MODEL-INDEPENDENT PRICING WITH INSIDER INFORMATION: A SKOROKHOD EMBEDDING APPROACH

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Abstract. In this paper, we consider the pricing and hedging of a financial derivative for an insider trader, in a model-independent setting. In particular, we suppose that the insider wants to act in a way which is independent of any modelling assumptions, but that she observes market information in the form of the prices of vanilla call options on the asset. We also assume that both the insider's information, which takes the form of a set of impossible paths, and the payoff of the derivative are time-invariant. This setup allows us to adapt recent work of Beiglböck, Cox, and Huesmann [BCH16] to prove duality results and a monotonicity principle, which enables us to determine geometric properties of the optimal models. Moreover, we show that this setup is powerful, in that we are able to find analytic and numerical solutions to certain pricing and hedging problems.

1. Introduction

It has long been recognised that information plays an extremely important role in the study of modern financial markets. This is most markedly true when two parties trading the same asset have access to different information sources, and then one can ask how the 'insider', who possesses additional information, should modify her behaviour to exploit her privileged position. Problems of this nature have a rich literature: the first work in the mathematical finance literature is Pikovsky and Karatzas [PK96], while important subsequent work includes [AIS98; BØ05; Cam05; GP98], and this topic is still a very active area of research.

In the past few years robust approaches to finance, where no underlying probability measure is assumed *a priori*, have become popular. Only very recently, additional/insider information has been considered in a robust framework. In both Acciaio and Larsson [AL15] and Aksamit, Hou, and Obłój [AHO16], the additional information is modelled by an enlargement of the filtration.

The goal of this paper is to consider the pricing and hedging problems for the insider in a continuous-time robust setting, where both the insider and the outsider (or uninformed agent) also observe future call prices at a fixed maturity T. Our analysis relies on two key assumptions. First, we only consider derivatives which are time invariant. Secondly, we assume that the insider's additional information is time invariant and such that it allows her to assume that a certain set of paths is impossible. The assumption of time invariance allows us to emulate the Skorokhod embedding (SEP) approach to

Date: October 28, 2016.

The authors are grateful for the excellent hospitality of the Hausdorff Research Institute for Mathematics (HIM), where the work was initiated. M.H. gratefully acknowledges the support of the CRC 1060 and financial support of the FWF-grant Y00782.

robust finance initiated in Hobson [Hob98] (see also [Hob11]), and includes examples such as lookback options, barrier options and variance options. The assumption on the form of the insider's information then translates the robust pricing problem into a constrained Skorokhod embedding problem, for which we are able to extend the framework developed in Beiglböck, Cox, and Huesmann [BCH16] for the unconstrained problem. In particular, by natural modifications of the results of [BCH16], we are able to prove our two main theoretical results which we believe to be of independent interest:

(i) a general superhedging and duality result for the insider/the constrained SEP (Theorem 3.3); and
(ii) a *monotonicity principle* for the insider/the constrained SEP (Theorem 3.8), which will lead to a systematic procedure by which extremal models may be constructed.

In particular, in Proposition 3.4, we are able to use Theorem 3.3 to provide simple necessary and sufficient conditions to exclude arbitrage for the insider in terms of solutions to the constrained SEP. The monotonicity principle will give a necessary condition, which the optimising probability measure for the insider/the constrained SEP must satisfy. The condition will take the form of *geometric conditions* on the support of the optimisers. In many cases, this condition turns out to be a characterizing condition leading to barrier type solutions, in a similar manner to the constructions of [BCH16]. In Section 4, we demonstrate this principle in the case of a variance option with insider information, which corresponds to a constrained Root-type embedding in the SEP setting. These ideas lead to many more examples in a similar manner to [BCH16, Section 6].

Moreover, in Section 4, we will show that it is possible not only to give abstract and general results in this framework, but also, partially relying on results of Cox, Obłój, and Touzi [COT15], to explicitly solve concrete examples, enabling us to compare the pricing and hedging problems for the insider and the outsider. Hence, we are able to calculate the value of the additional information in specific situations. More precisely:

(i) In Theorem 4.1 we link the question of arbitrage for the insider to easily checked conditions for certain types of information based on stopping times;
(ii) In Theorems 4.2 and 4.3 we show how, in certain special cases, Theorem 4.1 reduces the question of arbitrage for the insider to simple ordering properties of two explicit stopping times;
(iii) In Theorems 4.4 and 4.5, for the case of variance options, we demonstrate that one can explicitly construct an extremal model as well as an optimising superhedge;
(iv) In Section 4.2.2 we give numerical examples to show the impact of increasing information on the insider's primal and dual solutions (extremal model and superhedge). Thereby, we can nicely illustrate the impact of increasing information; see Figures 5, 6, and 7.

1.1. **Literature.** Problems involving financial insiders have a long history. In the economics literature the papers of Kyle [Kyl85] and Back [Bac92] are important early contributions which consider traders with privileged information. Subsequently there has been a significant literature on the problem, considering a wide range of related problems. In the mathematical finance literature, the usual approach is to consider the problem from the perspective of an insider trying to maximise utility.

In the robust approach to mathematical finance, the usual setting consists in having some assets available for dynamic trading, and some claims which are available at time zero for static, i.e. buy-and-hold, trading. The information at the disposal of the agent is the price of assets and claims at time zero, and the evolutions of the prices of the assets in time. In this framework, most of the literature so far has been devoted to showing pricing-hedging duality results, that is, that the minimal cost to superhedge pathwise a given derivative, equals its maximal price over calibrated martingale measures; see e.g. [ABPS16; BZ15; BHP13; BN15; FH16; Nut14] in discrete time, and [BCH16; BCHPP15; BNT15; BBKN15; DS14; DS15; GHT14; GTT15; HO15] in continuous time, among a rapidly growing literature.

The current literature on the insider problem in a robust setup is still in its infancy. The main relevant references are Acciaio and Larsson [AL15], and Aksamit, Hou, and Obłój [AHO16]. In both these papers, the additional information is modelled by an enlargement of the filtration. The informed agent has a richer information flow, which results in having more choices for trading strategies, and hence in cheaper robust (superhedging) prices. In [AL15] the authors study the models under which the market is complete in a semi-static sense, and through these models they compare the robust prices of the agents with and without additional information. In [AHO16], pricing-hedging duality results are given when the additional information is disclosed either at time zero or at a given future instant in time, and it is given by specific random variables.

Closer to the approach of the current work, are the papers by Cox, Hou, and Obłój [CHO16] and Hou and Obloj [HO15], even though they do not consider insider information. Both papers model beliefs in a robust setting by excluding some paths from the possible evolution of the asset's price process, thus specifying the set of feasible paths on which (super-)hedging arguments are required to work. In [CHO16] the authors consider a discrete-time setup and study pricing-hedging duality, showing that in some cases a duality gap may appear. In [HO15] a continuous-time setup is considered, and sufficient criteria are given so that duality holds asymptotically.

In the existing literature, the result which most closely resembles any of the conclusions of this paper is the duality result of [HO15], which is comparable to our main duality result. However, [HO15] consider derivatives with uniformly continuous payoff, so that the framework is orthogonal to the present one, where payoffs are assumed to be invariant to time change. Also in a similar spirit to our later results is the PhD thesis of Spoida [Spo14], which considers the situation where only finitely many options are available for static trading and, for specific kinds of derivatives, describes the optimal solutions for agents having beliefs on realised variance.

One notable contribution in this paper is that we introduce *constrained* Skorokhod Embedding problems (conSEP). To the best of our knowledge, the (conSEP) has not previously been systematically considered in the literature, and so there is little literature on this problem, although we believe that this constitutes an interesting new class of problems. The only papers which we are aware of which consider related problems are the papers [AS11] and [AHS15], who provide conditions under which a distribution may be embedded in Brownian motion or a diffusion in bounded time, which have some connections to the results in Section 4.

1.2. **Outline of the article.** In the present paper we will work in a continuous-time setup, under the assumption that the asset's price process S evolves continuously, and all call options for a given maturity T are traded at time zero in the market. By classical results ([BL78]), this is equivalent to knowing the marginal distribution, μ say, of S at time T under the pricing measure. In this context, calibration to market prices corresponds to considering probability measures on the path space such that the coordinate process S is a martingale and S_T has distribution μ. Emulating the analysis done in [BCH16] for the case without additional information, we perform a time-change to formulate the problem as a constrained optimal stopping problem in Wiener space, and resort to Skorokhod embedding techniques in order to study the primal (pricing) and dual (hedging) problems. For this approach to be effective, we need to restrict our attention to derivatives whose payoff function is invariant to time-changes in an appropriate sense. Similarly, the feasibility of paths needs to be invariant to time-changes, meaning that paths obtained as time-change of admissible paths, are still admissible. We will introduce the key concepts and definitions for this setup in Section 2.

In Section 3, we will prove that pricing-hedging duality holds for the informed agent, and we give the precise formulation both in the Wiener space, by relying on the analogous result in [BCH16], and in the original space, by extending results of [BCHPP15] for the case without additional information. Moreover, we show that the monotonicity principle of [BCH16] can be extended in a natural way to our setting, thus allowing us to give a geometric characterisation of the support of the optimisers in the primal problem. This turns out to be a powerful tool, enabling us to describe the optimal stopping times (and hence, up to time-change, the optimal models) as the hitting time of some barrier in a phase-space that depends on the derivatives' payoff.

