{(3\beta_1 + \beta_2 - 1)(2 + \beta_2)} < 1$. Then the Poincaré inequality $PI(\lambda)$ holds with constant $\lambda = \frac{2 - (\beta_2 + \sqrt{(3\beta_1 + \beta_2 - 1)(2 + \beta_2)})}{8(3\alpha_1 + \alpha_2)}$.

Combining the weak Poincaré inequality, Lemma 5.5 and this tightening result, after optimizing the constants, we immediately obtain

Theorem 5.7. *Assume that $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq -\kappa$ for some $\kappa \geq 0$ and that the diameter of (\mathcal{X}, d_W) is bounded by D . Assume that $e^{D^2\kappa/2} + \sqrt{(3\kappa D^2/2 + e^{D^2\kappa/2} - 1)(2 + e^{D^2\kappa/2})} < 2$. Then the Poincaré inequality $PI(\lambda)$ holds with a constant λ that only depends on κ and D . In particular, if $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq 0$, then $PI(cD^{-2})$ holds for a universal constant c .*

The best possible value of the constant c we obtain with this proof is hard to determine, but we can show that it satisfies $c \geq \frac{9 - \sqrt{62}}{80(45 + \ln(11/10))}$. In Section 5.3 we present an alternative argument that yield $PI(cD^{-2})$ with an explicit and probably better constant c .

Proof of Proposition 5.6. The proof essentially follows the argument of [2, Prop. 7.5.6], except that we use the Dirichlet form $\pi[\Gamma(f^2, \log f^2)]$, so some adaptation is required.

Consider f satisfying $\mathrm{med}(f) = 0$ and $\pi[f^2] = 1$ and fix $R > 0$. Without loss of generality, we may assume that $\pi[\{f = 0\}] = 0$, otherwise $\pi[\Gamma(f^2, \log f^2)] = \infty$ and there is nothing to prove. Let $f_R(x) = f(x)$ if $|f(x)| < R$, and R (resp. $-R$) if $f(x) \geq R$ (resp. $f(x) \leq -R$), and $d_R := \pm R + f - f_R$ (depending on the sign of $f(x)$). We have that both $\Gamma(f_R^2, \log f_R^2)$ and $\Gamma(d_R^2, \log d_R^2)$ are smaller than $\Gamma(f^2, \log f^2)$. Notice that f_R also has median 0.

Now we have

$$\begin{aligned} 1 &= \pi[f^2] = \pi[(f_R + (f - f_R))^2] \\ &= \pi[f_R^2] + 2\pi[f_R(f - f_R)] + \pi[(f - f_R)^2] \\ &= \pi[f_R^2] + \pi[d_R^2] - R^2 \end{aligned}$$

We also have

$$\begin{aligned} |\pi[f - f_R]| &\leq \pi[|f - f_R|] = \pi[|f| \mathbf{1}_{|f| \geq R}] - R\pi[\{|f| \geq R\}] \\ &\leq \sqrt{\pi[f^2]} \sqrt{\pi[\{|f| \geq R\}]} - R\pi[\{|f| \geq R\}] \leq \frac{1}{4R}. \end{aligned}$$

Moreover, since f has median 0, we have

$$|\pi[d_R]| = |\pi[f - f_R]| \leq \frac{1}{4R}. \quad (5.6)$$

Since f_R has median 0, we have $\pi[f_R^2] \leq 3 \operatorname{Var}_\pi(f)$ (see for instance [32, Lem. 2.1]). Applying (5.4) to f_R , we then get

$$\pi[f_R^2] \leq 3 \operatorname{Var}_\pi(f) \leq 3\alpha_1\pi[\Gamma(f^2, \log f^2)] + 3\beta_1 R^2. \quad (5.7)$$

We now seek to bound $\pi[d_R^2] - R^2$. Applying (5.5), we have

$$\begin{aligned} \pi[d_R^2] &\leq \pi[d_R]^2 + \alpha_2\pi[\Gamma(f^2, \log f^2)] + \beta_2\pi[|d_R|]^2 \\ &\leq \alpha_2\pi[\Gamma(f^2, \log f^2)] + \beta_2\pi[|d_R|]^2 + \frac{1}{16R^2}. \end{aligned}$$

Since

$$\begin{aligned} \pi[|d_R|]^2 &= R^2 + 2R\pi[|f - f_R|] + \pi[|f - f_R|]^2 \\ &\leq R^2 + \frac{1}{2} + \frac{1}{16R^2}, \end{aligned}$$

we get

$$\pi[d_R^2] - R^2 \leq \alpha_2\pi[\Gamma(f^2, \log f^2)] + \frac{\beta_2}{2} + (\beta_2 - 1)R^2 + \frac{1 + \beta_2}{16R^2}. \quad (5.8)$$

Combining this estimate with (5.7) we obtain

$$1 = \pi[f^2] \leq (3\alpha_1 + \alpha_2)\pi[\Gamma(f^2, \log f^2)] + \frac{\beta_2}{2} + (3\beta_1 + \beta_2 - 1)R^2 + \frac{2 + \beta_2}{16R^2}.$$

Optimizing in R then yields

$$1 \leq (3\alpha_1 + \alpha_2)\pi[\Gamma(f^2, \log f^2)] + \frac{\beta_2}{2} + \frac{1}{2}\sqrt{(3\beta_1 + \beta_2 - 1)(2 + \beta_2)}.$$

Since $\operatorname{Var}_\pi(f) \leq \pi[f^2] = 1$, this amounts to the Poincaré inequality by Lemma 5.4 as soon as

$$\frac{\beta_2}{2} + \frac{1}{2}\sqrt{(3\beta_1 + \beta_2 - 1)(2 + \beta_2)} < 1.$$

□

We can use the same arguments to treat the case where the diameter is not necessarily bounded, but the distance d_W has a square-exponential moment:

Theorem 5.8. *Assume that $\operatorname{Ric}(\mathcal{X}, Q, \pi) \geq 0$, and that there exists a constant $\alpha > 0$ such that for some $x_0 \in \mathcal{X}$*

$$D_\alpha := \pi \left[e^{\alpha d_W(\cdot, x_0)^2} \right] < \infty.$$

Then the Poincaré inequality $PI(\lambda)$ holds with some constant λ which depends on α and on the value D_α of the above expectation.

