A Benamou–Brenier formulation of martingale optimal transport

Martin Huesmann* and Dario Trevisan†

July 5, 2017

Abstract

We identify a Benamou–Brenier formulation for the continuous-time martingale optimal transport problem obtained as a weak length relaxation of its discrete-time counterpart. Using the general correspondence between classical martingale problem and Fokker-Planck equations, we obtain an equivalent PDE formulation for which basic properties such as existence, duality and geodesic equations can be analytically studied, yielding corresponding results for the stochastic formulation. Sufficient conditions for finiteness of the cost are also given, and, in the one dimensional case, a link between geodesics and porous medium equations is partially investigated.

Keywords: Martingale Optimal Transport, Benamou-Brenier formula, Fokker-Planck equations.

1 Introduction

Given two probability measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ in convex order and a cost function $\mathtt{c} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, the martingale optimal transport problem is the variational problem

$$\inf \left\{ \mathbb{E}\left[\mathtt{c}(X_0, X_1)\right] : \ (X_0, X_1) \text{ is a martingale, } (X_0)_\sharp \mathbb{P} = \mu, \, (X_1)_\sharp \mathbb{P} = \nu \right\}. \tag{1.1}$$

This variant of the classical Monge-Kantovorich problem [44, 1], with the extra martingale constraint, originates from intriguing questions of worst case bounds for derivate prices in model-independent finance, see e.g. [5] for discrete-time and [25] for continuous-time.

In classical optimal transport there is a one-to-one correspondence between discrete time couplings and continuous-time couplings, in the sense that any continuous-time solution induces a discrete time solution and any discrete-time solution can be optimally interpolated to a unique continuous-time solution, e.g. by using McCann's displacement interpolation [30]. For martingale optimal transport the link between the discrete and the continuous-time problem is less clear and only [28] provides, in dimension one, a continuous-time interpretation of the discrete-time transport problem using Skorokhod embedding techniques developed in [4]. However, this approach does not lead to time consistent couplings, i.e., such that the induced couplings between two arbitrary intermediate times will be optimal between their marginals.

The aim of this article is to narrow this gap by focusing on a certain class of continuous-time martingale transport problems that naturally appear via a weak length relaxation of the discrete-time problems.

*huesmann@iam.uni-bonn.de. Universität Bonn, Germany.
†dario.trevisan@unipi.it. Università di Pisa, Italy. Member of GNAMPA group (INdAM).

The main idea to identify the correct class of continuous-time problems and to link it with the discrete problems is to transfer to the martingale context the interpretation of the Lagrangian action functional in the Benamou–Brenier formula [10] as a length functional on the L^2 Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. More precisely, given a curve $(\rho_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\mathbb{R}^d)$ represented via the superposition principle [42] as a random curve $(X_t)_{t\in[0,1]}$ and a partition $\pi = \{t_0 = 0 < \ldots < t_n = 1\}$ of the unit interval we consider the discrete energy of X associated with π

$$\sum_{i=1}^{n}(t_{i+1} - t_i) \cdot \mathbb{E}_\rho\left[\left(\frac{|X_{t_{i+1}} - X_{t_i}|}{t_{i+1} - t_i}\right)^2\right] . \tag{1.2}$$

As the mesh of π goes to zero, (1.2) converges for nice curves X precisely to the Lagrangian action functional (written in probabilistic terms)

$$\int_0^1 \mathbb{E}_\rho[(\dot{X}_t)^2] \; \mathrm{d}t$$

of the Benamou–Brenier transport formulation, where $\dot{X}_t$ denotes the time derivative of X.

Following this idea and taking into account the scaling properties of martingales implied by the Burkholder–Davis–Gundy inequalities we arrive at our first main result (for a rigorous statement we refer to Theorem 3.1):

Theorem 1.1. *Let* c *be smooth and of bounded growth. Assume that* X *is a nice martingale and let* Z *be a* d*-dimensional standard normal random variable independent of* X*. Then, the following limit holds (where* $\langle \dot{X} \rangle = \mathrm{d}\langle X \rangle / \mathrm{d}t$*):*

$$\lim_{\|\pi\|\to 0} \sum_{t_i\in\pi} \mathbb{E}\left[\mathsf{c}\left(\frac{X_{t_i} - X_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}\right)\right](t_i - t_{i-1}) = \int_0^1 \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X} \rangle_s}\, Z\right)\right] \mathrm{d}s\,. \tag{1.3}$$

Therefore, by interpreting the discrete martingale transport cost as a non-symmetric distance function, the quantity on the right hand side of (1.3) can be seen as the "length" of the martingale measured in terms of c. As a direct consequence, there is a natural Benamou–Brenier type formulation of a continuous-time martingale transport problem as the martingale of "minimal length" connecting two given measures μ and ν which are increasing in convex order (with a slightly changed cost function c, where we also "absorb" the independent standard normal Z)

$$\mathsf{c}_{BB}(\mu, \nu) := \inf\left\{\int_0^1 \mathbb{E}\left[\mathsf{c}\left(\langle \dot{X} \rangle\right)\right] \mathrm{d}s\right\} \tag{1.4}$$

where the infimum runs over all martingales connecting μ and ν whose quadratic variation process $\langle X \rangle$ is absolutely continuous w.r.t. Lebesgue measure. Notably, this class of cost functions (as well as the bigger class of cost functions that additionally depend on time and space considered in the main part of this article) is precisely of the form considered in [41], but the control is restricted on the diffusion term, the drift being null. It could be of interest to provide an argument leading to the general costs considered in [41] as a relaxation of a semimartingale, perhaps first separating the martingale from the finite variation part via Doob–Meyer decomposition and by scaling differently the two parts.

In the second part of this article, we complement the results of [25, 41] by a new PDE perspective on this problem that on the one hand is closer to the original work of Benamou–Brenier [10] and its extension by Dolbeaut–Nazaret–Savaré [15] and on the other hand drastically reduces the complexity of the problem because we only have to deal with PDEs of the marginals and not with a stochastic process connecting these marginals.

By linking the optimization problem (1.4) to the classical martingale problem, we show that there is an equivalent formulation in terms of Fokker–Planck equations leading to the variational problem

$$\mathsf{c}_{FPE}(\mu,\nu) := \inf\left\{\int_0^1 \int_{\mathbb{R}^d} \mathsf{c}(2a_t(x))\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t \colon \partial_t\varrho = (a\colon\nabla^2)^*\varrho,\ \varrho_0 = \mu,\ \varrho_1 = \nu\right\}. \quad (1.5)$$

A direct consequence of this formulation as a Lagrangian action minimization problem is that any optimizer will have the time consistency property that all the continuous-time solutions constructed in [28] were lacking. In other words, any optimizer induces a natural interpolation or "geodesic" between its marginals.

Since the cost function c depends on the diffusion coefficient and not on the quadratic variation of the martingale the Burkholder–Davis–Gundy inequalities do not imply any bounds on the cost $\mathsf{c}_{FPE}(\mu,\nu) = \mathsf{c}_{BB}(\mu,\nu)$ in terms of moments if the growth of c can be controlled. However, assuming that $\mathsf{c}(a) \lesssim |a|^p$ using Skorokhod embedding techniques we show that $\mathsf{c}_{BB}(\mu,\nu) < \infty$ if ν has finite $2p+\varepsilon$-moments for some $\varepsilon > 0$.

Just as in classical transport, the decisive step to understand the optimizer of the martingale Benamou–Brenier problem (1.5) is the dual formulation of the variational problem, see Theorem 6.3. As a byproduct, we also prove existence of primal optimizers. An interesting consequence of this duality result is a "geodesic equation" (of Hamilton-Jacobi type) for the optimal potential function φ which allows us to recognize and construct optimizer. More precisely, given an optimal potential φ, i.e., a solution to the dual problem, assuming that all quantities are sufficiently smooth, we define the diffusion coefficient a via

$$a(t,x) = \nabla\mathsf{c}(\nabla^2\varphi(t,x))\,.$$

Then, solving the Fokker–Planck equation $\partial_t\varrho = (a : \nabla^2)^*\varrho$ with a given initial datum $\varrho_0 = \mu$, we obtain the solution to the martingale Benamou–Brenier problem between μ and ρ_1. In particular, if we find such a candidate curve connecting the measures μ and ν we have found an optimizer for the problem in (1.5). A particularly nice class of examples is given in dimension one by the cost functions $\mathsf{c}(a) = a^p$ for $p > 1$ (and $a \geq 0$). It then follows that the optimal diffusion coefficient has to solve the pressure equation corresponding to a degenerate porous medium equation (see Remark 7.3). Due to the rich literature on porous medium equations this allows us to construct various examples, see Theorem 7.2 and Example 7.4. Analogously, in the case $p < 1$, solutions should be related to the fast diffusion equation, providing us with another class of examples. In this article, however, we do not pursue this direction. We refer to the accompanying article [2] for an investigation of the case $p = 1/2$, which allows for an interesting probabilistic representation.

Finally, we observe that the diffusive structure of the optimizer to the continuous-time problems, together with our weak length relaxation result, is consistent with the natural interpretation of the discrete-time martingale transport problem as an infinitesimal version of the continuous-time problem. This may also explain why there cannot be a one-to-one correspondence between optimizers of the discrete-time and the continuous-time problems, since mass in the discrete-time problems is known to be split [6]. By contrast, in classical transport (at least on $\mathbb{R}^d$) one is not forced to split mass and one can follow the infinitesimal direction for one unit of time, yielding precisely the connection of the discrete and continuous-time optimizers.

Related literature. The one-dimensional discrete-time martingale optimal transport problem is by now well understood due to the seminal work of [6] for the geometric characterization of optimizers and [7] for a complete duality theory, see also the recent work [8]. This was recently extended to cover the discrete-time multi-marginal problem in [33]. In higher dimensions, a complete picture for the discrete-time problem is still missing. However, there is recent exciting progress in [26, 14, 35].

The continuous-time version of the martingale optimal transport problem has been studied in [41, 25, 17, 18] among others, where the main focus of the authors is to establish a duality result which can be interpreted as a robust super/subhedging result. The articles [41, 25] solve the problem by linking it to stochastic control theory, whereas [17, 18] use a careful discretization procedure of the space variables. Notably, these results imply numerical schemes to compute the value of the optimisation problem, e.g. [11, 41].

As Otto [36] observed, the Benamou–Brenier formula can be interpreted as a (formal) Riemannian metric on the space of probability measures with second moment. As such it is the basis for the famous Otto calculus with its striking applications in PDEs and numerics, e.g. [44, 1] for a detailed overview. Moreover, variants of the Benamou–Brenier formulation turned out to be a powerful tool for discrete probability, analysis and geometry [31, 32, 29, 22], quantum evolution [13], jump diffusions [20] and recently to a new approach to the Boltzmann equation [21].

Our duality result could be derived from the results of [41] which are established via stochastic control theory. However, we decided to use the PDE point of view to give a short and self-contained proof which we believe is another good example of the complexity reduction arising from such point of view.

The porous medium equation is a very well studied PDE and we refer to [43] for a comprehensive account. We also quote the recent preprint [3] which considers a degenerate class of porous medium equations in connection with stochastic optimal control problems.

Finally, we quote the accompanying article [2] for a nice probabilistic treatment of the special case $\mathsf{c}(a) = \mathrm{Trace}(\sqrt{a})$, in connection with stretched Brownian motion.

Outline. In Section 2 we introduce notation and review some technical results useful in the following. Section 3 is devoted to the proof of Theorem 1.1, connecting discrete and continuous problems. In Section 4 we introduce and study basic properties of the Benamou–Brenier formulation, in particular in connection with the PDE formulation, see Theorem 4.3. Sufficient conditions for finiteness of the transportation cost are studied in Section 5. In Section 6 the duality result is established, together with examples, while in Section 7 focuses in the one-dimensional case and connection with porous medium equations. Conclusions and open problems are stated in Section 8.

Acknowledgements. Both authors thank M. Beiglböck, J. Backhoff Veraguas and S. Källblad for interesting discussions on martingale optimal transportation. MH acknowledges support by the Deutsche Forschungsgemeinschaft through the CRC 1060 *The Mathematics of emergent effects* and the *Hausdorff Center for Mathematics*.

2 Notation and basic facts

For $d \geq 1$, let $\mathrm{Sim}^d \subseteq \mathbb{R}^{d\times d}$ denote the space of symmetric matrices $a = a^\tau$ and write $a \in \mathrm{Sim}^d_+$ or $a \geq 0$ if $a \in \mathrm{Sim}^d$ is non-negative definite. For $a, b \in \mathrm{Sim}^d$, we introduce the scalar product $a : b = \mathrm{Trace}(ab) \in \mathbb{R}$ and norm $|a| = \sqrt{a : a}$.