Then, in Section 4, we consider some specific sets of information, for which we can give explicit characterisations of the solutions and compare the pricing and hedging problems for agents with and without additional information. Specifically, we first consider the case where feasibility corresponds to being allowed to stop after a given stopping time (the hitting time of a barrier in a given phase-space) and determine necessary and sufficient conditions for the additional information not to introduce arbitrage possibilities. We then show how our monotonicity principle can be used in order to derive explicit formulae for the primal optimisers for derivatives such as variance options. Under some technical assumptions we can also provide explicit solutions to the dual problem, using results of [COT15]. We are then able to determine the value of the additional information, in terms of the change in price of options. The examples are illustrated with some numerical evidence.

2. Informed robust pricing

Throughout the paper, for $I \subset \mathbb{R}$, we write $\mathcal{C}(I)$ for the space of continuous functions $\omega : I \to \mathbb{R}$ endowed with the topology of uniform convergence on compacts. When $I \subset [0, \infty)$, we write $\mathcal{C}_x(I)$ for the subset of paths such that $\omega(0) = x$.

We consider a market consisting of a risk-free asset (bond), whose price is normalised to 1, and a risky asset (stock) which is assumed to have a continuous price evolution, though neither the dynamics nor a reference probability is specified. The assets are continuously traded on the fixed time-horizon $[0,T]$, $0 < T < \infty$. Let the initial price of the stock be s_0; in this way we can think of the stock price process S as the canonical process on $\mathcal{C}_{s_0}[0,T]$. We assume we observe the prices of call options with maturity T for all strikes, which corresponds to having the knowledge of the marginal distribution of S at time T, say μ, under any pricing measure by the Breeden-Litzenberger formula, [BL78]. In particular, $\int x\mu(dx) = s_0$. We assume $\int (x-s_0)^2\, \mu(dx) =: V < \infty$. This condition is introduced in order to simplify the presentation, and can be relaxed (see e.g. [BCH16, Section 7]). Given a derivative with payoff function F written on S, the robust pricing problem is to determine

$$\sup\{\mathbb{E}_{\mathbb{Q}}[F(S)] : \mathbb{Q} \in \mathcal{M}(\mu)\}, \tag{2.1}$$

where $\mathcal{M}(\mu)$ is the set of all martingale measures $\mathbb{Q}$ on $\mathcal{C}_{s_0}[0,T]$ such that $S_T \sim_{\mathbb{Q}} \mu$. This leads to the upper price bound for the derivative F related to the worst case scenario. Analogously, one can consider the infimum in (2.1), that is, the lower price bound for F. Mathematically the maximisation and minimisation problems are very similar, and in this article we concentrate on the former.

In practice, often not only the prices of call options with given maturity are available, but an agent may have other information or beliefs relating to the evolution of the asset price. Incorporating this information may rule out certain behaviour of the stock price S, and hence certain models for S, which in turn leads to potentially smaller price bounds. We model this by introducing an *informed agent*, also called the *insider*, possessing some additional information which enables her to only consider a subset $\mathcal{A} \subseteq \mathcal{C}_{s_0}[0,T]$ of *feasible paths* for S (precise assumptions on $\mathcal{A}$ will be given in (2.5)). All other paths in $\mathcal{C}_{s_0}[0,T] \setminus \mathcal{A}$ are deemed negligible due to the additional information held by the insider. Hence the robust pricing problem for the insider is

$$P_{\mathcal{A}} := \sup\{\mathbb{E}_{\mathbb{Q}}[F(S)] : \mathbb{Q} \in \mathcal{M}(\mu),\, \mathbb{Q}(\mathcal{A}) = 1\}. \tag{2.2}$$

To give a value to the additional information, we will talk of the *uninformed agent* or *outsider* when considering an agent who does not have any other information than the call prices. Hence, the outsider's pricing problem is the classical robust pricing problem in (2.1), which corresponds to $\mathcal{A} = \mathcal{C}_{s_0}[0,T]$ in (2.2).

In the rest of this section, we recall and adapt the setup and results from [BCH16] and [BCHPP15] relying on [Vov15] which will allow us to formulate and analyze (2.2) as a *constrained Skorokhod embedding problem.* In order to do so, we will first introduce a time-change, in Section 2.1, that is a different clock under which we want to observe the paths of S. Next we will show that the pricing problem (2.2) has an equivalent formulation as an optimal stopping problem for Brownian motion on some probability space (problem (2.6)), when the derivative and the additional information are invariant with respect to this time-change. Finally, we shall pass from this weak formulation of the problem to an optimization problem on a single probability space, the Wiener space, which will require more general stopping rules; see problem (conSEP).

2.1. **Time transformation.** The key tool to translate (2.2) into a constrained Skorokhod embedding problem is the Dambis-Dubins-Schwarz Theorem. However, we need to be careful in defining the time change since we want to be able to shift pathwise inequalities from $\mathcal{C}_{s_0}[0,T]$ to the Wiener space and back. Moreover, the time change will be a useful tool to precisely define the options we want to consider as well as the set of feasible paths for the insider.

For $\omega \in \mathcal{C}(\mathbb{R}_+)$ and $n \in \mathbb{N}$, we define the sequence of times

$$\sigma_0^n(\omega) := 0, \quad \sigma_{k+1}^n(\omega) := \inf\{t > \sigma_k^n(\omega) : |\omega(t) - \omega(\sigma_k^n)| \geq 2^{-n}\}, \ k \in \mathbb{N},$$

and we say that ω has quadratic variation if the sequence $(V_n(\omega))_{n\in\mathbb{N}}$ of functions

$$V_n(\omega)(t) := \sum_{k=0}^{\infty} (\omega(\sigma_{k+1}^n \wedge t) - \omega(\sigma_k^n \wedge t))^2, \qquad t \in \mathbb{R}_+$$

converges uniformly on compacts to some function in $\mathcal{C}_0(\mathbb{R}_+)$, that has the same intervals of constancy as ω. We denote this function by $\langle\omega\rangle$. We write Ω^{qv} for the space of all paths ω in $\mathcal{C}_{s_0}(\mathbb{R}_+)$ possessing such a quadratic variation process and such that either $\langle\omega\rangle$ diverges at infinity or $\langle\omega\rangle$ is bounded and ω has a well defined limit at infinity. These conditions are necessary in order for the map ntt given below to be well defined. It is not hard to show that Ω^{qv} is a measurable subset of $\mathcal{C}(\mathbb{R}_+)$.

We define the space of *stopped paths* as

$$\mathcal{S} := \{(f, s) : f \in \mathcal{C}_{s_0}[0, s], s \in \mathbb{R}_+\},$$

and equip it with the distance $d_{\mathcal{S}}$ defined for $s < t$ by

$$d_{\mathcal{S}}((f,s),(g,t)) = \max\Big\{t - s, \sup_{0\leq u\leq s} |f(u) - g(u)|, \sup_{s\leq u\leq t} |g(u) - f(s)|\Big\},$$

which turns $\mathcal{S}$ into a Polish space. The space $\mathcal{S}$ is a convenient way of encoding optionality of a process in our pathwise setup, e.g. [DM78, Theorem IV. 97] (using different notation). Note that optionality is equivalent to predictability, since we consider only continuous paths. More precisely, we put

$$r : \mathcal{C}_{s_0}(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathcal{S}, \quad (\omega, t) \mapsto (\omega|_{[0,t]}, t),$$

where $\omega|_{[0,t]}$ denotes the restriction of ω to $[0,t]$. Then a process X with $X_0 = s_0$ is optional if and only if there is a Borel function $H : \mathcal{S} \to \mathbb{R}$ such that $X = H \circ r$.

Denote by $B = (B_t)_{t\in\mathbb{R}_+}$ the coordinate process on Ω^{qv}, $B_t(\omega) = \omega(t)$, and by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the natural filtration given by $\mathcal{F}_t = \sigma(B_s : s \leq t)$, and set $\mathcal{F} = \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$. We call Ω_T^{qv} the set of paths in $\mathcal{C}_{s_0}[0,T]$ which have a continuation in Ω^{qv}, and for such paths we define the following new clock:

$$\tau_t(\omega) = \inf\{s \geq 0 : \langle\omega\rangle_s > t\}, \quad t \in \mathbb{R}_+. \tag{2.3}$$

We will work with the *normalising time transformation* introduced by Vovk [Vov12], which is defined by $\mathrm{ntt}_T : \Omega_T^{\mathsf{qv}} \to \mathcal{S}$ given by

$$\mathrm{ntt}_T(\omega) = ((\omega_{\tau_t})_{t\leq\langle\omega\rangle_T}, \langle\omega\rangle_T).$$

That is, $\mathrm{ntt}_T(\omega)$ is a version of the path ω run at a speed such that, for every t, its pathwise quadratic variation at time t is exactly t. It will also be notationally useful at times to 'forget' the time component, and consider the function $\mathrm{ntt}(\omega)$, which is equal to $(\omega_{\tau_t})_{t\leq\langle\omega\rangle_T} \in \mathcal{C}_{s_0}[0, \langle\omega\rangle_T]$. Of course, the two quantities are mathematically equivalent. The normalising time transformation will be the tool that will allow us to define the class of time-invariant derivatives and the kind of time-invariant additional information which are suitable in order to develop the SEP approach to robust pricing with insider information.