Note that the finiteness of the integral does not depend on the choice of x_0 , but the value does, which affects the value of the constant λ we obtain.

Proof. Since the proof follows the same lines as for the bounded diameter case, we shall only sketch it and point out the extra arguments required. Since we have a square-exponential moment, a Gaussian concentration property (and hence an exponential concentration property) holds, and Proposition 5.3 still applies. So all we need to do is to show that the conclusion of Lemma 5.5 still holds. Let f be a probability density. Note that by convexity of \mathcal{W}^2 we have the bound

$$\begin{aligned} \mathcal{W}(f, 1)^2 &\leq \sum_{x,y} d_{\mathcal{W}}(x, y)^2 f(x)\pi(x)\pi(y) \\ &\leq 2\pi [d_{\mathcal{W}}(\cdot, x_0)^2 f(\cdot)] + 2\pi [d_{\mathcal{W}}(x_0, \cdot)^2] . \end{aligned}$$

The second term is a constant that does not depend on f , and can be bounded using only the square-exponential moment. For the first term, we can use the bound

$$\pi [d_{\mathcal{W}}(\cdot, x_0)^2 f(\cdot)] \leq \frac{1}{\alpha} \log \pi [e^{\alpha d_{\mathcal{W}}(\cdot, x_0)^2}] + \frac{1}{\alpha} \pi [f \log f] .$$

Combining this inequality with $\mathcal{HWI}(0)$ and Young's inequality yields for any $\delta > 0$:

$$\begin{aligned} \pi [f \log f] &\leq \frac{1}{4\delta} \pi [\Gamma(f, \log f)] + 2\delta \mathcal{W}(f, 1)^2 \\ &\leq \frac{1}{4\delta} \pi [\Gamma(f, \log f)] + \frac{2\delta}{\alpha} \pi [f \log f] + C\delta \end{aligned}$$

with a constant C that depends on α and the square-exponential moment D_α , but not on f . Since we can make δ arbitrarily small, the second term on the right-hand side can be absorbed into the left-hand side. Then the proof continues by applying this inequality to f^2 with $\pi[f^2] = 1$ and arguing in the same way as for the bounded diameter case. \square

5.3. An alternative argument. We present an alternative derivation of the Poincaré inequality from diameter bounds in non-negative curvature following the approach of [11].

Proposition 5.9. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ is bounded by D . Then the Poincaré inequality $PI(\lambda)$ holds with*

$$\lambda = \frac{1}{eD^2} .$$

Proof. Recall that the optimal constant in the Poincaré inequality is (minus) the first non-zero eigenvalue of the generator L . Let f be an eigenfunction of the L with eigenvalue $-\lambda_1$. By scaling, we can assume without loss of generality that $\|f\|_\infty = 1$. Since f necessarily satisfies $\pi[f] = 0$, we have $\min f < 0 < \max f$.

Note that $P_t f = e^{-\lambda_1 t} f$. Thus, the reverse Poincaré inequality (3.3) implies that for any $\rho \in \mathcal{P}(\mathcal{X})$ we have

$$|\nabla f|_\rho^2 \leq \frac{e^{2\lambda t}}{2t} \|f\|_\infty^2 .$$

Optimizing in t and using $\|f\|_\infty = 1$ we find that

$$|\nabla f|_\rho^2 \leq e\lambda_1 .$$

Now, let x_0, x_1 be such that $f(x_0) = \min f$ and $f(x_1) = \max f$. Let $\varepsilon > 0$ and let $(\rho_s, \psi_s)_{s \in [0,1]}$ be a curve satisfying (2.1) such that $\int_0^1 |\nabla \psi_s|_{\rho_s}^2 ds \leq \mathcal{W}(\delta_{x_0}, \delta_{x_1})^2 + \varepsilon$. Then we estimate

$$\begin{aligned} 1 &\leq [f(x_1) - f(x_0)]^2 = \left| \sum f \rho_1 \pi - \sum f \rho_0 \pi \right|^2 = \left| \int_0^1 \langle \nabla f, \nabla \psi_s \rangle_{\rho_s} ds \right|^2 \\ &\leq (D^2 + \varepsilon) \int_0^1 |\nabla f|_{\rho_s}^2 ds \leq (D^2 + \varepsilon) \lambda_1 e \end{aligned}$$

and the result immediately follows. \square

This argument could be adapted to treat the case of negative entropic Ricci curvature as well, but we will not pursue this. We note that the proof just given is quite simpler than the previous one. However, it is not clear that we can use the same argument to cover the case where we only assume the invariant measure to have a square-exponential moment, or how to use it to prove a modified logarithmic Sobolev inequality. We will see in the next section that this is possible with the first method.

6. MODIFIED LOGARITHMIC SOBOLEV INEQUALITIES

In this section we will prove the third main result Theorem 1.3 establishing a modified logarithmic Sobolev inequality for Markov chains with non-negative entropic Ricci curvature under a diameter bound. We will then apply this to derive bounds on the total variation mixing time of the Markov chain. Finally, we formulate conjectures about possible improvements of the results replacing the bound on the diameter with a control on concentration properties.