Let $C_b([0,1]\times\mathbb{R}^d)$, $C_b^{1,2}([0,1]\times\mathbb{R}^d)$ be respectively the spaces of continuous functions $\varphi(t,x) = \varphi_t(x)$ on $[0,1]\times\mathbb{R}^d$ and of functions differentiable once with respect to t (and write $\partial_t\varphi$) and twice with respect to x ($\nabla\varphi$, $\nabla^2\varphi$) in $(0,1)\times\mathbb{R}^d$, unifermly bounded together with their derivatives. For our purposes, sometimes we need functions with linear or quadratic growth, and this can be obtained via suitable approximation procedures, e.g. as in [42, Remark 2.3]. For $r \in [0,+\infty)$, let B_r, $\overline{B_r} \subseteq \mathbb{R}^d$ denote respectively the open and closed balls centred at 0 in $\mathbb{R}^d$ and write $C_b^{1,2}([0,1]\times\overline{B_r})$ for the space of functions differentiable once with respect to t and twice with respect to x in $(0,1)\times B_r$ with uniformly continuous derivatives, so that they extend to functions in $C_b^{1,2}([0,1]\times\mathbb{R}^d)$.

Given $E \subseteq \mathbb{R}^k$ Borel we write $\mathcal{P}_2(E)$ for the set of probability measures on E with finite second moment, endowed with narrow topology (i.e., in duality with bounded continuous functions) together with convergence of second moments). We write $\mathcal{M}(E;\mathrm{Sim}^d_+)$ for the space of measures μ on E with values in the vector space Sim^d such that $\mu(A)\in\mathrm{Sim}^d_+$ for every $A\in\mathcal{E}$, or, equivalently, such that the matrix σ in the polar decomposition of $\mu=|\mu|\,\sigma$ belongs $|\mu|$-a.e. to Sim^d_+.

We say that $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$ are in convex order if, for every convex $\varphi:\mathbb{R}^d\to\mathbb{R}$ one has $\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu\le\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\nu$.

Cost functionals. Given $\mathsf{c}:\mathrm{Sim}^d_+\to\mathbb{R}\cup\{+\infty\}$, $a\mapsto\mathsf{c}(a)$ (or equivalently $\mathsf{c}:\mathrm{Sim}^d\to\mathbb{R}\cup\{+\infty\}$ with $\mathsf{c}(a)=+\infty$ if $a\notin\mathrm{Sim}^d_+$), its Legendre transform $\mathsf{c}^*:\mathrm{Sim}^d\to\mathbb{R}\cup\{+\infty\}$ is defined for $b\in\mathrm{Sim}^d$ as

$$\mathsf{c}^*(b):=\sup_{a\in\mathrm{Sim}^d_+}\{a\!:\!b-\mathsf{c}(a)\}\,.$$

The map c^* is then convex and lower semicontinuous (l.s.c.). If c is strictly convex, then [38, Theorem 26.3] c^* is continuously differentiable with

$$\mathsf{c}^*(b)=(\nabla\mathsf{c}^*(b))\!:\!b-\mathsf{c}(\nabla\mathsf{c}^*(b))\quad\text{for } b\in\mathrm{Sim}^d. \tag{2.1}$$

In particular, $\nabla\mathsf{c}^*(b)\in\mathrm{Sim}^d_+$. For $p\in(1,+\infty)$, we say that $\mathsf{c}:\mathrm{Sim}_d^+\to\mathbb{R}$ is p-coercive if there exists $\lambda>0$ such that $\mathsf{c}(a)\ge\lambda|a|^p$, for $a\in\mathrm{Sim}_d^+$ and that it has p-growth if $\mathsf{c}(a)\le\lambda|a|^p$, for $a\in\mathrm{Sim}_d^+$. The Legendre transform of a p-coercive function has q-growth and that of a function with p-growth is q-coercive, with $q=p/(p-1)$. If c is strictly convex, from (2.1) we obtain that if c is p-coercive then $|\nabla\mathsf{c}^*(b)|$ has $(q-1)$-growth. Finally, we say that c is p-admissible if it is strictly convex, p-coercive and has p-growth.

In this paper, we consider Borel cost functionals $\mathsf{c}:(0,1)\times\mathbb{R}^d\times\mathrm{Sim}^d_+\to\mathbb{R}\cup\{+\infty\}$, $\mathsf{c}(t,x,a)$ and their partial Legendre transform with respect to the variable $a\in\mathrm{Sim}^d_+$: all the definitions given above, i.e., (strict) convexity p-coercivity, p-growth and p-admissability are to be considered with respect to the variable a (or the dual variable $b\in\mathrm{Sim}^d_+$) and must hold with uniform constants with respect to $(t,x)\in(0,1)\times\mathbb{R}^d$.

Martingales. A ($\mathbb{P}$-a.s. continuous) real-valued stochastic process $M=(M_t)_{t\in[0,1]}$ defined on a filtered probability space $(\Omega,\mathcal{A},\mathbb{P},(\mathcal{F})_{t\in[0,1]})$ is a martingale if,for $t\in[0,1]$, M is progressively measurable, i.e., the restriction of M to $[0,t]\times\Omega$ is $\mathbb{B}(0,t)\times\mathcal{F}_t$-measurable, $\mathbb{E}[|M_t|]<\infty$, and, for $s<t\in[0,1]$, $\mathbb{E}[M_t|\mathcal{F}_s]=M_s$. Throughout this paper we consider only square integrable martingales M, i.e., such that $\mathbb{E}[|M_1|^2]<\infty$.

A martingale M has finite quadratic variation [37, Chapter I, §2] if there exists a non-negative (progressively measurable) process $\langle M\rangle=(\langle M\rangle_t)_{t\in[0,1]}$ such that, for any sequence of partitions $(\pi^n)_{n\ge1}$ of $[0,1]$, whose diameter $|\pi^n|\to0$ as $n\to+\infty$, then the limit in probability

$$\lim_{n\to+\infty}\sum_{t_i\in[0,t]\cap\pi^n}(M_{t_i}-M_{t_{i-1}})^2=\langle M\rangle_t$$

holds. By [37, Theorem 1.3] (if M is square integrable) then $\langle M\rangle$ exists and is the unique continuous, increasing, adapted process such that $\langle M\rangle_0=0$ and $(M_t^2-\langle M\rangle_t)_{t\in[0,1]}$ is a martingale (not necessarily square integrable).

Throughout this paper, we write that a martingale M has absolutely continuous quadratic variation if for some (progressively measurable) process $\langle\dot M\rangle=(\langle\dot M\rangle_t)_{t\in[0,1]}$, one has

$$\langle M\rangle_t=\int_0^t\langle\dot M\rangle_s\,\mathrm{d}s,\qquad\text{for } t\in[0,1],$$

and write $M\in\mathrm{AC}^p$ if $\mathbb{E}\left[\int_0^1\langle\dot M\rangle_t^p\,\mathrm{d}t\right]<\infty$. Similar definitions and properties hold for martingales taking values in $\mathbb{R}^d$, arguing componentwise: in particular, the processes $\langle M\rangle$, $\langle\dot M\rangle$ take values in Sim^d_+.

Fokker-Planck equations. Given $(\varrho_t)_{t\in[0,1]} \subseteq \mathcal{P}_2(\mathbb{R}^d)$ continuous and $a : (0,1)\times\mathbb{R}^d \to \mathrm{Sim}^d_+$ Borel, we say that the Fokker-Planck equation

$$\partial_t \varrho = (a\nabla^2)^* \varrho, \quad \text{in } (0,1)\times\mathbb{R}^d, \tag{FPE}$$

holds if $\int_0^1 \int_{\mathbb{R}^d} |a_t| \, d\varrho_t < \infty$ and for $\varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)$ one has

$$\int_0^1 \int_{\mathbb{R}^d} \left(\partial_t \varphi + \nabla^2 \varphi : a_t\right) d\varrho_t \, dt = \int_{\mathbb{R}^d} \varphi_1 \, d\varrho_1 - \int_{\mathbb{R}^d} \varphi_0 \, d\varrho_0. \tag{2.2}$$

It is technically useful to extend the notion of Fokker-Planck equation to general measures, essentially defining $\mathbf{a} := a\varrho$, so that the equation becomes linear. Precisely, given $\varrho \in \mathcal{P}_2((0,1)\times\mathbb{R}^d)$, $\mathbf{a} \in \mathcal{M}((0,1)\times\mathbb{R}^d; \mathrm{Sim}^d_+)$, we say that $\partial_t\varrho = (\nabla^2)^*\mathbf{a}$ holds in $(0,1)\times\mathbb{R}^d$ if one can disintegrate $\varrho = \int_0^1 \varrho_t \, dt$, $\mathbf{a} = \int_0^1 \mathbf{a}_t \, dt$, and the identity $\int_{(0,1)\times\mathbb{R}^d} \left(\partial\varphi \, d\varrho + \nabla^2\varphi : d\mathbf{a}\right) = 0$ holds for every $\varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)$ compactly supported in $(0,1)\times\mathbb{R}^d$. Notice that, when $\mathbf{a}$ is absolutely continuous with respect to ϱ, then the two given notions coincide, letting $a := \frac{d\mathbf{a}}{d\varrho}$ be the Radon-Nikodym derivative, up to providing a continuous representative for $(\varrho_t)_{t\in[0,1]}$, which can be always done, arguing analogously as in [1, Lemma 8.1.2], see also [42, Remark 2.3].

Martingale problems. We say that a continuous stochastic process $(X_t)_{t\in[0,1]}$, taking values in $\mathbb{R}^d$, and defined on some probability space endowed with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,1]}$, is a solution to the martingale problem [40, Chapter 6] associated to a Borel function $a : (0,1)\times\mathbb{R}^d \to \mathrm{Sim}^d_+$ (diffusion coefficient) if

$$\mathbb{E}\left[\int_0^1 |a_t(X_t)| \, dt\right] < \infty$$

and for every $\varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)$ the process

$$t \mapsto \varphi(t, X_t) - \int_0^t \left(\partial_t\varphi(s, X_s) + \nabla^2\varphi(s, X_s) : a_s(X_s)\right) ds \tag{2.3}$$

is a martingale. An application of Itô formula shows that its quadratic variation is

$$t \mapsto 2\int_0^t (a_s(X_s)\nabla\varphi_s(X_s)) \cdot \nabla\varphi_s(X_s) \, ds,$$

hence the martingale is in AC^1. Letting $\varphi(x) = x$ (more rigorously, using a suitable approximation via truncation [42, Remark 2.3]), then X itself is a martingale, with density of quadratic variation $\langle \dot{X} \rangle_t = a_t(X_t)$.

Since the expectation of the martingale (2.3) is constant, we see that the 1-marginal laws of X, i.e. the (continuous) curve $t \mapsto \varrho_t := (X_t)_\sharp\mathbb{P} \in \mathcal{P}_2(\mathbb{R}^d)$ solve the Fokker-Planck equation $\partial_t\varrho = \frac{1}{2}(a : \nabla^2)^*\varrho$ in $(0,1)\times\mathbb{R}^d$. Moreover, the curve $(\varrho_t)_{t\in[0,1]}$ is increasing with respect to convex ordering.

A converse result holds (see [23, Theorem 2.6] for a proof in case of bounded a, [42, Theorem 2.5] for the general case).

Theorem 2.1. *Let $(\varrho_t)_{t\in[0,1]} \subseteq \mathcal{P}_2(\mathbb{R}^d)$ be continuous and $a : (0,1)\times\mathbb{R}^d \to \mathrm{Sim}_d^+$ be Borel, solving $\partial_t\varrho = (a : \nabla^2)^*\varrho$ in $(0,1)\times\mathbb{R}^d$. Then, there exists a continuous process $(X_t)_{t\in[0,1]}$ solving the martingale problem associated to a, such that $\varrho_t = (X_t)_\sharp\mathbb{P}$, for every $t \in [0,1]$.*

3 From discrete to continuous costs

In this section we argue that the integral costs to be introduced in Section 4 naturally appear as length-type functionals with respect to the discrete-time martingale transport cost, providing a sort of (non-symmetric) metric on the space of probability measures. This is achieved by introducing suitably normalized "cumulative" costs associated to a partition $\pi = \{t_0 = 0 < \ldots < t_n = 1\} \subseteq [0,1]$, and investigating their limit an $\|\pi\| \to 0$, where $\|\pi\| = \sup_{i=1,\ldots,n} |t_i - t_{i-1}|$.