Remark 2.1. *Note that, for each $\mathbb{Q} \in \mathcal{M}(\mu)$, the support of $\mathbb{Q}$ is a.s. contained in Ω_T^{qv} (see Karandikar [Kar95]; Vovk [Vov12]), hence it will be sufficient to consider paths in this set when studying the pricing problem* (2.2)*.*

In this article, we consider payoff functions $F : \mathcal{C}_{s_0}[0,T] \to \mathbb{R}$ which on Ω_T^{qv} satisfy

$$F = \gamma \circ \mathrm{ntt}_T, \tag{2.4}$$

for some Borel measurable $\gamma : \mathcal{S} \to \mathbb{R}$. This means that the payoff function F is identical for all paths which are time-transformations of each other, that is, which coincide after normalising the speed at which they run.

A key additional component will be the information which is held by the insider, and which is not known to the market. We will model this by assuming that the insider knows a set of feasible paths $\mathcal{A} \subseteq \mathcal{C}_{s_0}[0,T]$. Thanks to Remark 2.1, we may assume without loss of generality that $\mathcal{A} \subseteq \Omega_T^{\mathsf{qv}}$. As with the payoff function, we will assume that the set $\mathcal{A}$ of feasible paths is time-invariant. More precisely, we will consider sets $\mathcal{A}$ given by

$$\mathcal{A} = \mathrm{ntt}_T^{-1}(\Lambda), \tag{2.5}$$

for some measurable subset $\Lambda \subset \mathcal{S}$, so that $1_{\mathcal{A}}(\omega) = 1_\Lambda \circ \mathrm{ntt}_T(\omega)$. We will call Λ the *feasibility set*. In this way, feasibility of a path $\omega \in \mathcal{C}_{s_0}[0,T]$ is shifted to admissibility of stopping the path $\mathrm{ntt}(\omega)$ at $\langle \omega \rangle_T$. This means that, if a path $\omega \in \mathcal{C}_{s_0}[0,T]$ is feasible, so is any other path which is a time transformation of ω.

2.2. **Informed robust pricing as constrained SEP.** The time transformation introduced above enables us to express the robust pricing problem (2.2) as a constrained optimal stopping problem for Brownian motion. To show this we follow [BCHPP15, Section 4]. Let $(\tau_t)_{t\in\mathbb{R}_+}$ be the time-change defined in (2.3), and $(\mathcal{F}_t^S)_{t\in[0,T]}$ the usual augmentation of the filtration generated by $(S_t)_{t\in[0,T]}$. It is easy to verify that $\langle S \rangle_T$ is a stopping time with respect to the filtration $(\mathcal{F}^S_{\tau_t\wedge T})_{t\in\mathbb{R}_+}$. Then, for any $\mathbb{Q} \in \mathcal{M}(\mu)$, the Dambis-Dubins-Schwarz theorem implies that the process $(X_t)_{t\in\mathbb{R}_+} = (\mathrm{ntt}(S)_{t\wedge\langle S\rangle_T})_{t\in\mathbb{R}_+}$ is a stopped Brownian motion under $\mathbb{Q}$ in the filtration $(\mathcal{F}^S_{\tau_t\wedge T})_{t\in\mathbb{R}_+}$. Moreover, it is uniformly integrable and satisfies $\mathrm{ntt}(S)_{\langle S\rangle_T} \sim \mu$. *Vice versa*, let W be a Brownian motion on some probability space $(\tilde{\Omega}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, and τ be a stopping time such that $W_{\cdot\wedge\tau}$ is uniformly integrable with $W_\tau \sim \mu$. Then, for $M = (M_t)_{t\in[0,T]}$ defined by $M_t := W_{\frac{t}{T-t}\wedge\tau}$, we have that $\mathbb{P}\circ M^{-1} \in \mathcal{M}(\mu)$. Hence, we have shown the following result.

Proposition 2.2. *Let F and $\mathcal{A}$ satisfy* (2.4) *and* (2.5)*. The pricing problem for the insider* (2.2) *can be formulated as*

$$P_\Lambda^* := \sup\left\{ \mathbb{E}[\gamma((W_t)_{t\leq\tau},\tau)] : \begin{array}{l} (\tilde{\Omega}, (\mathcal{G}_t)_{t\geq 0}, \mathcal{G}, \mathbb{P}) \textit{supporting Brownian motion } W, \\ W_0 = s_0, \tau \textit{ a } \mathcal{G}\textit{-stopping time s.t. } W_\tau \sim \mu, \\ (W_{t\wedge\tau})_{t\geq 0} \textit{ is u.i., and } \mathbb{E}[1_\Lambda((W_t)_{t\leq\tau},\tau)] = 1 \end{array} \right\}. \tag{2.6}$$

The condition $\mathbb{E}[1_\Lambda((W_t)_{t\le\tau},\tau)] = 1$ means that, when moving along a path of W, we can stop only at times such that the stopped path lies in Λ. This corresponds to the fact that informed agents only need to take into account the paths in the feasibility set, Λ.

To be able to analyse the optimisation problem (2.6) we introduce another optimisation problem living on a single probability space, the Wiener space $(\mathcal{C}_{s_0}(\mathbb{R}_+), \mathcal{F}, \mathbb{W})$. To this end we consider the set

$$M = \{\xi \in \mathcal{P}^{\le 1}(\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+) : \xi(d\omega, dt) = \xi_\omega(dt)\mathbb{W}(d\omega), \xi_\omega \in \mathcal{P}^{\le 1}(\mathbb{R}_+) \text{ for } \mathbb{W}\text{-a.e. } \omega\},$$

where $(\xi_\omega)_{\omega\in\mathcal{C}_{s_0}(\mathbb{R}_+)}$ is a disintegration of ξ with respect to the first coordinate ω. We equip M with the topology induced by the continuous bounded functions on $\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+$. Each $\xi \in M$ can be uniquely characterised by the cumulative distribution function $A^\xi(\omega,t) = \xi_\omega[0,t]$.

Definition 2.3. *We say that a measure $\xi \in M$ is a* randomised stopping time *if the corresponding increasing process A^ξ is optional, and write $\xi \in \mathsf{RST}$. For an optional process $X : \mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+ \to \mathbb{R}_+$ and $\xi \in \mathsf{RST}$ we define X_ξ as the pushforward of ξ under the mapping $(\omega,t) \mapsto X_t(\omega)$. We denote by $\mathsf{RST}(\mu)$ the set of all randomised stopping times such that $B_\xi = \mu$ and $\int t\ \xi(d\omega,dt) < \infty$.*

Considering the martingale $B_t^2 - t$ it follows from classical results on stopping times (e.g. [Hob11, Corollary 3.3], [BCH16, Lemma 3.12]) that, for $\xi \in \mathsf{RST}$ with $B_\xi = \mu$, the condition $\int t\ \xi(d\omega,dt) < \infty$ is equivalent to

$$\int t\ \xi(d\omega,dt) = \int (x-s_0)^2\ \mu(dx) = V, \tag{2.7}$$

which is assumed to be finite in our setup.

By [BCH16, Theorem 3.14], $\mathsf{RST}(\mu)$ is non-empty and compact with respect to the topology induced by the continuous and bounded functions on $\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+$. As a direct consequence we get

Corollary 2.4. *Let $\Lambda \subseteq \mathcal{S}$ be closed. Then the set of feasible randomised stopping times*

$$\mathsf{RST}(\mu;\Lambda) := \left\{\xi \in \mathsf{RST}(\mu) : \int_{\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+} 1_\Lambda \circ r(\omega,t)\xi(d\omega,dt) = 1\right\} \tag{2.8}$$

is convex and compact with respect to the topology induced by the continuous and bounded functions on $\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+$.

We highlight here the important feature that $\mathsf{RST}(\mu;\Lambda)$ might be empty, which can be understood as a robust arbitrage opportunity, see Propostion 3.4 and Section 4.

Proof. Since Λ is assumed to be closed, $\int_{\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+} 1_\Lambda \circ r(\omega,t)\xi(d\omega,dt) = 1$ is a closed condition. □

Another important property of the feasible randomised stopping times is that they are precisely the joint distributions on $\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+$ of pairs (W,τ) satisfying the constraints in (2.6). This is a straighforward extension of [BCH16, Lemma 3.11]. Putting everything together we have derived a formulation of our optimisation problem (2.2) resp. (2.6) on the Wiener space as a *constrained Skorokhod embedding problem*:

Proposition 2.5. *In the setting described above,*

$$P^*_\Lambda = \sup\left\{\int_{\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+} \gamma\circ r(\omega,t)\xi(d\omega,dt) : \xi\in\mathsf{RST}(\mu;\Lambda)\right\}. \tag{conSEP}$$

We will say that (conSEP) is well posed if $\int_{\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+}\gamma\circ r\; d\xi$ exists with values in $[-\infty,\infty)$ for all $\xi\in\mathsf{RST}(\mu;\Lambda)$ and is finite for one such ξ. In particular, (conSEP) is not well posed if $\mathsf{RST}(\mu;\Lambda)=\emptyset$ which has a pleasing financial interpretation (cf. Theorem 4.1).