6.1. Modified LogSobolev inequality from diameter bounds. We will show the following

Theorem 6.1. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that the diameter of $(X, d_{\mathcal{W}})$ is bounded by D . Then the modified logarithmic Sobolev inequality $\text{MLS}(\lambda)$ holds with constant $\lambda = \frac{c}{D^2}$ for some universal constant c .*

For convenience, we shall reformulate the modified logarithmic Sobolev inequality so that it applies to arbitrary non-negative functions instead of probability densities. To this end, given a measure ν and a function $g \in \mathbb{R}_+^{\mathcal{X}}$ we define

$$\text{Ent}_{\nu}(g) = \nu[g \log g] - \nu[g] \log \nu[g].$$

It is then immediate to check that the inequality $\text{MLS}(\lambda)$ defined in (2.5) is equivalent to

$$\text{Ent}_{\pi}(f^2) \leq \frac{1}{2\lambda} \pi [\Gamma(f^2, \log f^2)] \quad \forall f \in \mathbb{R}^{\mathcal{X}}.$$

The proof of Theorem 6.1 will again consist in first obtaining a weak version of the MLS via the \mathcal{HWI} inequality, and then tightening it. In the continuous setting, the corresponding result (and actually a much stronger one, as we shall discuss in the next section) was proven employing such a strategy in [15]. That work strongly relies on a self-tightening property of the logarithmic Sobolev inequality, which states that if a non-tight LSI of the form

$$\text{Ent}_{\pi}(f^2) \leq cI(f) + \alpha$$

holds and if α is small enough, then a tight LSI holds. It is not clear whether such a strong self-tightening property holds for the discrete modified logarithmic Sobolev inequality. To

bypass this issue, we have to rely on more involved arguments, inspired by a work of Barthe and Kolesnikov [4].

We shall need the following two lemmas.

Lemma 6.2. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that the diameter of (\mathcal{X}, d_W) is bounded by D . Then we have for any $\delta > 0$ and $f \in \mathbb{R}^{\mathcal{X}}$:*

$$\text{Ent}_\pi(f^2) \leq \delta D^2 \pi [\Gamma(f^2, \log f^2)] + \frac{1}{4\delta} \pi [f^2 \mathbf{1}_{\{f^2 > \pi[f^2]\}}] . \quad (6.1)$$

Proof. Since (6.10) is homogeneous in f^2 we can assume without restriction that $\pi[f^2] = 1$. From the $\mathcal{HWI}(0)$ inequality and Young's inequality we infer

$$\pi[f^2 \log f^2] \leq \delta D^2 \pi [\Gamma(f^2, \log f^2)] + \frac{1}{4\delta D^2} \mathcal{W}(f^2, 1)^2 . \quad (6.2)$$

We know that $\mathcal{W} \leq W_{2, d_W}$. It is not hard to construct a transport from $\mu = f^2 \pi$ to π that moves mass away only from points x where $\mu(x) > \pi(x)$, i.e. $f^2(x) > 1$. Hence we have that $W_{2, d_W}(\mu, \pi)^2 \leq D_{\mathcal{W}}^2 \pi [f^2 \mathbf{1}_{\{f^2 > 1\}}]$, and the result immediately follows. \square

Lemma 6.3 ([4, Lem. 2.5]). *For any $A > 1$ there exists $\gamma > 0$ such that for any $f \in \mathbb{R}^{\mathcal{X}}$ with $\pi[f^2] = 1$, we have*

$$\pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \leq \left(\frac{A}{A-1} \right)^2 \text{Var}_\pi(f) , \quad (6.3)$$

$$\pi [f^2 \log f^2] \leq \gamma \text{Var}_\pi(f) + \pi [f^2 \log f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] . \quad (6.4)$$

We can now give the proof of our third main result Theorem 1.3.

Proof of Theorem 6.1. Fix $A > 1$ and $f \in \mathbb{R}^{\mathcal{X}}$ with $\pi[f^2] = 1$. Set $f_A(x) := \max(f(x), A)$, and define the probability measure $\mu_A = f_A^2 / Z_A \pi$, where $Z_A := \pi[f_A^2]$. Note that $A^2 \leq Z_A \leq 1 + A^2$. From (6.4) in Lemma 6.3 we have

$$\text{Ent}_\pi(f^2) \leq \gamma \text{Var}_\pi(f) + \pi [f^2 \log f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] . \quad (6.5)$$

The first term can be estimated via the Poincaré inequality as

$$\gamma \text{Var}_\pi(f) \leq \gamma \frac{c}{4D^2} \pi [\Gamma(f^2, \log f^2)] \quad (6.6)$$

using Theorem 5.7 and Lemma 5.4. For the second term, we have

$$\begin{aligned} \pi [f^2 \log f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] &= \pi [f_A^2 \log f_A^2] - A^2 \log A^2 \pi[\{f < A\}] \\ &= Z_A \text{Ent}_\pi(\mu_A) + Z_A \log Z_A - A^2 \log A^2 \pi[\{f < A\}] . \end{aligned} \quad (6.7)$$

The entropy term in (6.7) can be handled using Lemma 6.2 and the fact that we have $\Gamma(f_A^2, \log f_A^2) \leq \Gamma(f^2, \log f^2)$.

$$\begin{aligned} \text{Ent}_\pi(\mu_A) &\leq \frac{\delta D^2}{Z_A} \pi [\Gamma(f^2, \log f^2)] + \frac{1}{Z_A} \frac{1}{4\delta} \pi [f^2 \mathbf{1}_{\{f^2 \geq Z_A\}}] \\ &\leq \frac{\delta D^2}{A^2} \pi [\Gamma(f^2, \log f^2)] + \frac{1}{4\delta A^2} \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] . \end{aligned}$$