Theorem 3.1. *Let $p \geq 1$, $\mathsf{c} \in C(\mathbb{R}^d)$ satisfy $|\mathsf{c}(y) - \mathsf{c}(x)| \leq \lambda(1 + |x|^{2p-1} + |y|^{2p-1})|y - x|$ for $x, y \in \mathbb{R}^d$, for some $\lambda \geq 0$. Let $X \in \mathrm{AC}^p$ with $t \mapsto \langle \dot{X} \rangle_t$ continuous ($\mathbb{P}$-a.s.). Then, if Z is a d-dimensional standard normal random variable independent of X, the following limits hold:*

$$\lim_{\|\pi\|\to 0} \sum_{t_i \in \pi} \mathbb{E}\left[\mathsf{c}\left(\frac{X_{t_i} - X_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}\right)\right](t_i - t_{i-1}) = \int_0^1 \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X} \rangle}\, Z\right)\right] ds, \tag{3.1}$$

$$\lim_{\|\pi\|\to 0} \sum_{t_i \in \pi} \left|\mathbb{E}\left[\mathsf{c}\left(\frac{X_{t_i} - X_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}\right)\right]\right|^{1/p} (t_i - t_{i-1}) = \int_0^1 \left|\mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X} \rangle}\, Z\right)\right]\right|^{1/p} ds\,. \tag{3.2}$$

Of course, there are many other "cumulative costs" that one can consider, but a common feature should be that, if $\mathsf{c}(x, y)$ depends on the difference $y - x$, then letting $x = X_t$, $y = X_s$ for a continuous martingale $(X_t)_{t\in[0,1]}$, a rescaling factor $\sqrt{t-s}$ should appear. Notice also that, if c is an odd function, the right hand side in (3.1) is identically zero, by independence of Z and $\sqrt{\langle \dot{X} \rangle}$.

Proof. We consider only the first limit in (3.1), the second being proved analogously. First, by using the continuity of $\langle \dot{X} \rangle$, the Riemann sums computed on a partition $\pi = \{t_0 = 0 < \ldots < t_n = 1\} \subseteq [0,1]$,

$$\sum_{t_i \in \pi} \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X} \rangle_{t_{i-1}}}\, Z\right)\right](t_i - t_{i-1}),$$

converge to the right hand side of (3.1). Therefore, it is sufficient to prove that the difference

$$\lim_{\|\pi\|\to 0} \sum_{t_i \in \pi} \left(\mathbb{E}\left[\mathsf{c}\left(\frac{X_{t_i} - X_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}\right)\right] - \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X} \rangle_{t_{i-1}}}\, Z\right)\right]\right)(t_i - t_{i-1}) = 0. \tag{3.3}$$

To this aim, we use [37, Theorem 3.9], so that there exists a predictable process $(\sigma_t)_{t\in[0,1]}$ with values in Sim^d_+ such that

$$X_t = X_0 + \int_0^t \sigma_s \, \mathrm{d}W_s, \quad \text{for } t \in [0,1],\ \mathbb{P}\text{-a.s.},$$

where $(W_t)_{t\in[0,1]}$ is a d-dimensional Wiener process, possibly on an enlarged probability space. In fact, the proof of [37, Theorem 3.9] gives that $\sigma_t = \sqrt{\langle \dot{X} \rangle_t}\, U_t$ holds, for a predictable process $(U_t)_{t\in[0,1]}$ with values in the $d \times d$ orthogonal matrices. Up to replacing $(W_t)_{t\in[0,1]}$ with the Wiener process $\int_0^t U_s \, \mathrm{d}W_s$, we can assume that U_t is the identity matrix for $t \in [0,1]$, hence $\sigma = \sqrt{\langle \dot{X} \rangle}$.

In this situation, for $t > 0$, one has, by the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\left|(X_t - X_0)\, t^{-1/2}\right|^{2p}\right] \leq \lambda_p \mathbb{E}\left[\left|\frac{1}{t}\int_0^t |\sigma_s|^2 ds\right|^p\right] \leq \lambda \mathbb{E}\left[\frac{1}{t}\int_0^t |\sigma_s|^{2p} \, \mathrm{d}s\right], \tag{3.4}$$

(here and below, λ represent different constants, possibly changing line to line). Similarly, starting from the identity

$$(X_t - X_0)\, t^{-1/2} - \sigma_0 W_t t^{-1/2} = \frac{1}{\sqrt{t}} \int_0^t (\sigma_s - \sigma_0)\, \mathrm{d}W_s$$

we obtain

$$\mathbb{E}\left[\left|(X_t - X_0)\, t^{-1/2} - \sigma_0 W_t t^{-1/2}\right|^{2p}\right] \le \lambda \mathbb{E}\left[\left|\frac{1}{t}\int_0^t |\sigma_s - \sigma_0|^2\, ds\right|^p\right] \le \lambda \mathbb{E}\left[\frac{1}{t}\int_0^t \left|\sigma_s^2 - \sigma_0^2\right|^p \,\mathrm{d}s\right], \tag{3.5}$$

where the second inequality follows by Jensen's inequality and $\left|a^{1/2} - b^{1/2}\right| \le \lambda\, |a - b|^{1/2}$, valid for $a, b \in \mathrm{Sim}^d_+$.

Using the assumption $|\mathsf{c}(y) - \mathsf{c}(x)| \le \lambda(1 + |x|^{2p-1} + |y|^{2p-1})|y - x|$, with $x = (X_t - X_0)t^{-1/2}$, $y = \sigma_0 W_t t^{-1/2}$, we have the inequality

$$\begin{aligned}
\left|\mathbb{E}\left[\mathsf{c}\left((X_t - X_0)\, t^{-1/2}\right)\right] - \mathbb{E}\left[\mathsf{c}\left(\sigma_0 W_t t^{-1/2}\right)\right]\right| &\le \mathbb{E}\left[\left|\mathsf{c}\left((X_t - X_0)\, t^{-1/2}\right) - \mathsf{c}\left(\sigma_0 W_t t^{-1/2}\right)\right|\right] \\
&\le \mathbb{E}\left[A\left|(X_t - X_0)t^{-1/2} - \sigma_0 W_t t^{-1/2}\right|\right] \\
&\quad \text{with } A := \lambda(1 + |(X_t - X_0)t^{-1/2}|^{2p-1} + |\sigma_0 W_t t^{-1/2}|^{2p-1}) \\
&\le \varepsilon \mathbb{E}\left[A^{2p/(2p-1)}\right] + \lambda(\varepsilon)\mathbb{E}\left[\left|(X_t - X_0)\, t^{-1/2} - \sigma_0 W_t t^{-1/2}\right|^{2p}\right] \\
&\quad \text{by Young inequality (with } \varepsilon > 0\text{),} \\
&\le \varepsilon \mathbb{E}\left[1 + \frac{1}{t}\int_0^t |\sigma_s|^{2p}\,\mathrm{d}s + |\sigma_0|^{2p}\, |W_t t^{-1/2}|^{2p}\right] + \lambda(\varepsilon)\mathbb{E}\left[\frac{1}{t}\int_0^t \left|\sigma_s^2 - \sigma_0^2\right|^p \,\mathrm{d}s\right] \\
&\quad \text{by (3.4) and (3.5).}
\end{aligned}$$

Moreover, since $W_t t^{-1/2}$ is a d-dimensional standard Gaussian independent of $\mathcal{F}_0$, one has

$$\mathbb{E}\left[\mathsf{c}\left(\sigma_0 W_t t^{-1/2}\right)\right] = \mathbb{E}\left[\mathsf{c}\left(\sigma_0 Z\right)\right].$$

Then, for a given partition $\pi = \{0 = t_0 < \ldots < t_n = 1\}$, for $i \in \{1, \ldots, n\}$, we apply (3.4) and (3.5) to each martingale $X^i_s := X_{(1-s)t_{i-1} + st_i}$, $s \in [0, 1]$ (with respect to the naturally reparametrized filtration), obtaining the inequality

$$\begin{aligned}
&\sum_{t_i \in \pi} \left|\left(\mathbb{E}\left[\mathsf{c}(X_{t_i} - X_{t_{i-1}})\right] - \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X}\rangle_{t_{i-1}}}\, Z\right)\right]\right)\right| (t_i - t_{i-1}) \le \\
&\le \sum_{t_i \in \pi} \varepsilon \mathbb{E}\left[(t_i - t_{i-1})(1 + |Z|^{2p}) + \int_{t_{i-1}}^{t_i} |\sigma_s|^{2p}\,\mathrm{d}s\right] + \lambda(\varepsilon)\mathbb{E}\left[\int_{t_{i-1}}^{t_i} |\sigma_s - \sigma_{t_{i-1}}|^{2p}\,\mathrm{d}s\right]
\end{aligned}$$

Letting $\|\pi\| \to 0$, we obtain by continuity of σ that

$$\limsup_{\|\pi\| \to 0} \sum_{t_i \in \pi} \left|\left(\mathbb{E}\left[\mathsf{c}(X_{t_i} - X_{t_{i-1}})\right] - \mathbb{E}\left[\mathsf{c}\left(\sqrt{\langle \dot{X}\rangle_{t_{i-1}}}\, Z\right)\right]\right)\right| (t_i - t_{i-1}) \le \varepsilon \mathbb{E}\left[1 + |Z|^{2p} + \int_0^1 |\sigma_s|^{2p}\,\mathrm{d}s\right],$$

and as $\varepsilon \to 0$ we conclude that (3.3) holds. □

4 A Benamou-Brenier type problem

In view of Theorem 3.1, we introduce the following minimization problem, as a continuous-time martingale optimal transport problem, for a cost functional $\mathsf{c} : [0, 1] \times \mathbb{R}^d \times \mathrm{Sim}^d_+ \to \mathbb{R} \cup \{+\infty\}$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\mathsf{c}_{BB}(\mu, \nu) := \inf\left\{\int_0^1 \mathbb{E}\left[\mathsf{c}(t, X_t, \langle \dot{X}\rangle_t)\right] \mathrm{d}t \colon X \in \mathrm{AC}, (X_0)_\sharp \mathbb{P} = \mu,\ (X_1)_\sharp \mathbb{P} = \nu\right\}, \tag{4.1}$$

which can be interpreted as the infimum over the length of martingale curves connecting μ to ν, where the length is the integral of the "speed" $\mathsf{c}(t, X_t, \langle \dot{X} \rangle_t)$. This can be seen as a "martingale" analogue of the dynamical formulation of optimal transport first studied in [9].

Remark 4.1 (time-changes and unit speed geodesics)**.** *Clearly, the choice of looking at martingales defined on the time interval* $[0,1]$ *is arbitrary. However, if the cost* c *is such that* $\mathsf{c}(t, x, \cdot)$ *is* p*-homogeneous for some* $p > 1$*, i.e.* $\mathsf{c}(t, x, \lambda a) = |\lambda|^p \mathsf{c}(t, x, a)$ *for* $\lambda \geq 0$*, one can perform a (deterministic) time-change* α*, so that the quadratic variation of* $X_{\alpha(t)}$ *satisfies* $\langle \dot{X}_\alpha \rangle_t = (\dot{\alpha}_t)^2 \langle \dot{X} \rangle_{\alpha(t)}$ *and the cost on different interval would change in a controlled way. Indeed, by choosing* α *as the inverse of* $t \mapsto \int_0^t \mathbb{E}\left[\mathsf{c}(r, X_r, \langle \dot{X} \rangle_r)\right] \mathrm{d}r$*, one can prove that for any* $T > 0$*, one has*

$$(\mathsf{c}_{BB}(\mu,\nu))^{1/(2p)} = \inf \left\{ \int_0^T \left| \mathbb{E}\left[\mathsf{c}(t, X_t, \langle \dot{X} \rangle_t)\right] \right|^{1/(2p)} \mathrm{d}t \colon X \in \mathrm{AC}, (X_0)_\sharp \mathbb{P} = \mu, \, (X_T)_\sharp \mathbb{P} = \nu \right\}.$$

We refer to [16, Theorem 5.4] for a complete derivation in the deterministic setting: the argument in this framework is the same up to using the different (quadratic) scaling for the quadratic variation in the time change. An interesting consequence of this identity and time rescaling is that that optimizers X *in* (4.1) *are then unit speed geodesics, i.e., letting* $\varrho_t := (X_t)_\sharp \mathbb{P}$*, one has* $(\mathsf{c}_{BB}(\varrho_s, \varrho_t))^{1/(2p)} = |t-s| \, (\mathsf{c}_{BB}(\varrho_0, \varrho_1))^{1/(2p)}$*.*

Concerning well-posedness of this variational problem, Theorem 6.3 in combination with Theorem 4.3 below ensures that, if c is p-admissible for some $p > 1$ and $\mathsf{c}_{BB}(\mu,\nu) < \infty$, then there exists an optimizer, for which one even knows that it is a solution to some martingale problem.