From an analytical point of view, the formulation (conSEP) is extremely useful since we are now dealing with a linear optimisation problem over a convex and compact set on a single probability space. A direct consequence is the following result:

Theorem 2.6. *Let $\gamma:\mathcal{S}\to\mathbb{R}$ be upper semi-continuous and bounded from above in the sense that, for some constants $a,b,c\in\mathbb{R}_+$,*

$$\gamma((\omega(s))_{s\le t},t)\le a+bt+c\sup_{s\le t}\omega(s)^2,\qquad (\omega,s)\in\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+. \tag{2.9}$$

Assume that $\Lambda\subseteq\mathcal{S}$ is closed and that $\mathsf{RST}(\mu;\Lambda)$ is non-empty. Then the optimisation problem (conSEP) *admits a maximiser.*

Proof. We claim that without loss of generality we can assume that γ is bounded from above. Indeed, by the pathwise version of Doob's inequality (see [ABPST13]),

$$\sup_{s\le t}\omega(s)^2\le M_t+4\omega(t)^2$$

for some martingale M_t starting in zero. Hence condition (2.9) implies that

$$\tilde\gamma\circ r(\omega,t):=\gamma\circ r(\omega,t)-a'-b't-c'(M_t+\omega(t)^2)$$

is bounded from above and the term $\int a'-b't-c'(M_t+\omega(t)^2)\;d\xi$ is independent of $\xi\in\mathsf{RST}(\mu)$ by (2.7) and the assumed second moment of μ. Therefore, we can assume γ to be bounded from above.

Finally, since $\mathsf{RST}(\mu;\Lambda)$ is compact and by the Portmanteau Theorem the map $\xi\mapsto\int\gamma\circ r\;d\xi$ is upper semi-continuous, we deduce the result. ☐

3. Super-replication and monotonicity principle

In this section, we establish our main abstract results, the duality or superhedging result, and the geometric characterisation of primal optimisers, the monotonicity principle for contrained Skorokhod embedding.

3.1. **Duality.**

The following duality result for the outsider was established in [BCH16, Theorem 4.2].

Theorem 3.1. *Let γ be upper semi-continuous and bounded from above in the sense of* (2.9)*. Set*

$$D^*_\mathcal{S}=\inf\left\{\int\psi d\mu:\begin{array}{l}\psi\in\mathcal{C}(\mathbb{R}),\exists\text{ an }\mathcal{S}\text{-continuous martingale }\phi,\ \phi_0=0\ s.t.\\ \phi_t(\omega)+\psi(\omega(t))\ge\gamma\circ r(\omega,t),\ (\omega,t)\in\mathcal{C}_{s_0}(\mathbb{R}_+)\times\mathbb{R}_+\end{array}\right\},$$

where ϕ, ψ satisfy $|\phi_t| \leq a + bt + cB_t^2$, $|\psi(y)| \leq a + by^2$ for some $a, b, c > 0$. Then we have

$$P_{\mathcal{S}}^* = D_{\mathcal{S}}^* \; .$$

A martingale ϕ is called $\mathcal{S}$-continuous if there exists a continuous $H : \mathcal{S} \to \mathbb{R}$ such that $\phi = H \circ r$. Note that any $\mathcal{S}$-continuous martingale is continuous but the other direction is in general not correct.

Theorem 3.2. *Let γ be upper semi-continuous and bounded from above in the sense of* (2.9) *and $\Lambda \subset \mathcal{S}$ be closed. Set*

$$D_\Lambda^* = \inf \left\{ \int \psi d\mu : \begin{array}{l} \psi \in \mathcal{C}(\mathbb{R}), \exists \text{ an } \mathcal{S}\text{-continuous martingale } \phi,\ \phi_0 = 0 \text{ s.t.} \\ \phi_t(\omega) + \psi(\omega(t)) \geq \gamma \circ r(\omega, t) \text{ for all } (\omega, t) \in r^{-1}(\Lambda) \end{array} \right\}, \tag{3.1}$$

where ϕ, ψ satisfy $|\phi_t| \leq a + bt + cB_t^2$, $|\psi(y)| \leq a + by^2$ for some $a, b, c > 0$. Then we have

$$P_\Lambda^* = D_\Lambda^* \; .$$

The proof will follow from Theorem 3.1 by an application of a Minmax Theorem.

Proof of Theorem 3.2. As in the proof of Theorem 2.6 we can assume γ to be bounded from above.

We claim that $D_\Lambda^* = \tilde{D}_\Lambda$ where $\tilde{D}_\Lambda$ is defined as

$$\inf \left\{ \int \psi d\mu : \psi \in \mathcal{C}(\mathbb{R}), \begin{array}{l} \exists \text{ an } \mathcal{S}\text{-continuous martingale } \phi, \phi_0 = 0, \alpha \geq 0, \\ \phi_t(\omega) + \psi(\omega(t)) - \alpha(1_\Lambda \circ r(\omega, t) - 1) \geq \gamma \circ r(\omega, t), \\ (\omega, t) \in \mathcal{C}_{s_0}(\mathbb{R}_+) \times \mathbb{R}_+ \end{array} \right\},$$

where ϕ, ψ are bounded.

It is immediate that $\tilde{D}_\Lambda \geq D_\Lambda^*$. To see the other direction, take (ψ, ϕ) satisfying the constraints in (3.1). By approximation, we can assume that $\phi + \psi \geq -c_1$ for some $c_1 \in \mathbb{R}_+$. Since γ is bounded from above by say c_2 it follows that

$$\phi_t(\omega) + \psi(\omega(t)) \geq \gamma \circ r(\omega, t) + (c_1 + c_2)(1_\Lambda(r(\omega, t) - 1),$$

implying (3.1). Hence, it is sufficient to show that $P_\Lambda^* = \tilde{D}_\Lambda$:

$$\begin{aligned}
P_\Lambda^* &= \sup \left\{ \int \gamma(\omega, t) \xi(d\omega, dt) : \xi \in \mathsf{RST}(\mu),\ \int 1_\Lambda \circ r(\omega, t) \xi(d\omega, dt) = 1 \right\} \\
&= \sup_{\xi \in \mathsf{RST}(\mu)} \inf_{\alpha \geq 0} \int \gamma(\omega, t) + \alpha(1_\Lambda \circ r(\omega, t) - 1) \xi(d\omega, dt) \\
&= \inf_{\alpha \geq 0} \sup_{\xi \in \mathsf{RST}(\mu)} \int \gamma(\omega, t) + \alpha(1_\Lambda \circ r(\omega, t) - 1) \xi(d\omega, dt) \\
&= \inf_{\alpha \geq 0} \inf \Big\{ \int \psi d\mu : \psi \in \mathcal{C}(\mathbb{R}),\ \exists \text{an } \mathcal{S}\text{-cont. mart. } \phi,\ \phi_0 = 0 \text{ s.t.} \\
&\qquad \phi_t(\omega) + \psi(\omega(t)) \geq \gamma \circ r(\omega, t) + \alpha(1_\Lambda \circ r(\omega, t) - 1) \text{ for all } (\omega, t) \in \mathcal{C}_{s_0}(\mathbb{R}_+) \times \mathbb{R}_+ \Big\} \\
&= \tilde{D}_\Lambda,
\end{aligned}$$

where we used the Minmax Theorem (see e.g. [Str85, Thm. 45.8], [AH96, Thm. 2.4.1] or [BCH16, Theorem 4.5]) in the third step, and Theorem 3.1 in the fourth step. □

This duality result is already of interest in its own right. However, to identify it as a superreplication result we need to recover the hedging strategies corresponding to the martingale. For this we need some kind of pathwise martingale representation theorem. In fact Theorem 6.2 of [Vov12] can be interpreted as such. Then, following the line of reasoning as in [BCHPP15] one can recover the hedging strategies to get the following result (for a proof of a basically identical result we refer to [BCHPP15]). We recall that $F = \gamma \circ \mathrm{ntt}_T$, and $\mathcal{A} = \mathrm{ntt}_T^{-1}(\Lambda)$.

Theorem 3.3. *Let γ be upper semi-continuous and bounded from above in the sense of* (2.9)*, and let $\Lambda \subset \mathcal{S}$ be closed. Set*

$$D_{\mathcal{A}} = \inf \left\{ \int \psi(y)\, d\mu(y) : \begin{array}{l} \psi \in \mathcal{C}(\mathbb{R}), \exists \text{ simple strategies } (H^n)_n \text{ s.t.} \\ \liminf_n (H^n \cdot S)_T(\omega) + \psi(\omega(T)) \geq F(\omega) \text{ for all } \omega \in \mathcal{A} \end{array} \right\},$$

where $|\psi(y)| \leq a + by^2$ and $(H^n \cdot S)_t \geq -a - bt$ for some $a, b > 0$ and all $t \in [0,T]$. Then, we have

$$P_{\mathcal{A}} = D_{\mathcal{A}}.$$

Theorem 3.3 is the analogue of the classical super-replication duality theorem, in the present robust insider setting. Moreover, like its classical counterpart, it additionally allows to give a version of the first fundamental theorem of asset pricing.

Proposition 3.4. *The following are equivalent:*

(i) $\exists\ \mathbb{Q} \in \mathcal{M}(\mu)$ *such that* $\mathbb{Q}(\mathcal{A}) = 1$*;*
(ii) $\mathsf{RST}(\mu; \Lambda) \neq \emptyset$*;*
(iii) $\nexists\ \varepsilon > 0$, *simple strategies* $(H^n)_n$, $\psi \in \mathcal{C}(\mathbb{R})$ *with* $\int \psi\, d\mu = 0$ *such that*

$$\liminf (H_n \cdot S)_T(\omega) + \psi(\omega(T)) \geq \varepsilon, \text{ for all } \omega \in \mathcal{A}. \tag{3.2}$$

Property (iii) means that one cannot make arbitrary profits by starting with zero capital. Indeed, if (3.2) holds for some $\hat{\varepsilon} > 0$, then so it does for any $\varepsilon > 0$.