Using further (6.3) and again the Poincaré inequality via Theorem 5.7 and Lemma 5.4 we arrive at

$$\begin{aligned} \text{Ent}_\pi(\mu_A) &\leq \frac{\delta D^2}{A^2} \pi [\Gamma(f^2, \log f^2)] + \frac{1}{4\delta(A-1)^2} \text{Var}_\pi(f) \\ &\leq \left(\frac{\delta D^2}{A^2} + \frac{1}{4\delta(A-1)^2} \frac{c}{4D^2} \right) \pi [\Gamma(f^2, \log f^2)] . \end{aligned} \quad (6.8)$$

So all that is left is to bound is the term $Z_A \log Z_A - A^2 \log A^2 \pi[\{f < A\}]$. We have

$$\begin{aligned} &Z_A \log Z_A - A^2 \log A^2 \pi[\{f < A\}] \\ &= \left(A^2 \pi[\{f < A\}] + \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \right) \log \left(A^2 \pi[\{f < A\}] + \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \right) \\ &\quad - A^2 \log A^2 \pi[\{f < A\}] \\ &= A^2 \pi[\{f < A\}] \log \left(\pi[\{f < A\}] + \frac{\pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}]}{A^2} \right) \\ &\quad + \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \times \log \left(A^2 \pi[\{f < A\}] + \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \right) \\ &\leq A^2 \log \left(1 + \frac{\pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}]}{A^2} \right) + \log(1 + A^2) \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] \\ &\leq (1 + \log(1 + A^2)) \pi [f^2 \mathbf{1}_{\{f^2 \geq A^2\}}] . \end{aligned}$$

We can then once more use (6.3) from Lemma 6.3 to bound this by the variance, and then the Poincaré inequality to arrive at

$$\begin{aligned} &Z_A \log Z_A - A^2 \log A^2 \pi[\{f < A\}] \\ &\leq (1 + \log(1 + A^2)) \left(\frac{A}{A-1} \right)^2 \frac{c}{4D^2} \pi [\Gamma(f^2, \log f^2)] . \end{aligned} \quad (6.9)$$

Combining (6.5) with (6.6), (6.7), (6.8) and (6.9) finishes the proof. \square

Remark 6.4. By the same method one can obtain a modified logarithmic Sobolev inequality under the assumption that $\text{Ric}(\mathcal{X}, Q, \pi) \geq -\kappa$ and for $\kappa > 0$ that the diameter is bounded by D , provided κ is sufficiently small compared to D . The only modification is an extra term $\kappa/2\mathcal{W}(f^2, 1)^2$ appearing in the application of the $\mathcal{HWI}(\kappa)$ inequality in (6.2). A similar remark applies to the next result.

As for the Poincaré inequality, we can replace the diameter bound by a finite square-exponential moment.

Theorem 6.5. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that there exists a constant $\alpha > 0$ such that*

$$D_\alpha = \pi \left[e^{\alpha d_{\mathcal{W}}(\cdot, x_0)^2} \right] < \infty$$

for some $x_0 \in \mathcal{X}$. Then the modified logarithmic Sobolev inequality $\text{MLSI}(\lambda)$ holds with some constant λ which depends on α and on the value D_α of the integral.

The proof proceeds in exactly the same way the proof of Theorem 6.1 except that we need the following replacement for Lemma 6.2.

Lemma 6.6. *Under the assumptions of Theorem 6.5 there exists a constant $C = C(\alpha, D_\alpha)$ depending only on α and D_α such that for any $\delta > 0$ and $f \in \mathbb{R}^{\mathcal{X}}$ we have:*

$$\left(1 - \frac{\delta}{\alpha}\right) \text{Ent}_\pi(f^2) \leq \frac{1}{4\delta} \pi [\Gamma(f^2, \log f^2)] + \delta C \pi [f^2 \mathbf{1}_{\{f > \pi[f^2]\}}]. \quad (6.10)$$

Proof. Without restriction we can assume that $\pi[f^2] = 1$. Arguing as in the proof of Lemma 6.2 we obtain the crude bound

$$\mathcal{W}(f^2, 1)^2 \leq \sum_{x,y} \delta_{\mathcal{W}}(x,y)^2 f^2(x) \mathbf{1}_{\{f > 1\}}(x) \pi(x) \pi(y).$$

From the $\mathcal{HWI}(0)$ inequality we thus infer that

$$\text{Ent}_\pi(f^2) \leq \frac{1}{4\delta} \pi [\Gamma(f^2, \log f^2)] + \delta \sum_{x,y} d_{\mathcal{W}}(x,y)^2 f^2(x) \mathbf{1}_{\{f > 1\}}(x) \pi(x) \pi(y).$$

From the triangle inequality we have

$$\sum d_{\mathcal{W}}(x,y)^2 f^2(x) \mathbf{1}_{\{f > 1\}} \pi(x) \pi(y) \leq 2\pi [d_{\mathcal{W}}(x_0, \cdot)^2 f^2 \mathbf{1}_{\{f > 1\}}] + 2\pi [f^2 \mathbf{1}_{\{f > 1\}}] \pi [d(\cdot, x_0)^2].$$