Remark 4.2 (strong relaxation)**.** *From a purely metric point of view, Theorem 3.1 shows that* (4.1) *is a "weak" relaxation of the discrete-time martingale transport problem associated to* c*. A much stronger result, more similar to the construction of a length metric would be the following one. Given* c *as in Theorem 3.1, for any partition* $\pi = \{0 = t_0 < \ldots < t_n = 1\} \subseteq [0,1]$*, introduce the rescaled cost* $\mathsf{c}^i(x,y) = \mathsf{c}((y-x)/\sqrt{t_i - t_{i-1}})$ *and let*

$$\mathsf{c}^\pi_{MK}(\mu,\nu) := \inf \left\{ \sum_{i=1}^n \mathsf{c}^i_{MK}(\varrho_{t_{i-1}}, \varrho_{t_i})(t_i - t_{i-1}) : \; (\varrho_{t_i})_{t_i \in \pi} \subseteq \mathcal{P}_2(\mathbb{R}^d), \, \varrho_0 = \mu, \, \varrho_1 = \nu. \right\}.$$

Then it seems plausible that $\liminf_{\|\pi\| \to 0} \mathsf{c}^\pi_{MK}(\mu,\nu) = \bar{\mathsf{c}}_{BB}(\mu,\nu)$*, where* $\bar{\mathsf{c}}(a) = \mathbb{E}\left[c(Z\sqrt{a})\right]$*, for a standard Gaussian random variable* Z*. In fact, it seems reasonable that one can obtain inequality* $\leq$*, up to approximating any* $X \in \mathrm{AC}$ *by martingales as in Theorem 3.1, but the validity of the converse inequality seems to be a much more difficult problem.*

Next, we notice that in the right hand side of (4.1), if X is a solution to the martingale problem associated to some diffusion coefficient a, then using $\langle \dot{X} \rangle_t = 2a_t(X_t)$ one can rewrite $\mathbb{E}[\mathsf{c}(t, X_t, \langle \dot{X} \rangle_t)] = \int_{\mathbb{R}^d} \mathsf{c}(t, x, 2a_t(x)) \, \mathrm{d}\varrho_t(x)$, letting $\varrho_t = (X_t)_\sharp \mathbb{P}$ be the marginal of X at time t. Hence the transportation cost depends on $(\varrho_t)_{t \in [0,1]}$ and a, for which we know that $\partial_t \varrho = (a \nabla^2)^* \varrho$ holds in $(0,1) \times \mathbb{R}^d$. Hence, we introduce the following variational problem, which has the advantage of being completely formulated in terms of partial differential equations.

$$\mathsf{c}_{FPE}(\mu,\nu) := \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} \mathsf{c}(t, x, 2a_t(x)) \, \mathrm{d}\varrho_t(x) \, \mathrm{d}t \colon \partial_t \varrho = (a \colon \nabla^2)^* \varrho, \, \varrho_0 = \mu, \, \varrho_1 = \nu \right\}, \tag{4.2}$$

Inequality $\mathsf{c}_{BB}(\mu,\nu) \leq \mathsf{c}_{FPE}(\mu,\nu)$ always holds. Indeed, by Theorem 2.1, any solution $(\varrho_t)_{t \in [0,1]}$ to $\partial_t \varrho = (a_t \colon \nabla^2)^* \varrho$ can be lifted to a solution $(X_t)_{t \in [0,1]}$ to the martingale problem associated to a. The converse inequality, hence $\mathsf{c}_{FPE} = \mathsf{c}_{BB}$, follows from the next result, which could be regarded as a variant of [1, Theorem 8.3.1] in the martingale setting.

Theorem 4.3. *For $p \in (1,+\infty)$ let $c : [0,1]\times\mathbb{R}^d\times\mathrm{Sim}_d^+ \to \mathbb{R}\cup\{+\infty\}$ be p-admissible. Then, for every $X \in \mathrm{AC}^p$, let $\varrho_t = (X_t)_\sharp\mathbb{P}$, for $t \in [0,1]$. Then, there exists $a : [0,1]\times\mathbb{R}^d \to \mathrm{Sim}_d^+$ such that $\partial_t\varrho = (a : \nabla^2)^*\varrho$ holds in $(0,1)\times\mathbb{R}^d$ and*

$$\int_0^1 \int_{\mathbb{R}^d} c(t,x,a_t(x))\,\mathrm{d}\varrho_t(x) \le \int_0^1 \mathbb{E}\left[c(t,X_t,\langle \dot{X}\rangle_t)\right]\mathrm{d}t. \tag{4.3}$$

Proof. Let $\mathcal{E}$ denote the right hand side in (4.3), which is finite by the assumption $X \in \mathrm{AC}^p$. We introduce the linear functional $\mathcal{L}$, on $C_b^{1,2}([0,1]\times\mathbb{R}^d)$,

$$\mathcal{L}\varphi := \int_{\mathbb{R}^d} \varphi(1,x)\,\mathrm{d}\varrho_1 - \int_{\mathbb{R}^d} \varphi(0,x)\,\mathrm{d}\varrho_0(x) - \int_0^1 \int_{\mathbb{R}^d} \partial_t\varphi(t,x)\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t, \tag{4.4}$$

and we notice that

$$\begin{aligned}
\mathcal{L}\varphi &= \mathbb{E}\left[\varphi(1,X_1) - \varphi(0,X_0) - \int_0^1 \partial_t\varphi(t,X_t)\,\mathrm{d}t\right] \quad \text{since } \varrho_t = (X_t)_\sharp\mathbb{P},\\
&= \mathbb{E}\left[\int_0^1 \nabla\varphi(t,X_t)\,\mathrm{d}X_t + \int_0^1 \langle \dot{X}\rangle_t : \nabla^2\varphi(t,X_t)\,\mathrm{d}t\right] \quad \text{by Itô formula,}\\
&= \mathbb{E}\left[\int_0^1 \langle \dot{X}\rangle_t : \nabla^2\varphi(t,X_t)\,\mathrm{d}t\right] \quad \text{for the stochastic integral is a martingale.}
\end{aligned}$$

By Hölder inequality, we deduce with $q = \frac{p}{p-1}$,

$$\begin{aligned}
|\mathcal{L}\varphi| &\le \mathbb{E}\left[\int_0^1 \left|\langle \dot{X}\rangle_t\right| \left|\nabla^2\varphi(t,X_t)\right|\,\mathrm{d}t\right] \le \mathbb{E}\left[\int_0^1 \left|\langle \dot{X}\rangle_t\right|^p\,\mathrm{d}t\right]^{1/p} \mathbb{E}\left[\int_0^1 \left|\nabla^2\varphi(t,X_t)\right|^q\,\mathrm{d}t\right]^{1/q}\\
&= \left\|\langle \dot{X}\rangle\right\|_{L^p} \left(\int_0^1 \int_{\mathbb{R}^d} \left|\nabla^2\varphi(t,x)\right|^q\,\mathrm{d}\varrho_t\,\mathrm{d}t\right)^{1/q} = \left\|\langle \dot{X}\rangle\right\|_{L^p} \left\|\nabla^2\varphi\right\|_{L^q(\varrho)}.
\end{aligned}$$

where $L^q(\varrho)$ denote the Lebesgue spaces with respect to $\int_0^1 \varrho_t\,\mathrm{d}t$. As a consequence, the linear functional $\tilde{\mathcal{L}}(\nabla^2\varphi) := \mathcal{L}\varphi$ is actually well-defined and continuous on the space

$$\mathcal{V} := \left\{\nabla^2\varphi\colon\ \varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)\right\} \subseteq L^q(\varrho;\mathrm{Sim}^d),$$

and extends by continuity to its closure $\overline{\mathcal{V}} \subseteq L^q(\varrho;\mathrm{Sim}^d)$. The inequality

$$\begin{aligned}
\mathcal{L}\varphi &= \mathbb{E}\left[\int_0^1 \langle \dot{X}\rangle_t : \nabla^2\varphi(t,X_t)\,\mathrm{d}t\right] \le \mathbb{E}\left[\int_0^1 c\left(t,X_t,\langle \dot{X}\rangle_t\right) + \mathsf{c}^*\left(t,X_t,\nabla^2\varphi(t,X_t)\right)\mathrm{d}t\right]\\
&= \mathcal{E} + \int_{(0,1)\times\mathbb{R}^d} \mathsf{c}^*\left(y,\nabla^2\varphi(y)\right)\mathrm{d}\varrho(y)
\end{aligned}$$

by continuity (for c^* has q-growth) extends to

$$\int_{[0,1]\times\mathbb{R}^d} \mathsf{c}^*(y,u(y))\,\mathrm{d}\varrho(y) - \tilde{\mathcal{L}}u \ge -\mathcal{E}, \quad \text{for every } u \in \overline{\mathcal{V}}. \tag{4.5}$$

Moreover, since c^* is q-coercive, the functional

$$\int_{[0,1]\times\mathbb{R}^d} \mathsf{c}^*(y,u(y))\,\mathrm{d}\varrho - \tilde{\mathcal{L}}u$$

is convex and coercive on $\overline{\mathcal{V}}$, hence it attains its minimum at some $\bar{u} \in \overline{\mathcal{V}}$. Using the fact that $\nabla_u\mathsf{c}^*$ exists, is continuous and has $(q-1)$-growth, optimality condition reads

$$\int_{[0,1]\times\mathbb{R}^d} \nabla_u\mathsf{c}^*(y,\bar{u}(y)) : u(y)\,\mathrm{d}\varrho(y) = \tilde{\mathcal{L}}u \quad \text{for every } u \in \overline{\mathcal{V}}.$$

Letting $u = \nabla^2\varphi$ for $\varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)$, recalling that $\tilde{\mathcal{L}}u = \mathcal{L}\varphi$ and (4.4), we deduce that $\partial_t\varrho = \left(a_t : \nabla^2\right)^* \varrho$ holds with

$$a_t(x) := \nabla_u \mathsf{c}^* \left(t, x, \bar{u}(t,x)\right).$$

Finally, choosing $u = \bar{u} \in \overline{\mathcal{V}}$ and using (4.5), we deduce

$$\int_{[0,1]\times\mathbb{R}^d} \nabla_u \mathsf{c}^* \left(y, \bar{u}(y)\right) : u(y)\, \mathrm{d}\varrho(y) \le \mathcal{E} + \int_{[0,1]\times\mathbb{R}^d} \mathsf{c}^* \left(y, \bar{u}(y)\right)\, \mathrm{d}\varrho(y),$$

hence, by (2.1), we conclude that

$$\int_{[0,1]\times\mathbb{R}^d} c\left(y, a(y)\right)\, \mathrm{d}\varrho = \int_{[0,1]\times\mathbb{R}^d} \nabla_u \mathsf{c}^* \left(y, \bar{u}(y)\right) : \bar{u}(y) - \mathsf{c}^* \left(y, \bar{u}(y)\right)\, \mathrm{d}\varrho(y) \le \mathcal{E}. \qquad \square$$

We discuss here two straightforward properties of the costs (4.1) and (4.2), namely localization and behaviour with respect to convolution.

To localize, e.g. on a ball B_r of radius $r > 0$, given a martingale X we introduce the stopping time (exit time)

$$\tau_r(X) := \inf\{t \ge 0 \,:\, X_t \notin B_r\}$$

We introduce then the "discounted" cost

$$\mathsf{c}_{BB}^r(\mu,\nu) := \inf\left\{ \mathbb{E}\left[\int_0^{\tau_r(X)} \mathsf{c}(t, X_t, \langle \dot{X}\rangle_t)\, \mathrm{d}t\right] : X \in \mathrm{AC}, (X_0)_\sharp\mathbb{P} = \mu,\ (X_1)_\sharp\mathbb{P} = \nu \right\}.$$

for which the following result holds.

Lemma 4.4 (localization)**.** *Let* $\mathsf{c} : (0,1)\times\mathbb{R}^d\times \mathrm{Sim}_+^d \to [0,\infty]$ *satisfy* $\mathsf{c}(t,x,0) = 0$ *and let* $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$*. Then*

$$\lim_{r\to\infty} \mathsf{c}_{BB}^r(\mu,\nu) = \sup_{r>0} \mathsf{c}_{BB}^r(\mu,\nu) = \mathsf{c}_{BB}(\mu,\nu),$$

Proof. If $X \in \mathrm{AC}$, then $\langle \dot{X}_t^r\rangle = \langle \dot{X}\rangle \chi_{t\le\tau_r(X)}$ hence the sequence $\mathsf{c}_{BB}^r(\mu,\nu)$ is increasing and bounded by $\mathsf{c}_{BB}(\mu,\nu)$, and the limit follows from monotone convergence. $\square$

Given a measure $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$, one can easily prove that, if $\mathsf{c}(t,x,a) = \mathsf{c}(t,a)$ does not depend upon $x \in \mathbb{R}^d$, then the convolution operation with σ is a contraction of the cost c_{BB}, i.e.,

$$\mathsf{c}_{BB}(\mu*\sigma, \nu*\sigma) \le \mathsf{c}_{BB}(\mu,\nu) \quad \text{for } \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d). \tag{4.6}$$

Indeed, given any martingale $X \in \mathrm{AC}$ with $(X_0)_\sharp\mathbb{P} = \mu$, $(X_1)_\sharp\mathbb{P} = \nu$, we may enlarge the filtration so that there exists a random variable Y independent of X and $\mathcal{F}_0$-measurable, with law σ. Then, the process $Z_t := X_t + Y$ is a martingale with $\langle Z\rangle = \langle X\rangle$ and 1-marginals as X after a convolution with σ, hence

$$\int_0^1 \mathbb{E}\left[\mathsf{c}(t, \langle \dot{Z}\rangle_t)\right]\, \mathrm{d}t = \int_0^1 \mathbb{E}\left[\mathsf{c}(t, \langle \dot{X}\rangle_t)\right]\, \mathrm{d}t,$$

and inequality (4.6) follows. Notice also that as $\sigma \to \delta_0$ we obtain that the costs converge. The fact that c is independent of x can be relaxed in a concavity assumption for $x \mapsto \mathsf{c}(t,x,a)$. A similar argument, in the formulation (4.2), gives the following result: for a proof, we refer to [42][Lemma A.1] and [1][Lemma 8.1.10]. The advantage with respect to (4.6) is that the diffusion coefficient become smooth, if σ is chosen appropriately.