Proof. The equivalence between (i) and (ii) follows from the arguments at the beginning of Section 2.
$(i) \Rightarrow (iii)$: Note that (i) implies $D_{\mathcal{A}} = P_{\mathcal{A}} = 0$ for any derivative F_0 s.t. $F_0 = 0$ on $\mathcal{A}$, by Theorem 3.3. Now suppose, by way of contradiction, that there exist $\varepsilon, (H^n)_n$ and ψ s.t. (3.2) is satisfied. Then $D_{\mathcal{A}} \leq -\varepsilon$ for F_0, which gives the desired contradiction.
$(iii) \Rightarrow (i)$: By Theorem 3.3, if there is no measure $\mathbb{Q} \in \mathcal{M}(\mu)$ such that $\mathbb{Q}(\mathcal{A}) = 1$, then $D_{\mathcal{A}} = P_{\mathcal{A}} = -\infty$ for all F. In particular, there exists ψ, H^n with $\int \psi\, d\mu = 0$ such that (3.2) holds. ☐

Remark 3.5. *In this paper, we have only considered the case where the option depends only on the path up to time T, and is invariant to time change. Using similar methods to those developed in [BCHPP15], it is also possible to extend Theorems 3.2 and 3.3 to the case where the options depend also on the value of the asset at times $0 \leq s_1 \leq s_2 \leq \cdots \leq s_N = T$, and provide a related formulation in the Brownian setup where the optimisation is over a sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N = \tau$. In this case, it is possible to consider both the cases where call price information fixes the distributions at the intermediate times, or it does not, or there is a mixture of some times having this information, and others lacking it. In this more general setup, it* *becomes possible to include a large class of options, for example, discretely monitored Asian options could be included in this setting.*

3.2. **Constrained monotonicity principle.** In this section, we provide a modified version of the monotonicity principle of [BCH16] giving necessary geometric conditions on the support set of an optimiser to (conSEP).

To this end, we denote the concatenation of two paths $(f,s),(g,t) \in \mathcal{S}$ by $f \oplus g$, i.e.

$$f \oplus g(u) = \begin{cases} f(u) & u \le s, \\ f(s) + g(u-s) - g(0) & s \le u \le t. \end{cases}$$

For $(f,s) \in \mathcal{S}$ we define the process $\gamma_t^{(f,s)\oplus}(\omega) := \gamma(f \oplus \omega|_{[0,t]}, s+t)$.

Definition 3.6. *A pair $((f,s),(g,t)) \in \mathcal{S} \times \mathcal{S}$ is called* feasible stop-go pair*, written $((f,s),(g,t)) \in \mathsf{SG}_\Lambda$, if $f(s) = g(t)$, $(f,s) \in \Lambda$ and*

$$\mathbb{E}[\gamma^{(f,s)\oplus}((W_u)_{u \le \sigma}, \sigma)] + \gamma(g,t) < \gamma(f,s) + \mathbb{E}[\gamma^{(g,t)\oplus}((W_u)_{u \le \sigma}, \sigma)] \ , \tag{3.3}$$

for every $(\mathcal{F}_t^W)_{t \ge 0}$ stopping time σ satisfying $0 < \mathbb{E}[\sigma] < \infty$ and $1_\Lambda \circ r(g \oplus W, t+\sigma) = 1$ a.s. where both sides of (3.3) are well defined and the right hand side is finite. Here, the probability space is assumed to be rich enough to support a Brownian motion W and an uniform random variable independent of W, and $(\mathcal{F}_t^W)_{t \ge 0}$ denotes the natural filtration generated by W.

The interpretation is that on average it is better to stop (f,s) and to let (g,t) go as long as this results in a feasible stopping rule. We remark here that – as a consequence of only considering $(\mathcal{F}_t^W)_{t \ge 0}$ stopping times – the definition of feasible stop-go pairs is independent of the probability space on which σ lives as long as it is rich enough to support the Brownian motion W and a uniform random variable independent of W. This is reminiscent of the fact that we have to consider randomised stopping times in (conSEP).

Denote by $\Gamma^<$ the set of all stopped paths which have a proper extension in Γ:

$$\Gamma^< := \{(f,s) \in \mathcal{S} : \exists (g,t) \in \Gamma,\ s < t,\ g|_{[0,s]} = f\}.$$

Definition 3.7. *A set $\Gamma \subset \Lambda$ is called* feasible γ-monotone *if*

$$\mathsf{SG}_\Lambda \cap (\Gamma^< \times \Gamma) = \emptyset \ .$$

Theorem 3.8. *Let $\gamma : \mathcal{S} \to \mathbb{R}$ be Borel. Assume that* (conSEP) *is well posed and that $\xi \in \mathsf{RST}(\mu;\Lambda)$ is an optimiser. Then, there exists a feasible γ-monotone set $\Gamma \subset \mathcal{S}$ such that*

$$\xi(r^{-1}(\Gamma)) = 1.$$

Proof. Put

$$\overline{\gamma}(f,t) = \begin{cases} \gamma(f,t) & (f,t) \in \Lambda, \\ -\infty & (f,t) \in \mathcal{S} \setminus \Lambda, \end{cases}$$

and note that the well posedness assumption of (conSEP) for γ implies that (conSEP) is still well posed for $\overline{\gamma}$ and $\Lambda = \mathcal{S}$. Hence, the result follows from [BCH16, Theorem 5.7]. $\square$

4. No-arbitrage, pricing and hedging in specific information settings

In this section we consider some natural choices for the insider's information set, Λ, and show that under additional assumptions, we are able to prove a variety of new results about the insider's setting. Notably, we are able to exploit our setup and the richness of the Brownian framework to give examples where we can provide explicit characterisation of the solutions, and further our understanding of the market behaviour in different scenarios.

We use the notation $\prec$ to denote convex order relation between probability measures; specifically, we say that $\lambda \prec \mu$ if $\int c(x)\lambda(dx) \leq \int c(x)\mu(dx)$ for any convex function c.

In the examples we consider, we will typically address two related questions:

(1) Given a pair (μ, Λ), when does there exist any consistent model for the insider agent? Specifically, is $\mathsf{RST}(\mu; \Lambda)$ non-empty? We address these points in Theorems 4.1, 4.2, and 4.3.
(2) Given a pair (μ, Λ) such that $\mathsf{RST}(\mu; \Lambda) \neq \emptyset$, and a derivative with payoff F, what is the value of P^*_Λ, and how does this differ from $P^*_{\mathcal{S}}$, the price of the uninformed agent? If the insider were to superhedge the option, how would this differ from the hedge of the uninformed agent? Can we determine the 'value' for the insider of this information? We consider these questions in Section 4.2.

In investigating the questions above, we will consider the three following natural examples where the additional information translates into stopping the Brownian motion after and/or before given stopping times. Let $\underline{\tau}, \overline{\tau}$ be stopping times such that $\underline{\tau} \leq \overline{\tau}$, $(B_{t\wedge\underline{\tau}})_{t\geq 0}, (B_{t\wedge\overline{\tau}})_{t\geq 0}$ are uniformly integrable, and consider the sets

$$\Lambda_1 = \{r(\omega, t) : t \leq \overline{\tau}\},\ \Lambda_2 = \{r(\omega, t) : t \geq \underline{\tau}\},\ \Lambda_3 = \{r(\omega, t) : \underline{\tau} \leq t \leq \overline{\tau}\}. \tag{4.1}$$

As a consequence of Strassen's Theorem [Str65] with the notation $B_{\underline{\tau}} \sim \underline{\mu}, B_{\overline{\tau}} \sim \overline{\mu}$, a solution to the constrained problem (2.6) exists for Λ_2 if and only if $\underline{\mu} \prec \mu$. Similarly, in the case of Λ_1, then the condition $\mu \prec \overline{\mu}$ is a necessary condition for the existence of a stopping time $\tau \leq \overline{\tau}$ for the Brownian motion such that $B_\tau \sim \mu$, but it is not sufficient unless $\overline{\mu}$ is supported on two points; this result is due to Meilijson [Mei82] and van der Vecht [Vec86]. (A simple example can be constructed by considering the measures $\overline{\mu} = N(0, 1)$, with stopping time $\overline{\tau} = 1$ and $\mu = \frac{\varepsilon}{2}(\delta_1 + \delta_{-1}) + (1 - \varepsilon)\delta_0$. For ε sufficiently small, it is easily checked that $\mu \prec \overline{\mu}$, but there is no bounded stopping time embedding μ.) To the best of our knowledge, necessary and sufficient conditions for the existence of $\xi \in \mathsf{RST}(\mu; \Lambda_1)$ are unknown. We are able to provide them in specific settings (see Section 4.1), while the existence of general criteria remains an interesting open problem.

We summarise these results as the following theorem.

Theorem 4.1. *Suppose that the insider has information given by* (4.1). *Then:*

(1) $\Lambda = \Lambda_1$*: the set* $\mathsf{RST}(\mu; \Lambda) = \emptyset$ *if* $\mu \not\prec \overline{\mu}$*;*
(2) $\Lambda = \Lambda_2$*: the set* $\mathsf{RST}(\mu; \Lambda) = \emptyset$ *if and only if* $\underline{\mu} \not\prec \mu$*;*
(3) $\Lambda = \Lambda_3$*: the set* $\mathsf{RST}(\mu; \Lambda) = \emptyset$ *if* $\underline{\mu} \not\prec \mu$ *or* $\mu \not\prec \overline{\mu}$*.*

In particular, if any of the conditions on the measures $\underline{\mu}, \mu, \overline{\mu}$ *above hold, then the insider can make unlimited profit in the sense of* (3.2).