The bound on the exponential moment immediately leads to a bound of the form $C\pi [f^2 \mathbf{1}_{\{f > 1\}}]$ for the second term on the right-hand side, with C only depending on α and D_α . We thus consider the first term. From the Young-type inequality $ab \leq a \log a + e^b$ for $a \geq 0$ and $b \in \mathbb{R}$ we deduce, setting $Z = \pi [f^2 \mathbf{1}_{\{f > 1\}}]$, that

$$\begin{aligned} \pi [d_{\mathcal{W}}(x_0, \cdot)^2 f^2 \mathbf{1}_{\{f > 1\}}] &= \frac{Z}{\alpha} \pi \left[\alpha d_{\mathcal{W}}(x_0, \cdot)^2 \frac{f^2}{Z} \mathbf{1}_{\{f > 1\}} \right] \\ &\leq \frac{Z}{\alpha} \text{Ent}_{\pi \mathbf{1}_{\{f > 1\}}} \left(\frac{f^2 \mathbf{1}_{\{f > 1\}}}{Z} \right) + \frac{Z}{\alpha} \pi \left[e^{\alpha d_{\mathcal{W}}(\cdot, x_0)^2} \mathbf{1}_{\{f > 1\}} \right] \\ &\leq \frac{1}{\alpha} \text{Ent}_{\pi \mathbf{1}_{\{f > 1\}}} (f^2 \mathbf{1}_{\{f > 1\}}) + \frac{D_\alpha}{\alpha} Z. \end{aligned}$$

Hence the proof is finished once we note that $\text{Ent}_{\pi \mathbf{1}_{\{f > 1\}}} (f^2 \mathbf{1}_{\{f > 1\}}) \leq \text{Ent}_\pi(f^2)$. This is a consequence of the duality formula

$$\text{Ent}_\nu(g) = \sup_h \left[\nu[hg] - \log \nu [e^h] + \log \nu[\mathcal{X}] \right]$$

for any non-negative function g with $\nu[g] = 1$. \square

6.2. Total variation mixing time for Markov chains with non-negative curvature.

Both the Poincaré inequality and the logarithmic Sobolev inequality yield bounds on the rate of convergence to equilibrium for the Markov chain, respectively in the $L^2(\pi)$ norm and in relative entropy, see Section 2.3 and in particular (2.7). Another relevant way of measuring closeness to equilibrium, often used in practice, is the total variation norm. In particular, there is a lot of interest in obtaining bounds on the total variation mixing time, defined as follows.

Definition 6.7. *The total variation mixing time is defined for $\varepsilon > 0$ as*

$$\tau_{mix}(\varepsilon) := \sup \left\{ t > 0; \|P_t^* \delta_x - \pi\|_{\text{TV}} < \varepsilon \forall x \in \mathcal{X} \right\}.$$

Here P_t^* denotes the dual Markov semigroup acting on probability measures. We refer to the book [24] for an introduction and overview of the many works on mixing times.

Since the Pinsker inequality states that $2\|\nu - \pi\|_{\text{TV}}^2 \leq \text{Ent}_\pi(\nu)$, the modified logarithmic Sobolev inequality is a useful tool to obtain upper bounds on the mixing time. However, since the estimate must hold uniformly in the initial data, it is not enough. In the continuous setting, since the relative entropy functional is unbounded, an extra argument is always needed. In the finite setting, since we always have $\text{Ent}_\pi(\delta_x) = -\log \pi(x)$, the inequality $\text{MLSI}(\rho)$ implies the estimate

$$\tau_{\text{mix}}(\varepsilon) \leq \frac{1}{2\rho} [-\log(2\varepsilon^2) + \log \log \pi_*^{-1}]$$

where $\pi_* = \inf\{\pi(x) : x \in \mathcal{X}\}$.

One of the flaws of this bound is that π_* is quite small when the space has many points. In particular, it does not behave well when studying continuous limits. In the context of Markov chains with non-negative curvature, we can give a general estimate on the mixing time that does not involve π_* .

Theorem 6.8. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that the diameter of (\mathcal{X}, d_W) is bounded by D . If $\text{MLSI}(\rho)$ holds then we have*

$$\tau_{\text{mix}}(\varepsilon) \leq \frac{D^2}{4} + \frac{\log \varepsilon}{\rho}.$$

In particular, we obtain that for a universal constant c

$$\tau_{\text{mix}}(\varepsilon) \leq D^2(1/4 + c \log \varepsilon).$$

Proof. The second bound immediately follows from the first using that Theorem 6.1 yields the validity of $\text{MLSI}(cD^{-2})$ for a suitable constant c . To show the first bound we first note the estimate

$$\mathcal{H}(P_t f) \leq \frac{\mathcal{W}(f, 1)^2}{4t}$$

which is an immediate consequence of the Evolution Variational Inequality established in [13, Thm. 4.5]. Hence $\mathcal{H}(P_t f) \leq 2$ for all $t \geq D^2/4$ and all $f \in \mathcal{P}(\mathcal{X})$. The result then follows using the exponential convergence $\mathcal{H}(P_t f) \leq e^{-2\rho t} \mathcal{H}(f)$ implied by $\text{MLSI}(\rho)$ and Pinsker's inequality. \square

6.3. A conjecture. If we apply the abstract results to a simple random walk on the discrete torus $(\mathbb{Z}/L\mathbb{Z})^d$, we get a spectral gap and a modified LSI with constant $O(d^2 L^2)$. However, the optimal constant behaves like dL^2 , so our estimate is off by a dimensional factor. This was to be expected: if we consider a product space, both the Poincaré inequality and the modified LSI tensorize (up to a scaling of the time), while the squared diameter grows linearly with the dimension. This shows that diameter estimates should not allow one to capture the sharp behavior of functional inequalities for dynamics in high dimension.

To have any hope of obtaining good estimates in high dimension, we should therefore rely on a different kind of assumption. In a series of contributions [32, 31, 33], Milman showed that for Riemannian manifolds, we can effectively use assumptions on the concentration profile to derive functional inequalities for manifolds of non-negative Ricci curvature. This improves on the diameter assumption, since concentration estimates may be dimension-free (although not always). Moreover, it is a strictly weaker assumption, since when the diameter is bounded

we automatically have Gaussian and exponential concentration, with constants controlled by the diameter.