Lemma 4.5 (convolution)**.** *Let $\sigma \in C^2(\mathbb{R}^d)$ be a probability density, positive everywhere and with $|\nabla^i \sigma| \le \lambda\sigma$ for $i \in \{1,2\}$ (for some $\lambda > 0$). Then, if $(\varrho_t)_{t\in[0,1]}$, $a : (0,1)\times\mathbb{R}^d \to \mathrm{Sim}^d_+$ solve (FPE), then $\tilde\varrho_t := \varrho_t * \sigma$ and*

$$\tilde a := \frac{\mathrm{d}\,(\varrho a * \sigma)}{\mathrm{d}(\varrho * \sigma)} \in C^2(\mathbb{R}^d; \mathrm{Sim}^d_+)$$

solve $\partial_t\tilde\varrho = (\tilde a\nabla^2)^*\tilde\varrho$ in $(0,1)\times\mathbb{R}^d$. Moreover, if $\mathsf{c} = \mathsf{c}(t,a)$ does not depend on $x \in \mathbb{R}^d$ and is such that $\mathsf{c}(t,x,\cdot)$ is convex, then

$$\int_0^1 \int_{\mathbb{R}^d} \mathsf{c}(t,x,\tilde a(t,x))\,\mathrm{d}\tilde\varrho_t(x) \le \int_0^1 \int_{\mathbb{R}^d} \mathsf{c}(t,x,a(t,x))\,\mathrm{d}\varrho_t(x).$$

5 Upper bounds

As already remarked, if c is p-admissible and $\mathsf{c}_{BB}(\mu,\nu)$ is finite, then Theorem 6.3 guarantees existence of minimizers. Therefore, if c has p-growth for some $p \ge 1$, one may ask for conditions, besides the convex ordering between μ and ν, ensuring that $\mathsf{c}_{BB}(\mu,\nu)$ is finite. When $p = 1$, Itô isometry implies that, for any martingale X,

$$\int_0^1 \mathbb{E}\left[\mathsf{c}(t,X_t,\langle\dot X\rangle_t)\right]\mathrm{d}t \le \lambda\mathbb{E}\left[\int_0^1 \langle\dot X\rangle_t\,\mathrm{d}t\right] = \lambda\mathbb{E}\left[(X_1 - X_0)^2\right] < \infty$$

using the assumptions on the second moments of μ and ν. For different values of p, one cannot rely e.g., on Burkholder-Davis-Gundy inequality, since one should also apply Jensen's inequality, but one then fails to obtain an upper bound (but one could obtain a lower bound). Nevertheless, relying on the solution to the Skorokhod embedding problem, we have the following result.

Proposition 5.1. *Let c have p-growth, for some $p > 1$, let μ and ν be in convex order with $\int_{\mathbb{R}^d}|x|^q\,d\nu(x) < \infty$ for some $q > 2p$. Then, there is a martingale $X \in \mathrm{AC}$ with $(X_0)_\sharp\mathbb{P} = \mu$, $(X_1)_\sharp\mathbb{P} = \nu$ and*

$$\int_0^1 \mathbb{E}\left[c\left(t,X_t,\langle\dot X\rangle_t\right)\right]dt < \infty\,.$$

In particular, $\mathsf{c}_{BB}(\mu,\nu) < \infty$.

Proof. Clearly, it is sufficient to show the thesis with $\mathsf{c}(t,x,a) = |a|^p$. Assume first $d = 1$, and let τ be any solution to the Skorokhod embedding problem (see [34] and [27] for comprehensive surveys) for ν with start in μ such that $(B_{t\wedge\tau})_{t\ge0}$ is uniformly integrable, i.e. $B_0 \sim \varrho$, $B_\tau \sim \nu$. The finite q-th moment of ν together with the Burkholder-Davis-Gundy inequality, and the uniform integrability of $(B_{t\wedge\tau})_{t\ge0}$, imply $\mathbb{E}[\tau^{q/2}] \le \lambda\int|x|^q\,d\nu < \infty$, for some $\lambda = \lambda(q)$. Next, we perform a change of time introducing the martingale $(X_t)_{t\in[0,1]}$,

$$X_t := B_{\tau\wedge\beta_t}$$

with $\beta_t := \left(\frac{t}{1-t}\right)^{1/r}$, for some $r \in (0,(q-2p)/(2p-2)]$. Clearly, $X \in \mathrm{AC}$ with $\langle\dot X\rangle_t = \chi_{\{\beta_t\le\tau\}}\beta'_t =: a_t$. Moreover, we can calculate

$$\begin{aligned}\int_0^1 a_t^p\,\mathrm{d}t &= \int_0^{\beta^{-1}(\tau)} (\beta'_t)^{p-1}\beta'_t\,\mathrm{d}t = \int_0^\tau \left((\beta^{-1})'_s\right)^{1-p}\,\mathrm{d}s\\ &= \int_0^\tau \frac{1}{r^{p-1}} s^{(1-r)(p-1)}(1+s^r)^{2(p-1)}\,\mathrm{d}s \le \lambda(r,p)\left(\tau^{(1-r)(p-1)+1} + \tau^p + \tau^{(1+r)(p-1)+1}\right).\end{aligned}$$

The choice of r ensures that $(1+r)(p-1)+1 \le q/2$ so that

$$\mathbb{E}\left[\int_0^1 a_t^p\right] \mathrm{d}t \le \lambda(r,p)\mathbb{E}\left[1+\tau^{q/2}\right] \le \lambda\left(1+\int_{\mathbb{R}^d} |x|^q \, d\nu\right) < \infty,$$

where we also applied Burkholder-Davis-Gundy inequality once more.

In the general case of $d > 1$, first we notice that, in the argument above, changing β to $\tilde{\beta}(t) = \left(\frac{t}{T-t}\right)^{1/r}$ time changes the Brownian motion to a martingale on the interval $[0,T]$ on the cost of a factor $1/T^{p-1}$ in the estimate of the cost. Then, we observe that any martingale coupling π between ϱ and ν allows one to decompose a d-dimensional transport problem into d one-dimensional martingale transport problems by considering the disintegration $\pi_x(\,\mathrm{d}y)$ of π with respect to μ. Using the one-dimensional case established above, given $x = (x_1, \dots, x_d)$ there is a martingale $X = (X_t)_{t\in[0,1]} \in \mathrm{AC}$ interpolating in the interval $[(i-1)/d, i/d]$ between δ_{x_i} and $q_x(dy_i)$ such that the i-th marginal $(X^i_{(i-1)/d}, X^i_{i/d})$ has law $(\delta_{x_i}, q_x(\,\mathrm{d}y_i))$, $\mathbb{E}[\int_{(i-1)/d}^{i/d} \langle \dot{X}^i\rangle_t^p \,\mathrm{d}t] < \infty$ and $\langle \dot{X}^i\rangle_t = 0$ for $t \notin [(i-1)/d, i/d]$. Putting these pieces together yields a martingale $(X^x_t)_{t\in[0,1]}$ such that X^x_0 has law δ_x and X^x_1 has law $\pi_x(\,\mathrm{d}y)$ and $\mathbb{E}[\int_0^1 \langle \dot{X}\rangle_t^p \,\mathrm{d}t] < \infty$. By convexity of $a \mapsto |a|^p$, the martingale $t \mapsto \int_{\mathbb{R}^d} X^x_t \,\mathrm{d}\mu(x)$ gives the thesis. □

When $d = 1$ and both μ and ν have densities with respect to Lebesgue measure, one can rely on the following variant of the Dacorogna-Moser interpolation technique. See [19] for applications of related ideas to the Skorohod embedding problem.

Proposition 5.2 (Dacorogna-Moser interpolation)**.** *Let $\mu = m(x)\,\mathrm{d}x$, $\nu = n(x)\,\mathrm{d}x \in \mathcal{P}_2(\mathbb{R}^d)$ have strictly positive densities, be in convex order and let $f := k * (n-m)$, with $k(x) = x^+$. Then, $f \ge 0$ and, if $\varrho_t := (1-t)\mu + t\nu$, $a_t := f/((1-t)m + tn)$, for $t \in [0,1]$, then (FPE) holds. Moreover, for any $p \ge 1$,*

$$\int_0^1 \int_{\mathbb{R}} |a_t|^p \,\mathrm{d}\varrho_t \,\mathrm{d}t \le \|n-m\|_{L^1}^{p-1} \int_{\mathbb{R}} |n(x)-m(x)| \int_0^x M_p\left(m(y), n(y)\right) |y-x|^p \,\mathrm{d}y \,\mathrm{d}x, \quad (5.1)$$

where, for $u, v > 0$,

$$M_p(u,v) := \int_0^1 ((1-t)u + tv)^{1-p} = \begin{cases} \frac{v^p - u^p}{(p-2)(v-u)} & \text{if } p \neq 2, \\ \frac{\log(v/u)}{v-u} & \text{if } p = 2. \end{cases}$$

Proof. We have that $f \ge 0$ directly from the assumption of convex ordering, the representation, for $x \in \mathbb{R}$,

$$f(x) = \int_{\mathbb{R}} k(y-x)(n(y)-m(y))\,\mathrm{d}y = \int_{\mathbb{R}} (y-x)^+ \,\mathrm{d}\nu(y) - \int_{\mathbb{R}} (y-x)^+ \,\mathrm{d}\mu(y)$$

and convexity of $y \mapsto (y-x)^+$. To show that (FPE) holds, we use instead the fact that x^+ is a fundamental solution to the Laplacian (in dimension $d = 1$), hence we formally write

$$\partial_t \varrho = \nu - \mu = \Delta f = \Delta(a\varrho) = (a\Delta)^* \varrho.$$

Rigorously, the identity

$$\Delta f = n - m$$

holds in duality with functions in $C^2(\mathbb{R})$. Let then $\varphi \in C^{1,2}([0,1]\times\mathbb{R}^d)$ and argue by duality,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\varphi(t,x)\,\mathrm{d}\varrho_t(x) &= \int_{\mathbb{R}}\partial_t\varphi(t,x)\,\mathrm{d}\varrho_t(x) + \int_{\mathbb{R}}\varphi(t,x)(n(x)-m(x))\,\mathrm{d}x\\ &= \int_{\mathbb{R}}\partial_t\varphi(t,x)\,\mathrm{d}\varrho_t(x) + \int_{\mathbb{R}}\varphi(t,x)\Delta f\,\mathrm{d}x\\ &= \int_{\mathbb{R}}\partial_t\varphi(t,x)\,\mathrm{d}\varrho_t(x) + \int_{\mathbb{R}}a_t(x)\,(\Delta\varphi)\,(t,x)\,((1-t)m(x)+tn(x))\,\mathrm{d}x\\ &= \int_{\mathbb{R}}\partial_t\varphi(t,x)\,\mathrm{d}\varrho_t(x) + \int_{\mathbb{R}}a_t(x)\,(\Delta\varphi)\,(t,x)\,\mathrm{d}\varrho_t(x),\end{aligned}$$

hence integrating with respect to $t \in (0,1)$, we obtain (2.2).