It is clear that, in general, not all information processes are of the form (4.1). However the constraint $\tau \in \mathsf{RST}(\mu)$ may impose additional conditions that are not immediate from the construction of Λ. Consider for example the case where

$$\Lambda = \Big\{(f,t) \in \mathcal{S} : \sup_{s \le t} f(s) - c \le f(t)\Big\}, \tag{4.2}$$

for some fixed $c \in \mathbb{R}_+$. Minimality implies that an admissible stopping time must occur before $\overline{\tau} := \inf\{t \ge 0 : \sup_{s \le t} \omega(s) - c > \omega(t)\}$, by a simple martingale argument. Hence, although there exist feasible paths in Λ which live longer than $\overline{\tau}$, any τ which is in $\mathsf{RST}(\mu)$ must, with probability one, be before $\overline{\tau}$. Therefore, the set of feasible stopped paths in this case is $\Lambda' = \{r(\omega, t) : t \le \overline{\tau}\}$. Then, from the argument above, $\mu \prec \overline{\mu} \sim B_{\overline{\tau}}$ must hold in order to have a solution to the constrained embedding problem. The fact that the measures μ and $\overline{\mu}$ are in convex order does not imply, per se, that there is an order between the respective barycenter functions, b_μ and $b_{\overline{\mu}}$. However, we will show that in order to have an embedding by stopping before $\overline{\tau}$, the relation $b_\mu \le b_{\overline{\mu}}$ must hold; see the discussion after Theorem 4.2.

On the other hand, if the set of admissible evolutions for the asset is

$$\Lambda = \Big\{(f,t) \in \mathcal{S} : \sup_{s \le t} f(s) - c \le \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s\Big\},$$

then we are not able to give an equivalent set $\Lambda' \subseteq \Lambda$ as above with a 'nice' form. Here the class of admissible stopping times is certainly bounded above by a stopping time $(\inf\{t \ge 0 : \max\{t^{-1}\int_0^t f(s)\,\mathrm{d}s, f(t)\} \le \sup_{s \le t} f(s) - c\})$, but it is easily seen that there are inadmissible paths which occur before this time.

In what follows we will consider the cases in Theorem 4.1 separately, analysing them in specific settings. In particular, in Sections 4.1.1 and 4.1.2 we present two frameworks where the additional information Λ is of the kind Λ_1 in (4.1), and we are able to give necessary and sufficient conditions for the set $\mathsf{RST}(\mu;\Lambda)$ to be non-empty, hence strengthening the result in case (1) of Theorem 4.1. Moreover, in Section 4.2 we consider the additional information Λ to be of the kind Λ_2 in (4.1) and, for options on variance, we determine the primal optimizers by means of our constrained monotonicity principle (Theorem 3.8), as well as the dual optimizers.

We remark that the first two cases imply results and constraints for the third case also, e.g. Theorems 4.2 and 4.3 directly imply necessary conditions for the third case. More generally, using the monotonicity principle Theorem 3.8 one can derive the corresponding versions of Root and Azéma-Yor embedding with a general time-space starting law (cf. Section 4.2 for the case of Root). Using similar arguments as in the proof of Theorem 4.2 and 4.3 with slightly more notation one can derive the corresponding versions of these results keeping also track of the condition $\underline{\tau} \le t$.

4.1. **Information as barrier in a certain phase space.** We now consider the case where the additional information is of the kind of Λ_1 in (4.1) and translates in having a barrier in a certain phase space. We will see how in this situation the No-Arbitrage condition (cf. Proposition 3.4, Theorem 4.1) imposes an order between such a barrier, and the barrier characterising the unique optimal stopping for the uninformed agent in such a phase space.

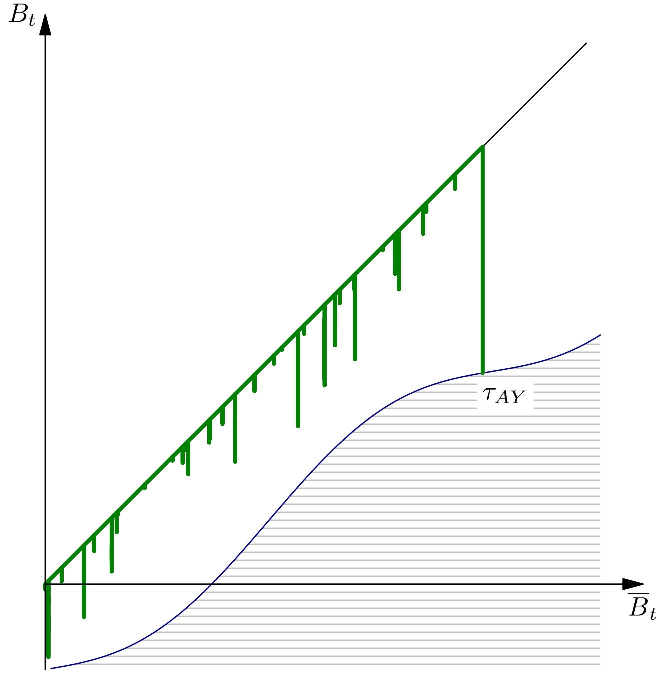

FIGURE 1. The Azéma-Yor construction.

4.1.1. *The Azéma-Yor phase space.* Let $\bar{B}_t := \sup_{0\le s\le t} B_s$. We recall that the Azéma-Yor solution of the Skorokhod Embedding Problem for the distribution μ is given by

$$\tau_{AY}(\mu) = \inf\{t \ge 0 : B_t \le \beta(\bar{B}_t)\},$$

where β is the inverse of the barycenter function associated to μ, that is, β is the right-continuous inverse of the function

$$b_\mu(x) := \frac{\int_{[x,\infty)} y\,\mu(\mathrm{d}y)}{\mu([x,\infty))}.$$

We illustrate the Azéma-Yor solution in Figure 1.

For the informed agent, we assume that

$$\Lambda = \{r(\omega, t) : t \le \overline{\tau}\}, \tag{4.3}$$

where the stopping time $\overline{\tau}$ is the hitting time of a barrier in the phase space $(\bar{B}, B)$:

$$\overline{\tau} = \inf\{t \ge 0 : (\bar{B}_t, B_t) \in \mathcal{H}\},$$

where $\mathcal{H}$ is a Borel set $\mathcal{H} \subseteq \{(x,y) \in \mathbb{R}_+ \times \mathbb{R} : y \le x\}$ induced by some increasing left-continuous Borel function $h : \mathbb{R}_+ \to \mathbb{R}$ via

$$\mathcal{H} = \{(x,y) : y \le h(x)\},$$

so that $(x,y) \in \mathcal{H}$ and $z > x$ imply $(z,y) \in \mathcal{H}$. Note that this implies

$$\overline{\tau} = \inf\{t \ge 0 : B_t \le h(\bar{B}_t)\}. \tag{4.4}$$

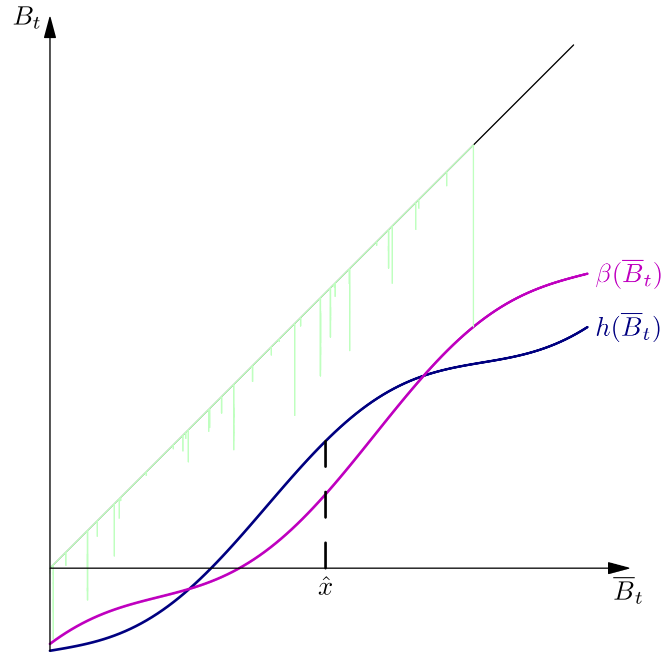

FIGURE 2. Proof of Theorem 4.2

We now give a result which shows that, when the informed agent's information is given by Λ as in (4.3), with $\overline{\tau}$ of the form in (4.4), then we can provide a simple necessary and sufficient condition for case (1) of Theorem 4.1.

Theorem 4.2. *Suppose that the set Λ is given by (4.3), with $\overline{\tau}$ of the form in (4.4). Then the set $\mathsf{RST}(\mu;\Lambda)$ is non-empty if and only if:*

$$h(x) \leq \beta(x) \quad \textit{for all } x \in \mathbb{R}_+, \tag{4.5}$$

which yields $\tau_{AY}(\mu) \leq \overline{\tau}$. In particular, if (4.5) holds, the stopping rule $\tau_{AY}(\mu)$ is admissible for the informed agent, in the sense that $((B_t)_{t\leq\tau_{AY}(\mu)}, \tau_{AY}(\mu)) \in \Lambda$ a.s.