More precisely, what Milman showed is the following:

- If curvature is bounded from below by $-\kappa$ for some $\kappa > 0$, then a strong enough Gaussian concentration implies a Gaussian isoperimetric inequality, and hence both a logarithmic Sobolev inequality and a Poincaré inequality. The constant only depends on κ and on the constant appearing in the Gaussian concentration property.
- If curvature is non-negative, exponential concentration implies a Cheeger isoperimetric inequality, and hence a Poincaré inequality. The constant only depends on the constant appearing in the exponential concentration property.

Since Gaussian concentration is equivalent to finiteness of a square-exponential moment, qualitatively the first result at first glance may not appear so different from Wang’s theorem. The important difference (in addition to the isoperimetric inequality) is that the constant does not depend anymore on the value of the square-exponential moment. This makes a significant difference in high dimensional situations, where the square exponential moment depends on the dimension, but the Gaussian concentration constant often does not.

Milman’s work relies on tools of Riemannian geometry (concavity of isoperimetric profiles and the Heinz-Karcher theorem), so it does not seem like his arguments can be adapted to the discrete case. An alternative proof by Ledoux [22] also relied on concavity of isoperimetric profiles.

As we have seen in the previous sections, the alternative approach of Gozlan, Roberto and Samson [15], based on functional inequalities, is more easily adapted to the discrete setting. While unlike Milman, they do not recover the Gaussian isoperimetric inequality, they nonetheless show that when curvature is bounded from below, a strong enough Gaussian concentration implies a logarithmic Sobolev inequality. However, we have not been able to adapt a key step in their approach, which is that Gaussian concentration implies a weak transport-entropy inequality. In the discrete setting, the analogous inequality we would need would be

$$\mathcal{W}(\mu, \pi)^2 \leq c_1 \text{Ent}_\pi(\mu) + c_2.$$

To establish it, we would need to better understand the relationship between bounds on \mathcal{W} and concentration. An important difference between the discrete and the continuous situation is that lack of a dual Kantorovich formulation for the distance \mathcal{W} .

Nonetheless, we state as conjectures the discrete analogues of the results of [32, 31, 33, 22, 15]:

Conjecture 6.9. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ and that the invariant measure π satisfies a concentration property w.r.t. the distance $d_{\mathcal{W}}$ with profile $\alpha(r) = Me^{-\rho r}$. Then there exists a constant $C(M)$ such that $PI(C(M)\rho^{-2})$ holds.*

Conjecture 6.10. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq -\kappa$ for some $\kappa > 0$, and that a concentration property with respect to the distance $d_{\mathcal{W}}$ holds with profile $\alpha(r) = Me^{-\rho r^2}$. Then there exists a constant $\tau(M)$ and $\lambda(\kappa, M, \rho)$ such that if $\frac{\kappa}{\rho} < \tau(M)$ then $MLSI(\lambda(\kappa, M, \rho))$ holds. If moreover $\text{Ric}(\mathcal{X}, Q, \pi) \geq 0$ then $MLSI(cM\rho)$ holds for some universal constant c .*

In the Riemannian setting, these results hold with no dependence on M , but for non-smooth geodesic spaces the proof of [15] has an extra dependence on M of the form we use in the statements of these conjectures.

As in the continuous setting, Theorem 6.5 already tells us that under these assumptions a mLSI holds. The open problem in Conjecture 6.10 is the value of the constant.

As we shall see in the next section, if these conjectures are indeed true, then we could use curvature to better understand the behavior of some interacting particle systems with degenerate rates.

7. APPLICATION TO THE ZERO-RANGE PROCESS WITH CONSTANT RATES

In this section, we shall discuss functional inequalities for a system of K interacting particles on the complete graph with L sites, namely the zero range process.

The state space is $\mathcal{X}_{K,L} = \{\eta \in \mathbb{N}^L : \sum_{i=1}^L \eta_i = K\}$. The dynamics we are interested in is defined as follows. With rate 1, we select a site i uniformly at random. If $\eta_i = 0$ (no particles on site i), we do nothing. Else we choose a second site j uniformly at random, and move a single particle from i to j . We shall denote by $\eta^{i,j}$ the new configuration obtained after such a move. More precisely, the transition rates of the corresponding continuous time Markov chain for $\eta \neq \eta'$ are thus given by

$$Q_{K,L}(\eta, \eta') = \begin{cases} \frac{1}{L} & \eta' = \eta^{i,j} \text{ for some } i, j, \\ 0 & \text{else.} \end{cases}$$

The invariant measure is the uniform measure on $\mathcal{X}_{K,L}$ denoted by $\pi_{K,L}$.

This model constitutes a degenerate version of the classical zero range process, where particles on site i jump at rate $f(\eta_i)$ for some rate function f . For example, independent particles correspond to the case $f(n) = \lambda n$ for a constant λ . Our situation corresponds to the case where the jump rate f is constant.

In [14], entropic Ricci curvature lower bounds for the zero range process were established, in the situation where the jump rate is strictly increasing: If the rate satisfies $0 < c \leq f(n+1) - f(n) \leq c + \delta$ for all n and some constants c, δ and δ is small enough compared to c , then curvature is bounded from below by a strictly positive constant. It is easy to check that the proof can be straightforwardly adapted to show that the zero range process with constant rates has non-negative curvature, i.e. $\text{Ric}(\mathcal{X}_{K,L}, Q_{K,L}, \pi_{K,L}) \geq 0$. We can thus use the abstract results of the previous section together with the following diameter estimate to obtain the mLSI for the degenerate zero range process.