To show (5.1), we notice first that, since both m and n are probability densities and the first moment of m and n coincide (by convex order), we have the identity, with $k'(x) = \chi_{\{x>0\}}$,

$$f(x) = \int_{\mathbb{R}}(k(x-y)-k(x)-k'(x)y)\,(n(y)-m(y))\,\mathrm{d}y,$$

hence the upper bound, by Hölder inequality,

$$\begin{aligned}|f(x)|^p &= \left|\int_{\mathbb{R}}(k(x-y)-k(x)-k'(x)y)\,(m(y)-n(y))\,\mathrm{d}y\right|^p\\ &\le \|m-n\|_{L^1}^{p-1}\int_{\mathbb{R}}\left|k(x-y)-k(x)-k'(x)y\right|^p|n(y)-m(y)|\,\mathrm{d}y.\end{aligned}$$

We have then

$$\begin{aligned}\int_0^1\int_{\mathbb{R}}|a_t|^p\,\mathrm{d}\varrho_t\,\mathrm{d}t &= \int_{\mathbb{R}}|f|^p\,(x)\int_0^1((1-t)m(x)+tn(x)))^{p-1}\,\mathrm{d}t\\ &= \int_{\mathbb{R}}|f|^p\,(x)M_p\,(m(x),n(x))\,\mathrm{d}x\\ &\le \|n-m\|_{L^1}^{p-1}\int_{\mathbb{R}}|n(y)-m(y)|\int_0^y M_p\,(m(x),n(x))\,|y-x|\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

since $|k(x-y)-k(x)-k'(x)y| = |y-x|\left(\chi_{\{y<x<0\}}+\chi_{\{0<x<y\}}\right)$. □

Remark 5.3 (Strassen theorem). *As an application of the technique of Proposition 5.2 and Theorem 2.1, we may also obtain a "PDE proof" of the following fundamental result [39]: if μ, $\nu \in \mathcal{P}_2(\mathbb{R})$ are in convex order, then there exists a (discrete-time) martingale (X_0, X_1) such that $(X_0)_\sharp\mathbb{P} = \mu$, $(X_1)_\sharp\mathbb{P} = \nu$.*

For $\varepsilon > 0$, let $\sigma^\varepsilon(x)$ be a smooth, positive everywhere, mollification kernel on $\mathbb{R}$ converging to δ_0 as $\varepsilon \to 0$ (e.g., the heat kernel) and define $\varrho^\varepsilon := \mu*\sigma^\varepsilon$, $\nu^\varepsilon := \nu*\sigma^\varepsilon$, which are in convex order and have strictly positive densities m^ε, n^ε with respect to Lebesgue measure, and let ϱ^ε, a^ε be as in Proposition 5.2. By Theorem 2.1, there exist solutions $(X_t^\varepsilon)_{t\in[0,1]}$ to the martingale problem associated with $a_t^\varepsilon\Delta$, with 1-marginals $(\varrho_t^\varepsilon)_{t\in[0,1]}$, which are in particular martingales. Moreover, since the marginals at time 0 and time 1 converge as $\varepsilon \to 0$ (respectively to μ and ν), we have that the discrete martingale $(X_0^\varepsilon, X_1^\varepsilon)$ is a compact sequence with respect to convergence in law. Any limit point provides a required martingale.

Notice however that, in general, with this technique one cannot find an interpolating martingale $(X_t)_{t\in[0,1]}$ with continuous paths. For example, when $\mu = \delta_0$, $\nu = \frac{1}{2}(\delta_{-1}+\delta_1)$, one would obtain a martingale with marginals $\varrho_t = (1-t)\varrho + t\nu$, for $t \in [0,1]$, which must have discontinuous paths.

6 Duality

In this section, we introduce a dual problem to (4.2) which allows us to write optimality conditions for the primal problem. A key role is played by the following (backward) Hamilton-Jacobi-Bellman PDE,

$$\partial_t \varphi(t,x) = -\mathsf{c}^*\left(t,x,\nabla^2\varphi(t,x)\right). \tag{HJB}$$

To keep technicalities at minimum, we choose to consider only classical solutions and avoid the use of viscosity solutions, which is a standard tool in such optimal control problems [24], see also Remark 6.5. We also work on the domains $[0,1]\times\overline{B_r}$ ($r>0$) and $[0,1]\times\mathbb{R}^d$, specifying boundary conditions in the former situation.

Definition 6.1 (solutions to (HJB))**.** *Let $\Omega = B_r$ or $\Omega=\mathbb{R}^d$. We say that $\varphi \in C^{1,2}_b([0,1]\times\overline{\Omega})$ is a solution to (HJB) if identity holds in (HJB) for every $(t,x)\in(0,1)\times\Omega$. We say that $\varphi\in C^{1,2}_b([0,1]\times\overline{B_r})$ is a super-solution (respectively, sub-solution) to (HJB) if inequality $\le$ (respectively, $\ge$) holds, instead of equality, at every $(t,x)\in(0,1)\times\Omega$. When $\Omega=B_r$, we say that boundary conditions $\nabla^2\varphi=0$ holds if (the continuous extension of $\nabla^2\varphi$ satisfies) $\nabla^2\varphi(t,x)=0$, for $(t,x)\in[0,1]\times\partial B_r$.*

Remark 6.2 (comparison principle)**.** *The terms super-solution and sub-solution are justified by the validity of a (standard) comparison result. For $r>0$, let $\varphi,\psi\in C^{1,2}_b([0,1]\times\overline{B_r})$ be respectively a sub-solution and a super-solution to (HJB), with $\varphi(1,x)\le\psi(1,x)$ for every $x\in B_r$, and boundary condition $\nabla^2\varphi=0$. Then, $\varphi\le\psi$ on $[0,1]\times\overline{B_r}$.*

As a first observation, we notice that, if $\varphi\in C^{1,2}_b([0,1]\times\mathbb{R}^d)$ is a super-solution to (HJB), then given any solution $(\varrho_t)_{t\in[0,1]}\subseteq\mathcal{P}_2(\mathbb{R}^d)$ to $\partial_t\varrho = (a_t:\nabla^2)^*\varrho$, for some $a:(0,1)\times\mathbb{R}^d\to\mathrm{Sim}^d_+$, then

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(1,x)\,\mathrm{d}\varrho_1(x) - \int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\varrho_0(x) &= \int_0^1\int_{\mathbb{R}^d}(\partial_t\varphi(t,x)+a_t(x):\nabla^2\varphi(t,x))\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t\\ &\le \int_0^1\int_{\mathbb{R}^d}(-\mathsf{c}^*(t,x,\nabla^2\varphi(t,x))+a_t(x):\nabla^2\varphi)\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t\\ &\le \int_0^1\int_{\mathbb{R}^d}\mathsf{c}(t,x,a_t(x))\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t,\end{aligned} \tag{6.1}$$

and minimizing over the choice of ϱ and a gives

$$\int_{\mathbb{R}^d}\varphi(1,x)\,\mathrm{d}\varrho_1(x) - \int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\varrho_0(x) \le \mathsf{c}_{FPE}(\varrho_0,\varrho_1).$$

The following result shows that optimizing the left hand side yields equality.

Theorem 6.3 (existence and duality)**.** *Let $\Omega=B_r$, for $r>0$ or $\Omega=\mathbb{R}^d$, $p\in(1,+\infty)$ and $\mathsf{c}:(0,1)\times\mathbb{R}^d\times\mathrm{Sim}^+_d\to\mathbb{R}\cup\{+\infty\}$ be p-admissible. Then, for every $\mu,\ \nu\in\mathcal{P}_2(\Omega)$, if $\mathsf{c}_{FPE}(\mu,\nu)<\infty$ then*

$$\mathsf{c}_{FPE}(\mu,\nu) = \sup\left\{\int_\Omega\varphi(1,x)\,\mathrm{d}\nu(x) - \int_\Omega\varphi(0,x)\,\mathrm{d}\mu(x)\right\}, \tag{6.2}$$

the supremum running over all super-solutions $\varphi\in C^{1,2}_b([0,1]\times\overline{\Omega})$ to (HJB) with boundary condition $\nabla^2\varphi=0$ if $\Omega=B_r$. Moreover, the infimum in (4.2) (or in (4.1)) is actually a minimum.

Remark 6.4 (uniqueness)**.** *Being $\mathsf{c}(t,x,\cdot)$ strictly convex, one has that the minimum in* (4.2) *is unique (see also the proof below). In particular, by Theorem 4.3, one has that* (4.1) *is also a minimum, and a minimizer solves the martingale problem associated to the diffusion coefficient of the minimum in* (4.2)*. However, the problem of (probabilistic) uniqueness of minimizers in* (4.1) *remains open, as it seems to rely on regularity of minimizers, which implies uniqueness for the martingale problem.*

The proof is an application of the Fenchel-Rockafellar duality theorem, e.g. [44, Theorem 1.9], following closely e.g. [12, Section 3.2].

Proof. We give the proof in the case of $\Omega = B_r$, the case $\Omega = \mathbb{R}^d$ being along the same lines (a crucial difference being remarked below). First, we may assume that $\int_{B_r} \ell \cdot x \, d\mu = \int_{B_r} \ell \cdot x \, d\nu$, for every $\ell \in \mathbb{R}^d$, otherwise both terms are $+\infty$: the left hand side because μ and ν would not be in convex order, and the right hand side by letting $\varphi(t,x) := \lambda \ell \cdot x$, which solves (HJB) (with appropriate boundary conditions) and letting $\lambda \to \pm\infty$, depending on the sign of the difference.

Write then $K := [0,1] \times \overline{B_r}$ and $E = C(K; \mathbb{R} \times \mathrm{Sim}^d)$, which equipped with the uniform norm is a Banach space, with continuous dual $E^* = \mathcal{M}(K; \mathbb{R} \times \mathrm{Sim}^d)$, and write the dual pairing as $\varrho(F) + \mathbf{a}(\Phi)$ for $(F,\Phi) \in E$, $(\varrho, \mathbf{a}) \in E^*$. In case $\Omega = \mathbb{R}^d$, since the dual $E^* \supseteq \mathcal{M}(K; \mathbb{R} \times \mathrm{Sim}^d)$ one has to argue that the linear functional $(\varrho, \mathbf{a})$ that we obtain below is tight, hence induced by a signed measure.

Define $\alpha : E \to (-\infty, \infty]$ by

$$\alpha(F,\Phi) = \begin{cases} 0 & \text{if } F(t,x) + \mathsf{c}^*(t,x,\Phi(t,x)) \leq 0 \text{ for every } (t,x) \in K \\ \infty & \text{else.} \end{cases}$$

Being $a^* \mapsto c^*(t,x,a^*)$ convex, it follows that α is a convex function. Its Legendre-Fenchel transform, defined by

$$\alpha^*(\varrho, \mathbf{a}) = \sup \{ \varrho(F) + \mathbf{a}(\Phi) \ : \ F + \mathsf{c}^*(\Phi) \leq 0 \}$$

is explicitly given by the (strictly) convex function

$$\alpha^*(\varrho, \mathbf{a}) = \begin{cases} \int_K \mathsf{c}\,(t,x,a(t,x)) \, d\varrho(t,x) & \text{if } \varrho \in \mathcal{M}^+(K) \text{ and } \mathbf{a} = a\varrho \text{ with } a \geq 0, \\ \infty & \text{else.} \end{cases} \tag{6.3}$$

Indeed, if $\mathbf{a}$ is not absolutely continuous with respect to ϱ, we would like to let $F = -\mathsf{c}^*(\lambda I)\chi_A$ (here we use that $\mathsf{c}^* < \infty$) and $\Phi = \lambda \chi_A$, where A is such that $\varrho(A) = 0$ and $\mathbf{a}(A) \neq 0$, so that

$$\alpha^*(\varrho, \mathbf{a}) \geq \lambda \mathbf{a}(A) \to +\infty$$

letting $\lambda \to \pm\infty$, depending on the sign of $\mathbf{a}(A)$. However, such a choice of (F,Φ) is not immediately possible, but it is sufficient to approximate χ_A, by density of continuous functions in $L^1(|\varrho| + |\mathbf{a}|)$. Hence, we may assume $\mathbf{a} = a\varrho$ with $a \in L^1(|\varrho|)$, so that

$$\varrho(F) + \mathbf{a}(\Phi) = \int_K (F + a\Phi) \, d\varrho \leq \int_K (a\Phi - \mathsf{c}^*(\Phi)) \, d\varrho \leq \int_K \mathsf{c}\,(a) \, d\varrho,$$

and by optimizing among (F,Φ) one obtains (6.3).

Next, we say that a $(F,\Phi) \in E$ is *represented* by $\varphi \in C^{1,2}(K)$ if $F = -\partial_t \varphi$, $\Phi = -\nabla^2 \varphi$ and $\nabla^2 \varphi = 0$ on $[0,1] \times \partial \mathbb{R}^d$. We define $\beta : E \to (-\infty, \infty]$ by

$$\beta(F,\Phi) = \begin{cases} \int_{\mathbb{R}^d} \varphi(0,x) \, d\mu(x) - \int_{\mathbb{R}^d} \varphi(1,x) \, d\nu(x) & \text{if } (F,\Phi) \text{ is represented by } \varphi \in C^{1,2}(K), \\ \infty & \text{else.} \end{cases}$$

Notice that β is well-defined, i.e. it does not depend on the choice of φ, since if both φ and ψ represent (F, Φ), then $\partial_t(\varphi - \psi) = F - F = 0$ and $\nabla^2(\varphi - \psi) = \Phi - \Phi = 0$ on K, hence $\varphi - \psi = b + \ell \cdot x$ for some $b \in \mathbb{R}$, $\ell \in \mathbb{R}^d$, and the argument at the beginning of the proof yields $\int_{\mathbb{R}^d} (\ell \cdot x + b) \,\mathrm{d}(\nu - \mu) = 0$.