Proof. We first observe that if (4.5) holds, then we immediately have $\tau_{AY}(\mu) \leq \overline{\tau}$, and since $\tau_{AY}(\mu) \in \mathsf{RST}(\mu;\Lambda)$, then $\mathsf{RST}(\mu;\Lambda) \neq \emptyset$.

For the reverse implication, we suppose that there exists $\hat{x} \in \mathbb{R}_+$ such that $h(\hat{x}) > \beta(\hat{x})$, as in Figure 2. Then we fix $\tau' \in \mathsf{RST}(\mu;\Lambda)$, and argue as follows. Define a measure

$$\eta(A) := \mathbb{P}(B_{\tau'} \in A, \overline{B}_{\tau'} \geq \hat{x})$$

and note that, by the martingale property, $\int y\,\eta(\mathrm{d}y) = \hat{x}\cdot\eta(\mathbb{R})$. Moreover, $\eta(A \cap [\hat{x},\infty)) = \mu(A\cap[\hat{x},\infty))$, and $\eta(A) \leq \mu(A)$ for all Borel sets A.

Define functions $\Phi_\eta, \Phi_\mu : (-\infty,\hat{x}] \to \mathbb{R}$ by:

$$\Phi_\eta(x) = \int_{[x,\infty)} y\,\eta(\mathrm{d}y) - \hat{x}\cdot\eta([x,\infty)) = \int_{[x,\infty)} (y-\hat{x})\,\eta(\mathrm{d}y),$$

and similarly for μ. Then Φ_μ, Φ_η are both increasing on $(-\infty,\hat{x}]$, $\Phi_\mu(\hat{x}) = \Phi_\eta(\hat{x})$, and $\Phi_\mu(x) - \Phi_\eta(x)$ is increasing in x for $x \in (-\infty,\hat{x}]$ since $\mu(\mathrm{d}y) \geq \eta(\mathrm{d}y)$. Hence we deduce that $\Phi_\mu(x) \leq \Phi_\eta(x)$ for $x \leq \hat{x}$.

Now we observe that $\eta((-\infty, h(\hat{x}))) = 0$, so $\Phi_\eta(h(\hat{x})) = 0$. On the other hand, by the definition of the barycentre function,

$$\beta(\hat{x}) := \sup\{y < \hat{x} : \Phi_\mu(y) \le 0\}.$$

It follows from $\Phi_\mu(x) \le \Phi_\eta(x)$ that $h(\hat{x}) \le \beta(\hat{x})$, contradicting our original assumption. □

For example, the case considered in (4.2) corresponds to $\Lambda = [\![0, \overline{\tau}]\!]$, with $\overline{\tau}$ given as in (4.4) for $h(x) = x - c$. Theorem 4.2 then implies that $x - c \le \beta(x) = b_\mu^{-1}(x)$ must hold in order to have a solution for the informed agent. Note that h is the inverse of the barycenter function associated to $\overline{\mu} \sim B_{\overline{\tau}}$, hence we have established the relation $b_{\overline{\mu}}^{-1} \le b_\mu^{-1}$, that is, $b_\mu \le b_{\overline{\mu}}$.

4.1.2. *The Root phase space.* We recall that the Root solution of the Skorokhod Embedding Problem for the distribution μ is given by

$$\tau_{Root}(\mu) = \inf\{t \ge 0 : (t, B_t) \in \mathcal{R}\},$$

where $\mathcal{R}$ is a closed *barrier*, that is, $(t, x) \in \mathcal{R}$ implies $(s, x) \in \mathcal{R}$ for $s > t$. See Figure 3. To avoid trivialities, we assume that our barriers are *regular* (see [COT15]), that is, they are closed and $\{x : (0, x) \notin \mathcal{R}\}$ is an open interval, containing the origin; any barrier which is not regular can be replaced by a regular barrier without changing the hitting time. Any regular barrier can be described by its lower semi-continuous barrier function $\mathcal{R}(x) = \inf\{t : (t, x) \in \mathcal{R}\}$.

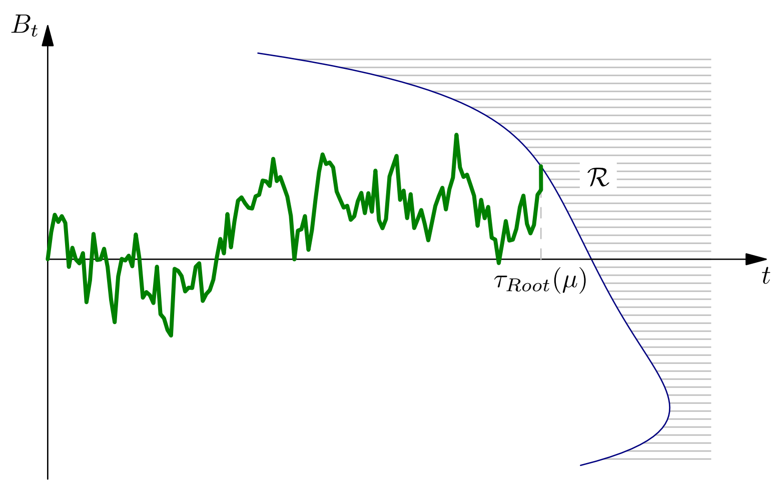

FIGURE 3. The Root solution to the SEP.

For the informed agent we assume that

$$\Lambda = \{r(\omega, t) : t \le \overline{\tau}\},$$

where the stopping time $\overline{\tau}$ is the hitting time of a regular barrier $\mathcal{B}$ in the phase space (t, B), i.e. a Root-type barrier:

$$\overline{\tau} = \inf\{t \ge 0 : (t, B_t) \in \mathcal{B}\}. \tag{4.6}$$

As in Theorem 4.2, we are able to determine whether $\mathsf{RST}(\mu;\Lambda)$ is empty, and hence whether there is an arbitrage for the informed agent, through properties of the barriers.

Theorem 4.3. *Suppose that the set Λ is given by (4.3), with $\overline{\tau}$ of the form in (4.6). Then the set $\mathsf{RST}(\mu;\Lambda)$ is non-empty if and only if:*

$$\mathcal{B} \subseteq \mathcal{R}, \tag{4.7}$$

which yields $\tau_{Root}(\mu) \leq \overline{\tau}$. In particular, if (4.7) holds, the stopping rule $\tau_{Root}(\mu)$ is admissible for the informed agent, in the sense that $((B_t)_{t\leq\tau_{Root}(\mu)}, \tau_{Root}(\mu)) \in \Lambda$ a.s.

Proof. As previously, it is immediate that when (4.7) holds, then $\mathsf{RST}(\mu;\Lambda)$ is non-empty. So suppose, for contradiction, that $\mathsf{RST}(\mu;\Lambda)$ is non-empty and $\mathcal{B} \not\subseteq \mathcal{R}$. This means that there exist pairs $(t,x) \in \mathcal{B}\setminus\mathcal{R}$. Among those pairs, we consider a fixed $(\hat{t},\hat{x})$ such that there are no $(t,\hat{x}) \in \mathcal{B}\setminus\mathcal{R}$ with $t < \hat{t}$, as in Figure 4.

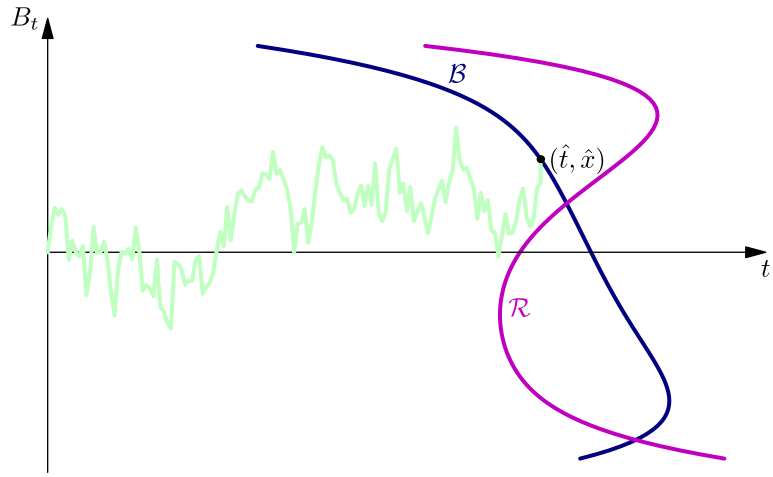

FIGURE 4. Proof of Theorem 4.3.