Lemma 7.1. *There exists a constant $c > 0$ such that for any L, K and the diameter of $(\mathcal{X}_{K,L}, d_W)$ is bounded by $cK\sqrt{L \log L}$.*

Theorem 7.2. *For the zero-range process with constant rate 1 on the complete graph with L sites, K particles the modified logarithmic Sobolev inequality $MLSI\left(\frac{c}{K^2 L \log L}\right)$ for a universal constant c .*

We do not believe this constant to be optimal. Morris [34] showed that the spectral gap is of order L/K^2 , so for a fixed density of particles K/L our estimate is off by a factor $K^2 \log L$. For the mLSI, no better result seems to be known, but we believe that it should behave like $1/L$ at fixed density, by analogy with the situation for gamma distributions studied in [5]. As mentioned in Section 6.3, one source of error is that we expect that when curvature is non-negative the mLSI constant is controlled by the Gaussian concentration constant, and that in high dimension the diameter is much larger than the Gaussian concentration constant. Since $\pi_{K,L}$ is the uniform measure on all admissible configurations, the distribution of the number of particles on a given site is a binomial distribution, with parameters K and $1/L$, so that it satisfies an exponential concentration property with a constant that only depends on the particle density K/L (which matches well with the result of Morris). For fixed density

$\rho = K/L$ and large K and L , the binomial law approximates a Poisson law with parameter ρ , so that the invariant measure looks like a product of Poisson measures, with an added constraint of fixed total sum (which is K). The results of [5] then suggest that we should expect the Gaussian concentration constant to behave like $1/L$. With the way we defined the rates of the Markov chain (that differs with the rate used in [34] by a factor $1/L$, this leads us to expect the mLSI constant to behave like $1/L^2$ at fixed density K/L , and suggests that our result is off by a factor $1/(K \log L)$ (since at fixed density, the asymptotic behavior of K and L is the same).

Proof of Lemma 7.1. We need to show that

$$\mathcal{W}(\delta_\eta, \delta_{\tilde{\eta}}) \leq cK\sqrt{L \log L}$$

for any $\eta, \tilde{\eta} \in \mathcal{X}_{K,L}$ and a suitable constant c . For each pair $\eta, \tilde{\eta}$ we can find a sequence $\eta = \eta_1, \dots, \eta_n = \tilde{\eta}$ of length at most K such that η_i and η_{i+1} differ only by the position of a single particle. From the triangle inequality for \mathcal{W} , it is enough to show that $\mathcal{W}(\delta_{\eta_i}, \delta_{\eta_{i+1}}) \leq c\sqrt{L \log L}$. But when looking at the movement of a single particle, the situation is the same as for a random walk on the complete graph with rate $1/L$. More precisely, we claim that

$$\mathcal{W}(\delta_{\eta_i}, \delta_{\eta_{i+1}}) \leq \mathcal{W}(\delta_x, \delta_y),$$

where the right-hand side is the transport distance between Dirac masses in point x, y on the complete graph with L sites and rates $1/L$. To see this, we can lift an optimal solution to the continuity equation (ρ_t, ψ_t) on the complete graph connecting δ_x, δ_y to a solution to the continuity equation $(\bar{\rho}_t, \bar{\psi}_t)$ on the state space of the zero range process connecting $\delta_{\eta_i}, \delta_{\eta_{i+1}}$ (see [29, Lem. 3.14], where such a lifting is carried out in detail for a comparison to the two-point space). So it is enough to show that the distance on the complete graph induced by the simple random walk with unit rate has diameter bounded by $c\sqrt{\log L}$ (the change in speed changes the diameter by a factor \sqrt{L}). This diameter bound will follow from a general diameter bound in Proposition 7.3 below. For simple random walk on the complete graph, the minimal mass of a point is given by $\pi_* = 1/L$ and curvature is bounded from below by $1/2$. \square

We conclude with a general estimate on the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ that can be seen as a discrete analogue to the Bonnet-Myers theorem in Riemannian geometry.

Proposition 7.3. *Assume that $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ for $\kappa > 0$. Then for any $x, y \in \mathcal{X}$ we have*

$$d_{\mathcal{W}}(x, y) \leq 2\sqrt{\frac{-\log \pi(x) - \log \pi(y)}{\kappa}}.$$

Thus, the diameter of $(\mathcal{X}, d_{\mathcal{W}})$ is bounded by $2\sqrt{\frac{-2\log \pi_}{\kappa}}$, where $\pi_* := \inf\{\pi(x) : x \in \mathcal{X}\}$.*

The dependence on π_* might seem undesirable, but since we used no upper bound on the dimension, we cannot expect the diameter bound to depend only on κ . In the case of the discrete hyper-cube of dimension n , we have $-\log \pi_* = n \log 2$, which is the correct dependence on the dimension.

Proof of Proposition 7.3. From the convexity of the entropy (2.3), we have

$$0 \leq \mathcal{H}(\rho_{\frac{x,y}{2}}) \leq \frac{1}{2}\mathcal{H}(\delta_x) + \frac{1}{2}\mathcal{H}(\delta_y) - \frac{\kappa}{8}d_{\mathcal{W}}(x, y)^2,$$

where $(\rho_t^{x,y})_{t \in [0,1]}$ is the \mathcal{W} -geodesic connecting δ_x to δ_y . We then use that $\mathcal{H}(\delta_x) = -\log \pi(x)$ to conclude. \square