Moreover, since the set of represented functions $(F, \Phi) \in E$ is a linear subspace, and the expression defining β, when it is finite, is linear with respect to φ, the function β is convex, with Legendre transform

$$\beta^*(\varrho, \mathbf{a}) = \sup \left\{ \varrho(F) + \mathbf{a}(\Phi) + \int_{\mathbb{R}^d} \varphi(1, x) \,\mathrm{d}\nu(x) - \int_{\mathbb{R}^d} \varphi(0, x) \,\mathrm{d}\mu(x) \ : \ (F, \Phi) \text{ is represented} \right\},$$

which in fact takes values in $\{0, +\infty\}$ and it is zero if and only if, for every $\varphi \in C^{1,2}(K)$ with $\nabla^2 \varphi = 0$ on $[0, 1] \times \partial\mathbb{R}^d$, one has

$$\int_K \partial_t \varphi \,\mathrm{d}\varrho + \int_K \nabla^2 \varphi \colon \mathrm{d}\mathbf{a} = \int_{\mathbb{R}^d} \varphi(1, x) \,\mathrm{d}\nu(x) - \int_{\mathbb{R}^d} \varphi(0, x) \,\mathrm{d}\mu(x). \tag{6.4}$$

At the point $(-1, 0) \in E$, represented by $\varphi(t, x) = -t$, we see that α is continuous, for c^* is continuous, and β is bounded. Therefore the Fenchel-Rockafellar duality [44, Theorem 1.9] implies

$$\inf \{\alpha^*(\varrho, \mathbf{a}) + \beta^*(\varrho, \mathbf{a}) \ : \ (\varrho, \mathbf{a}) \in E^*\} = \sup \{-\alpha(-F, -\Phi) - \beta(F, \Phi) \ : \ (F, \Phi) \in E\} \tag{6.5}$$

and that the left hand side is actually a minimum. Since the right hand side in (6.5) is immediately seen to coincide with the right hand side in (6.2), to conclude we argue that the left hand side above coincides with (4.2).

Indeed, if $(\varrho, \mathbf{a}) \in E^*$ is such that $\alpha^*(\varrho, \mathbf{a}) + \beta^*(\varrho, \mathbf{a}) < \infty$, we claim that $\varrho := \varrho_t \,\mathrm{d}t$ where $(\varrho_t)_{t \in [0,1]} \subseteq \mathcal{P}_2(B_r)$ solves the Fokker-Planck equation $\partial_t \varrho = (a\nabla^2)^* \varrho$ in $(0, 1) \times \mathbb{R}^d$. From (6.3) and (6.4), letting $\varphi(t, x) = \int_0^t g(s) \,\mathrm{d}s - \int_0^1 g(s) \,\mathrm{d}s$, with $g \in C([0, 1])$, we deduce the identity

$$\int_K g(t) \,\mathrm{d}\varrho = \int_0^1 g(t) \,\mathrm{d}t.$$

Letting $g(t) = 1$, it follows that $\varrho \in \mathcal{P}_2(K)$. Moreover, a density argument implies that the t-marginal of ϱ is Lebesgue measure, and by abstract disintegration of measures we have $\varrho = \varrho_t \,\mathrm{d}t$ for some Borel curve $(\varrho_t)_{t \in (0,1)} \subseteq \mathcal{P}_2(B_r)$. Moreover, from (6.5) we have that the FPE holds in the extended sense of measure-valued solutions, see Section 2. However, since $\alpha^*(\varrho, \mathbf{a}) < \infty$ implies that $\mathbf{a}$ is absolutely continuous with respect to ϱ, we conclude that the infimum is actually running on the set of weak solutions to (FPE) (and in particular, the minimum exists in such set). Moreover, by strict convexity of α we deduce that the minimum is unique. □

Remark 6.5 (viscosity solutions)**.** *As a general consequence of the comparison principle, Remark 6.2, solutions to (HJB) always increase the right hand side in (6.2) with respect to super-solutions. Indeed, if $\varphi, \psi \in C^{1,2}([0, 1] \times \overline{B_r})$ are respectively a solution and a super-solution to (HJB) (with appropriate boundary conditions) and $\varphi(1, x) = \psi(1, x)$ for every $x \in B_r$, then the comparison principle entails $\varphi(0, x) \leq \psi(0, x)$ for every $x \in \mathbb{R}^d$, hence*

$$\int_{\mathbb{R}^d} \varphi(1, x) \,\mathrm{d}\nu - \int_{\mathbb{R}^d} \varphi(0, x) \,\mathrm{d}\mu \leq \int_{\mathbb{R}^d} \psi(1, x) \,\mathrm{d}\nu - \int_{\mathbb{R}^d} \psi(0, x) \,\mathrm{d}\mu.$$

Then, formally, one could restate the duality (6.5) by maximizing among solutions, but this comes with the price of introducing viscosity solutions, in order to obtain solutions for any initial datum. This also gives a precise link with the work [41], where all the theory relies from the very beginning on viscosity solutions given by "explicit" formulas of Hopf-Lax type.

From Theorem 6.3, we also obtain sufficient conditions for optimality.

Corollary 6.6. *Let c be p-admissible and $\varphi \in C_b^{1,2}([0,1]\times\mathbb{R}^d)$ solve (HJB), set*

$$a_t(x) := \nabla_a \mathsf{c}^*(t,x,\nabla^2\varphi(t,x)) \quad \text{for } (t,x)\in(0,1)\times\mathbb{R}^d,$$

and let $(\varrho_t)_{t\in[0,1]}\subseteq\mathcal{P}(\mathbb{R}^d)$ be a solution to $\partial_t\varrho = (a_t : \nabla^2)^*\varrho$. Then, $(\varrho_t)_{t\in[0,1]}\subseteq\mathcal{P}(\mathbb{R}^d)$ is a minimizer in (4.2), i.e., for every $s\le t\in[0,1]$,

$$\mathsf{c}_{FPE}(\varrho_0,\varrho_1) = \int_0^1\int_{\mathbb{R}^d} \mathsf{c}(r,x,a_r(x))\,\mathrm{d}\varrho_r(x)\,\mathrm{d}r.$$

Proof. Indeed, for any super-solution $\psi\in C^{1,2}([0,1]\times\mathbb{R}^d)$ to (HJB), arguing as in (6.1), one has

$$\int_{\mathbb{R}^d}\psi(1,x)\,\mathrm{d}\varrho_1(x) - \int_{\mathbb{R}^d}\psi(0,x)\,\mathrm{d}\varrho_0(x) \le \int_0^1\int_{\mathbb{R}^d}\mathsf{c}(t,x,a_t(x))\,\mathrm{d}\varrho_t(x)\,\mathrm{d}t.$$

with equality if $\psi=\varphi$, hence Theorem 6.3 yields the thesis. □

Example 6.7 (transporting Gaussian measures)**.** *Let $\gamma_{0,Q}\in\mathcal{P}_2(\mathbb{R}^d)$ be a Gaussian measure with mean 0 and covariance matrix Q. For any $\mu\in\mathcal{P}(\mathbb{R}^d)$, define $\nu := \mu * \gamma_{0,Q}$ and let $(B_t)_{t\in[0,1]}$ be a d-dimensional Brownian motion with law of B_0 equal to μ (hence necessarily the law of B_1 equals ν). Then, $(M_t)_{t\in[0,1]} := (\sqrt{Q}\cdot B_t)_{t\in[0,1]}$ is an optimizer for any cost function of the form $\mathsf{c}(t,x,a)=\mathsf{c}(a)$. Indeed, by (2.1), there exists $R\in\mathrm{Sim}_+^d$ such that $Q=\nabla_a\mathsf{c}^*(R)$ and letting*

$$\varphi(t,x) := -t\mathsf{c}^*(R) + \frac{1}{2}(Rx)\cdot x, \quad \text{for } (t,x)\in[0,1]\times\mathbb{R}^d,$$

one has that φ solves (HJB) and $\nabla_a\mathsf{c}^*(R)=Q$. We are in a position to apply Corollary 6.6, which implies that the curve $(\mu*\gamma_{0,tQ})_{t\in[0,1]}$, which coincides with that of the 1-marginals of $(M_t)_{t\in[0,1]}$ is a minimizer.

Remark 6.8. *Assuming that c is sufficiently smooth, one can directly write an equation for the optimal diffusion coefficient a. Assume that $\nabla_a\mathsf{c}(t,x,\cdot)$ is invertible with differentiable inverse $(\nabla_a\mathsf{c})^{-1}$ and Hessian $\nabla_a^2\mathsf{c}(t,x,\cdot)$, and put $u(t,x) := \nabla_a\mathsf{c}(t,x,a(t,x))$. If $a:[0,1]\times\mathbb{R}^d\to\mathrm{Sim}_d^+$ satisfies (in a strong sense)*

$$\partial_t a_t = (\partial_t(\nabla_a\mathsf{c})^{-1})(t,x,u(t,x)) - (\nabla_a^2\mathsf{c})^{-1}(t,x,a(t,x)))\cdot\nabla_x^2(\mathsf{c}^*(t,x,u(t,x))), \tag{6.6}$$

and $\varrho = (\varrho_t)_{t\in[0,1]}$ satisfies $\partial_t\varrho = (a\nabla^2)^*\varrho$, then it is a minimizer for the problem of transporting ϱ_0 to ϱ_1.

7 One-dimensional case

In this section, we specialize to the case $d=1$, so that $\nabla^2=\Delta$ and the equation (HJB) can be written as

$$\partial_t\varphi(t,x) = -\mathsf{c}^*(t,x,\Delta\varphi(t,x)).$$

The main idea is that by formally taking the Laplacian Δ on both sides, the variable $u:=\Delta\varphi$ is a solution to the following backwards generalized porous medium type equation

$$\partial_t u(t,x) = -\Delta\mathsf{c}^*\,(t,x,u(t,x))\,. \tag{PME}$$

Moreover, the boundary conditions $\Delta\varphi=0$ on ∂B_r become Dirichlet boundary conditions for u. Hence, we may rely on the theory of porous medium equations [43] to obtain solutions u to (PME) and via integration recover solutions φ. Unfortunately, the complete picture is not definitely clear, due to the fact that the duality result Theorem 6.3 is formulated in terms of classical super-solutions, and even formally we need solutions to pass to the variable $u = \Delta\varphi$. General viscosity solutions (see Remark 6.5) may also be not sufficiently regular to provide a weak notion of Laplacian solving (PME). Let us notice that, in the literature of porous media equation, the variable φ is known as a potential to the porous medium equation and (HJB) is also called *dual filtration equation*.

Definition 7.1 (weak solutions to (PME))**.** *Let $r > 0$. We say that $u \in C([0,1]\times[-r,r])$ is a solution to (PME) if, for every $g \in C^{1,2}_c((0,1)\times(-r,r))$ it holds*

$$\int_0^1 \int_{-r}^r u(t,x)\partial_t g(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_{-r}^r \mathsf{c}^*\left(t,x,u(t,x)\right)\Delta g(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

The one-dimensional theory of such porous medium type equations is well understood, at least in the case of $\mathsf{c}^*(t,x,a) = |a|^q$, see [43, Chapter 15]. Thus, we rely on such results to prove the following theorem.

Theorem 7.2 (existence of solutions to (HJB))**.** *Let $\mathsf{c}^*(t,x,a) = (a^+)^q$, $q > 1$. $r \in [0,\infty)$, $\bar{u} \in C([-r,r])$, $\bar{u} \geq 0$ and $\bar{u}(-r) = \bar{u}(r) = 0$. Then, there exists a unique $\varphi \in C^{1,2}_b([0,1]\times[-r,r])$ solving (HJB) in $(0,1)\times(-r,r)$ with boundary condition $\Delta\varphi = 0$ on $(0,1)\times\{-r,r\}$ and $\Delta\varphi_1 = \bar{u}$. Moreover, one has $\Delta\varphi(t,x) \geq 0$ for $(t,x) \in (0,1)\times(-r,r)$ and for every $t \in [0,1)$, $\Delta\varphi(t,x)$ is Lipschitz continuous.*

Proof. By the results in [43, Chapter 15], there exists a weak solution u to (PME) with $\mathsf{c}^*(t,x,a) = |a|^q$, Dirichlet boundary conditions and $u_1 = \bar{u}$. Moreover, the maximum principle ensures that $u_t \geq 0$ for $t \in [0,1]$ hence u is also a solution with respect to $\mathsf{c}^*(t,x,a) = (a^+)^q$. Moreover, for $t \in [0,1)$, $u_t(x)$ is Lipschitz continuous is space and in time [43, Theorem 15.6]. For every $t \in [0,1]$ let φ_t solve $\Delta\varphi_t = u_t$ in $(-r,r)$ with Dirichlet boundary conditions $\varphi_t(x) = 0$ for $x \in \{-r,r\}$. Then, $\varphi \in C^{1,2}([0,1)\times[-r,r])$ and one has the identity

$$\Delta\partial_t\varphi = \partial_t\Delta\varphi = \partial_t u = -\Delta|u|^q.$$

Since both $\partial_t\varphi$ and $-|u|^q$ agree on the boundary (both are null), we deduce that φ solves (HJB). Moreover, since the right hand side in (HJB) is continuous up to $t = 1$, we deduce that $\varphi \in C^{1,2}_b([0,1)\times[-r,r])$. □

Remark 7.3 (Pressure equation)**.** *The connection between (PME) and (HJB) is even more intriguing if looking directly at the equation for the optimal diffusion coefficient (6.6). Indeed, letting $\mathsf{c}^*(t,x,a) = |a|^q$, one formally obtains*

$$\partial_t a(t,x) = -a(t,x)\Delta a(t,x) - (q-1)(\nabla a(t,x))^2, \tag{PMPE}$$

which is precisely the pressure equation associated to (PME). The interplay between equation (PME) and (PMPE) is very well understood, and may be useful to provide examples, e.g., using explicit solutions such as Barenblatt profiles. Let us also notice that combining the regularity theory for (PMPE) and the well-known Watanabe criterion [37, Chapter IX, §3], one has Lipschitz continuity of $x \mapsto a_t(x)$, hence $1/2$-Hölder continuity of $\sigma_t(x) = \sqrt{a_t(x)}$, hence uniqueness of solutions to the martingale problem. Thus, minimizers obtained via Corollary 6.6 are unique in the probabilistic sense.