REFERENCES

- [1] D. Bakry and M. Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.
- [2] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2014.
- [3] D. Bakry and Z. Qian. Some new results on eigenvectors via dimension, diameter, and Ricci curvature. *Adv. Math.*, 155(1):98–153, 2000.
- [4] F. Barthe and A.V. Kolesnikov. Mass transport and variants of the logarithmic Sobolev inequality. *J. Geom. Anal.*, 18(4):921–979, 2008.
- [5] F. Barthe and P. Wolff. Remarks on non-interacting conservative spin systems: the case of gamma distributions. *Stochastic Process. Appl.*, 119(8):2711–2723, 2009.
- [6] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.*, 163(1):1–28, 1999.
- [7] A.-I. Bonciocat and K.-Th. Sturm. Mass transportation and rough curvature bounds for discrete spaces. *J. Funct. Anal.*, 256(9):2944–2966, 2009.
- [8] P. Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.
- [9] F. Cavaletti and A. Mondino. Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *arXiv:1502.06465*, 2015.
- [10] F. Cavaletti and A. Mondino. Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. *arXiv:1505.02061*, 2015.
- [11] F. Chung, Y. Lin, and S.-T. Yau. Harnack inequalities for graphs with non-negative Ricci curvature. *J. Math. Anal. Appl.*, 415(1):25–32, 2014.
- [12] M. Erbar, M. Fathi, J. Maas, A. Schlichting, and P. Tetali. *in preparation*.
- [13] M. Erbar and J. Maas. Ricci curvature of finite Markov chains via convexity of the entropy. *Arch. Ration. Mech. Anal.*, 206(3):997–1038, 2012.
- [14] M. Fathi and J. Maas. Entropic Ricci curvature bounds for discrete interacting systems. *Ann. Appl. Probab.*, 26(3):1774–1806, 2016.
- [15] N. Gozlan, C. Roberto, and P.-M. Samson. From concentration to logarithmic Sobolev and Poincaré inequalities. *J. Funct. Anal.*, 260(5):1491–1522, 2011.
- [16] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Displacement convexity of entropy and related inequalities on graphs. *Probability Theory and Related Fields*, 160:47–94, 2014.
- [17] F. Hang and X. Wang. A remark on Zhong-Yang’s eigenvalue estimate. *Int. Math. Res. Not. IMRN*, (18):Art. ID rnm064, 9, 2007.
- [18] B. Klartag, G. Kozma, P. Ralli, and P. Tetali. Discrete curvature and abelian groups. *Canad. J. Math.*, 68(3):655–674, 2016.
- [19] M. Ledoux. A simple analytic proof of an inequality by P. Buser. *Proc. Amer. Math. Soc.*, 121(3):951–959, 1994.
- [20] M. Ledoux. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.
- [21] M. Ledoux. Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in differential geometry. Vol. IX*, Surv. Differ. Geom., IX, pages 219–240. Int. Press, Somerville, MA, 2004.
- [22] M. Ledoux. From concentration to isoperimetry: semigroup proofs. In *Concentration, functional inequalities and isoperimetry*, volume 545 of *Contemp. Math.*, pages 155–166. Amer. Math. Soc., Providence, RI, 2011.
- [23] C. Léonard. On the convexity of the entropy along entropic interpolations. *preprint*, 2013.
- [24] D.A. Levin, Y. Peres, and E.L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.
- [25] P. Li. A lower bound for the first eigenvalue of the Laplacian on a compact manifold. *Indiana Univ. Math. J.*, 28(6):1013–1019, 1979.

- [26] P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.
- [27] Y. Lin and S.-T. Yau. Ricci curvature and eigenvalue estimate on locally finite graphs. *Math. Res. Lett.*, 17(2):343–356, 2010.
- [28] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. Math. (2)*, 169(3):903–991, 2009.
- [29] J. Maas. Gradient flows of the entropy for finite Markov chains. *J. Funct. Anal.*, 261(8):2250–2292, 2011.
- [30] A. Mielke. Geodesic convexity of the relative entropy in reversible Markov chains. *Calc. Var. Partial Differential Equations*, 48(1-2):1–31, 2013.
- [31] E. Milman. On the role of convexity in functional and isoperimetric inequalities. *Proc. Lond. Math. Soc. (3)*, 99(1):32–66, 2009.
- [32] E. Milman. On the role of convexity in isoperimetry, spectral gap and concentration. *Invent. Math.*, 177(1):1–43, 2009.
- [33] E. Milman. Isoperimetric and concentration inequalities: equivalence under curvature lower bound. *Duke Math. J.*, 154(2):207–239, 2010.
- [34] B. Morris. Spectral gap for the zero range process with constant rate. *Ann. Probab.*, 34(5):1645–1664, 2006.
- [35] Y. Ollivier. Ricci curvature of Markov chains on metric spaces. *J. Funct. Anal.*, 256(3):810–864, 2009.
- [36] L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292 (1960), 1960.
- [37] M. Schmuckenschläger. Curvature of nonlocal markov generators. In *Convex geometric analysis (Berkeley, CA, 1996)*, volume 34 of *Math. Sci. Res. Inst. Publ.*, pages 189–197. Cambridge Univ. Press, 1999.
- [38] K.-Th. Sturm. On the geometry of metric measure spaces. I and II. *Acta Math.*, 196(1):65–177, 2006.
- [39] C. Villani. *Optimal transport, Old and new*, volume 338 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2009.
- [40] F.-Y. Wang. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Probab. Theory Related Fields*, 109(3):417–424, 1997.
- [41] J.Q. Zhong and H.C. Yang. On the estimate of the first eigenvalue of a compact Riemannian manifold. *Sci. Sinica Ser. A*, 27(12):1265–1273, 1984.

UNIVERSITY OF BONN, INSTITUTE FOR APPLIED MATHEMATICS, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

E-mail address: `erbar@iam.uni-bonn.de`

UNIVERSITY OF CALIFORNIA, BERKELEY, DEPARTMENT OF MATHEMATICS, EVANS HALL, BERKELEY, USA

E-mail address: `maxf@berkeley.edu`