It is even possible to consider "explosive" initial data $\bar{u}$ in Theorem 7.2, leading to non-trivial interpolations, as the next example shows.

Example 7.4. *Let $d = 1$ and $\nu = \frac{1}{2}(\delta_{-1} + \delta_1)$ and μ in convex order with respect to ν. Let $\mathsf{c}(t, x, a) = a^p$ for $p > 1$ and let $q = \frac{p}{p-1}$ and $u = (u_t)_{t\in[0,1]}$ be the solution to the backwards porous medium equation*

$$\partial_t u = -\Delta u^q$$

with terminal condition $u_1 = \infty\chi_{(-1,1)}$, or more precisely, we let u be the so-called friendly giant (backward), so that [43, Theorem 5.20] gives $u_t(x) = (1-t)^{-\frac{1}{q-1}} g(x)$ where g is the unique (positive everywhere) solution to

$$\Delta g^q + \frac{1}{q-1} g = 0, \quad g^q \in H_0^1((-1, 1)).$$

Defining $a_t = qu_t^{q-1} = q\frac{1}{1-t}g^{q-1}$, which is the corresponding pressure variable, then a solves (6.6). Let $\varrho = (\varrho_t)_{t\in[0,1)}$ be a solution to the corresponding Fokker-Planck equation with initial condition $\varrho_0 = \mu$. For any $t < 1$, a solution exists since a is bounded and continuous.

We argue that necessarily $\lim_{t\uparrow 1} \rho_t = \nu$. Let $X = (X_t)_{t\in[0,1)}$ be a continuous martingale with marginals $(\varrho_t)_t$ and $\langle \dot{X}\rangle_t = a_t(X_t)$ on some probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$. Observe that X can only stop diffusing at the boundary $\{-1, 1\}$, for $a_t(x) > 0$ on $[0, 1) \times (-1, 1)$. For $y \in (0, 1)$ let

$$\tau_y = \inf\{t \geq 0 : |X_t| \geq y\}.$$

We claim that $\mathbb{P}[\tau_y < 1] = 1$. Indeed, put $F^*(y) = \inf\{F(x) : x \in [-y, y]\} > 0$ by positivity of F inside $(-1, 1)$. By the Dambis-Dubins-Schwarz Theorem, possibly enlarging our probability space, X is a time change of Brownian motion B, i.e. $X_t = B_{\langle X\rangle_t}$. Hence, for any $s < 1$ we have

$$\mathbb{P}[\tau_y > s] = \mathbb{P}[\max_{0\leq r\leq s} |X_r| < y] = \mathbb{P}[\max_{0\leq r\leq s} \left|B_{\langle X\rangle_r}\right| < y] \leq \mathbb{P}[\max_{0\leq r\leq s} \left|B_{\int_0^r \frac{1}{1-t} F^*(y)dt}\right| < y],$$

which goes to zero for s tending to 1. Hence, for any $y \in (0, 1)$ we have $\mathbb{P}[\tau_y < 1] = 1$ which implies by continuity of X that $\mathbb{P}[\tau_1 \leq 1] = 1$. This is in turn implies our claim. As a consequence of Corollary 6.6, X is a minimizer in (4.1).

Remark 7.5 (localization does not preserve optimality)**.** *We also notice that, differently from the classical optimal transport, "localization" does not preserve optimality, in general. Indeed, considering the previous example with $\mu = \delta_0$, by stopping X upon leaving the interval $(-y, y)$ will force $X_1^\tau \sim \frac{1}{2}(\delta_{-y} + \delta_y)$. However, $\langle \dot{X}_t^\tau\rangle$ is not induced by the corresponding friendly giant on $(-r, r)$, hence it cannot be optimal.*

8 Conclusion

In this article we have identified a class of Benamou–Brenier type martingale optimal transport problem via a weak length relaxation procedure of discrete martingale transport problems. By linking this class of optimization problems to the classical field of martingale problems, we establish an equivalent formulation gaining a complexity reduction, shifting the problem from stochastic processes to PDE's.

This approach as well as our results lead to a number of interesting questions that we leave for future work. The first one is whether a "strong relaxation" holds as described in Remark 4.2, which would provide a closer connection between the discrete- and continuous-time problems.

As a second problem, one could ask whether in duality (6.2) the supremum is actually a maximum, after a suitable relaxation of the notion of solution to (HJB), e.g. arguing with viscosity solutions. If this is true, then natural questions such as regularity of such optimal "potentials" should be addressed, leading to deeper understanding of the structure of optimizers of the primal problem.

Finally, in the one-dimensional case, by performing the change of variable $u = \Delta\varphi$, Theorem 6.3 seems to be equivalent to

$$\sup\left\{\int_{\mathbb{R}} u(1,x)\pi_\nu(x)\,\mathrm{d}x - \int_{\mathbb{R}} u(0,x)\pi_\mu(x)\,\mathrm{d}x\right\}, \tag{8.1}$$

the supremum running over all solutions to (PME), and $\pi_\varrho(x) := \int_{\mathbb{R}} |x-y|\,\mathrm{d}\varrho(x)$ being the one-dimensional Newtonian potential of a measure ϱ. Rigorously, such alternative duality is equivalent to Theorem 6.3 if we ask u to be a weak super-solution to (PME) in duality against convex functions only: is such an extremely weak notion to (PME) sufficient to obtain a reasonable theory of well-posedness? This alternative formulation of the dual problem would provide us with strong structural results, as it automatically takes the irreducible components into account (see [6] for the definition and importance of irreducible components for martingale transport, and [7, 8] for an application to duality).

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[2] J. Backhoff, M. Beiglboeck, M. Huesmann, and S. Källblad. Martingale Benamou–Brenier: A probabilistic perspective. *preprint*, 2017.

[3] V. Barbu, C. Benazzoli, and L. Di Persio. Mild solutions to the dynamic programming equation for stochastic optimal control problems. *arXiv:1706.06824*, 2017.

[4] M. Beiglböck, A. M. G. Cox, and M. Huesmann. Optimal transport and Skorokhod embedding. *Invent. Math.*, 208(2):327–400, 2017.

[5] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-Independent Bounds for Option Prices: A Mass Transport Approach. *Finance and Stochastics*, 17(3):477–501, 2013.

[6] M. Beiglböck and N. Juillet. On a Problem of Optimal Transport under Marginal Martingale Constraints. *Ann. Probab.*, 44(1):42–106, 2016.

[7] M. Beiglböck, M. Nutz, and N. Touzi. Complete Duality for Martingale Optimal Transport on the Line. *arXiv:1507.00671*, 2015. to appear in Ann. Probab.

[8] M. Beiglboeck, T. Lim, and J. Obłój. Dual attainment for the martingale transport problem. *arXiv:1705.04273*, 2017.

[9] J. Benamou and Y. Brenier. A numerical method for the optimal time-continuous mass transport problem and related problems. In *Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, volume 226 of *Contemp. Math.*, pages 1–11. Amer. Math. Soc., Providence, RI, 1999.

[10] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.

[11] J. F. Bonnans and X. Tan. A model-free no-arbitrage price bound for variance options. *Appl. Math. Optim.*, 68(1):43–73, 2013.

[12] Y. Brenier. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math.*, 52(4):411–452, 1999.

[13] E. Carlen and J. Maas. An Analog of the 2-Wasserstein Metric in Non-Commutative Probability Under Which the Fermionic Fokker–Planck Equation is Gradient Flow for the Entropy. *Communications in mathematical physics*, 331(3):887–926, 2014.

[14] H. De March and N. Touzi. Irreducible convex paving for decomposition of multi-dimensional martingale transport plans. *arXiv:1702.08298*, 2017.

[15] J. Dolbeault, B. Nazaret, and G. Savaré. A new class of transport distances between measures. *Calc. Var. Partial Differ. Equ.*, 34(2):193–231, 2009.

[16] J. Dolbeault, B. Nazaret, and G. Savaré. A new class of transport distances between measures. *Calc. Var.*, 34(2):193–231, February 2009.

[17] Y. Dolinsky and H. M. Soner. Martingale Optimal Transport and Robust Hedging in Continuous Time. *Probab. Theory Relat. Fields*, 160(1-2):391–427, 2014.

[18] Y. Dolinsky and H. M. Soner. Martingale Optimal Transport in the Skorokhod Space. *Stochastic Process. Appl.*, 125(10):3893–3931, 2015.

[19] L. Döring, L. Gonon, D. J. Prömel, and O. Reichmann. On Skorokhod Embeddings and Poisson Equations. *arXiv:1703.05673 [math]*, March 2017. arXiv: 1703.05673.

[20] M. Erbar. Gradient flows of the entropy for jump processes. *Annales de l'Institut Henri Poincaré-Probabilités et Statistiques*, 50(3):920–945, 2014.

[21] M. Erbar. A gradient flow approach to the boltzmann equation. *arXiv:1603.00540*, 2016.

[22] M. Erbar and J. Maas. Ricci curvature of finite markov chains via convexity of the entropy. *Archive for Rational Mechanics and Analysis*, pages 1–42, 2012.

[23] A. Figalli. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.*, 254(1):109–153, 2008.

[24] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[25] A. Galichon, P. Henry-Labordère, and N. Touzi. A Stochastic Control Approach to No-Arbitrage Bounds Given Marginals, with an Application to Lookback Options. *Ann. Appl. Probab.*, 24(1):312–336, 2014.

[26] N. Ghoussoub, Y.-H. Kim, and T. Lim. Structure of Optimal Martingale Transport in General Dimensions. *arXiv:1508.01806*, 2015.

[27] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, Berlin, 2011.

[28] M. Huesmann and F. Stebegg. Monotonicity preserving transformations of MOT and SEP. *arXiv:1609.04084*, 2016.

[29] J. Maas. Gradient flows of the entropy for finite markov chains. *Journal of Functional Analysis*, 261(8):2250–2292, 2011.

[30] R. J. McCann. *A convexity theory for interacting gases and equilibrium crystals*. PhD thesis, Princeton University, 1995.

[31] A. Mielke. A gradient structure for reaction–diffusion systems and for energy-drift-diffusion systems. *Nonlinearity*, 24(4):1329, 2011.

[32] A. Mielke. Geodesic convexity of the relative entropy in reversible markov chains. *Calculus of Variations and Partial Differential Equations*, 48(1-2):1–31, 2013.

[33] M. Nutz, F. Stebegg, and X. Tan. Multiperiod Martingale Transport. *arXiv:1703.10588*, 2017.

[34] J. Obłój. The Skorokhod embedding problem and its offspring. *Probab. Surv.*, 1:321–390, 2004.

[35] J. Obłój and P. Siorpaes. Structure of martingale transports in finite dimensions. *arXiv:1702.08433*, 2017.

[36] F. Otto. The geometry of dissipative evolution equations: The porous medium equation. *Commun. Partial Differ. Equations*, 26(1-2):101–174, 2001.

[37] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[38] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[39] V. Strassen. The existence of probability measures with given marginals. *Ann. Math. Statist.*, 36:423–439, 1965.

[40] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[41] X. Tan and N. Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab.*, to appear, 2011.

[42] D. Trevisan. Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. *Electron. J. Probab.*, 21:Paper No. 22, 41, 2016.

[43] J. L. Vázquez. *The porous medium equation: mathematical theory*. Oxford University Press, 2007.

[44] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.