Now consider $\tau' \in \mathsf{RST}(\mu;\Lambda)$. Denote the local time of Brownian motion in z by L^z. Since the Root embedding maximises $\mathbb{E}[L^x_{\tau\wedge t}]$ among all stopping times τ which are minimal embeddings of μ, simultaneously for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ (e.g. by [GOR15, Theorem 3]) then in particular

$$\mathbb{E}\left[L^{\hat{x}}_{\tau'\wedge\hat{t}}\right] \leq \mathbb{E}\left[L^{\hat{x}}_{\tau_{Root}\wedge\hat{t}}\right].$$

On the other hand, the path stopped at τ' cannot accumulate any more local time at $\hat{x}$ after $\hat{t}$, i.e. $\mathbb{E}\left[L^{\hat{x}}_{\tau'\wedge\hat{t}}\right] = \mathbb{E}\left[L^{\hat{x}}_{\tau'\wedge t}\right]$ for all $t \geq \hat{t}$, while the Root stopping rule will do so ($\mathbb{E}\left[L^{\hat{x}}_{\tau_{Root}\wedge\hat{t}}\right] < \mathbb{E}\left[L^{\hat{x}}_{\tau_{Root}\wedge t}\right]$ when $t > \hat{t}$), because the barrier is assumed to be regular. Therefore,

$$\mathbb{E}\left[L^{\hat{x}}_{\tau'}\right] = \mathbb{E}\left[L^{\hat{x}}_{\tau'\wedge\hat{t}}\right] \leq \mathbb{E}\left[L^{\hat{x}}_{\tau_{Root}\wedge\hat{t}}\right] < \mathbb{E}\left[L^{\hat{x}}_{\tau_{Root}}\right].$$

This gives the desired contradiction, since, for any $x \in \mathbb{R}$ and any stopping time $\tau \in \mathsf{RST}(\mu)$,

$$\mathbb{E}\left[L^x_\tau\right] = \mathbb{E}\left[|B_\tau - x|\right] - |x| = -u_\mu(x) - |x|,$$

where u_μ is the potential function associated to μ, i.e., $u_\mu(x) = -\int |y-x|\mu(dy)$. □

4.2. **Option pricing in the presence of insider information: Variance options.** In this section, we consider the impact on the insider's pricing bounds which come from additional information. Specifically, we suppose that the information is of the kind Λ_2 of (4.1):

$$\Lambda = \{r(\omega,t) : t \geq \underline{\tau}\}, \tag{4.8}$$

where the stopping time $\underline{\tau}$ is an arbitrary stopping time (in particular, we do not assume any specific barrier-like structure, as in the previous examples) such that $(B_{t\wedge\underline{\tau}})_{t\geq 0}$ is u.i. From Theorem 4.1, we know that the set $\mathsf{RST}(\mu;\Lambda)$ is non-empty if and only if $\mu \succeq \lambda$, where $B_{\underline{\tau}} \sim \lambda$.

Consider an asset which follows a model of the form: $\mathrm{d}S_t = S_t\sigma_t\,\mathrm{d}W_t$, where S_t is the discounted asset price, and W_t a Brownian motion. The process σ_t is the volatility, and $\int_0^t \sigma_r^2\,\mathrm{d}r$ is known as the *integrated variance*. A variance option is then a contract which pays the holder $F\left(\int_0^t \sigma_r^2\,\mathrm{d}r\right)$. The most common example is the variance call, where $F(v) = (v-K)_+$. Note that the integrated variance process can be determined as $\langle \ln S\rangle_t$, the quadratic variation of the logarithm of the asset price. For further details, we refer the reader to [CL10; CW13a; CW13b; Lee10].

The standard method for pricing such options is to time-change the process S_t by a time change τ_t such that $X_t := S_{\tau_t}$ is a *geometric* Brownian motion. With this time change, $(X_t,t) = (S_{\tau_t}, \langle \ln S\rangle_{\tau_t})$, that is, the time-scale in the transformed picture corresponds to the integrated variance process. In particular, the problem of finding a model S_t which minimises $\mathbb{E}[F(\langle \ln S\rangle_T)]$ subject to $S_T \sim \mu$ is equivalent to finding a stopping time τ for X to minimise $\mathbb{E}[F(\tau)]$ subject to $X_\tau \sim \mu$.

We would therefore like to compare the minimal (model-independent) price of the variance option for the insider, to that for the uninformed agent. To keep things simple, we consider a variance call on the *arithmetic variance*, $V_t := \int_0^t S_r^2\sigma_r^2\,\mathrm{d}r$, which corresponds to choosing a time-change τ_t so that $X_t = S_{\tau_t}$ is a Brownian motion. This places us trivially in the setup of the rest of this paper, and additionally allows us to use directly results from [COT15], however we emphasise that *mutatis mutandis*, both sets of results extend to the case of standard variance options.

Our problem of interest now may be posed as follows: consider an agent who holds a variance option (and e.g. for regulatory reasons, is unable to sell this option) with payoff $F(V_T)$, where F is a non-negative, convex, increasing function. To hedge their risk, they may look to sell the most expensive, model-independent sub-hedging strategy, guaranteeing that they do not lose anything, and minimise their risk. If the agent also knows the feasibility set Λ given by (4.8), then their problem becomes to find the corresponding version of (2.6),

$$\begin{aligned} P_\Lambda^* &= \inf\{\mathbb{E}_\mathbb{P}[\gamma((B_t)_{t\leq\tau},\tau)] : B_\tau \sim \mu,\ B_{\cdot\wedge\tau} \text{ is u.i.},\ \mathbb{E}_\mathbb{P}[1_\Lambda((B_t)_{t\leq\tau},\tau)] = 1\} \\ &= \inf\{\mathbb{E}_\mathbb{P}[F(\tau)] : B_\tau \sim \mu,\ B_{\cdot\wedge\tau} \text{ is u.i.},\ \tau \geq \underline{\tau} \text{ a.s.}\}. \end{aligned} \tag{4.9}$$

By Theorem 3.3, if we can identify the solution to this problem, then we will be able to find a corresponding sub-hedging strategy.

Observe that we can consider this problem as an optimal Skorokhod Embedding Problem, with a general time-space starting law: let ζ be a measure on $\mathbb{R}_+ \times \mathbb{R}$ given by $\zeta(C\times D) = \mathbb{P}((\underline{\tau}, B_{\underline{\tau}}) \in C\times D)$. Then the problem above is to find a stopping time τ for the space-time Markov process (t, B_t) started with law ζ, such that $B_\tau \sim \mu$, which minimises $\mathbb{E}^\zeta[F(\tau)]$. We observe that problems of this form have (indirectly) been considered in [COT15], and we will use these results below to derive explicit formulae, however we first show informally how the monotonicity principle, Theorem 3.8, can be used to derive the form of the optimiser.

Our argument is essentially identical to the proof of Theorem 2.1 of [BCH16]. Assume that we are given an optimiser τ^* which attains the infimum in (P^*_Λ). For instance this follows when Λ is closed by an application of Theorem 2.6, since in that case $\mathsf{RST}(\Lambda,\mu)$ is compact by Corollary 2.4. Note that Λ in (4.8) is closed if $\underline{\tau}$ is a Root stopping time or an Azéma Yor type stopping time induced by an increasing right continuous barycenter function.

By Theorem 3.8, there exists a feasible γ-monotone Borel set Γ which supports τ^* (although note that we reverse the direction of the inequality in Definition 3.6 since we are minimising, not maximising). Now consider $(f,s),(g,t) \in \mathcal{S}$ with $f(s) = g(t)$, and such that $(g,t) \in \Lambda$. From the convexity of the function F, and since $\gamma((f,s)) = F(s)$, we see immediately that $((f,s),(g,t)) \in \mathsf{SG}_\Lambda$ if and only if $(f,s) \in \Lambda$ and $s > t$. The remainder of the argument runs exactly as in Theorem 2.1 of [BCH16], by defining the sets

$$\mathcal{R}_{\mathsf{CL}} := \{(s,x) : \exists (g,t) \in \Gamma, g(t) = x, t \le s\}$$
$$\mathcal{R}_{\mathsf{OP}} := \{(s,x) : \exists (g,t) \in \Gamma, g(t) = x, t < s\}$$

and observing that τ^* must lie between the respective first hitting times of these sets, after $\underline{\tau}$, which are almost surely equal.

Moreover, we can recover the barrier in terms of a related optimal stopping problem. Define

$$v^{\underline{\tau}}(t,x) := -\mathbb{E}\left|B_{t\wedge\underline{\tau}} - x\right|,$$

for $t \ge 0, x \in \mathbb{R}$, and set $w^\mu(x) := u^\mu(x) - u^\lambda(x)$, difference of the potential functions (see end of proof of Theorem 4.3) associated to μ and λ. Write $\mathcal{T}_t$ for the set of stopping times which are bounded by t. Then:

Theorem 4.4 (Theorem 4.1, [COT15])**.** *Let*

$$v(t,x) := \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^x\left[v^{\underline{\tau}}(t-\tau, W_\tau) + w^\mu(W_\tau)\mathbf{1}_{\{\tau<t\}}\right], \quad t \ge 0,\ x \in \mathbb{R}, \tag{4.10}$$

be an optimal stopping problem, with stopping region

$$\mathcal{R} := \{(t,x) : v(t,x) = v^{\underline{\tau}}(t,x) + w^\mu(x)\}.$$

Then $\mathcal{R}$ is a barrier such that

$$\tau := \inf\{t \ge \underline{\tau} : (t, B_t) \in \mathcal{R}\}$$

satisfies $B_\tau \sim \mu$, and $B_{\cdot\wedge\tau}$ is u.i. Moreover, this stopping time attains the infimum in (4.9)*.*

Proof. The main substance of the proof is contained exactly in Theorem 4.1 of [COT15]. The fact that the resulting stopping time is the optimiser follows from the fact that there is a unique barrier embedding μ, see for example the arguments following the proof of Theorem 2.1 in [BCH16]. □

We finish this section with two further consequences. The first is that, under some additional technical assumptions, we can also provide explicit solutions to the dual problem, (D^*_Λ), using results of [COT15]. The second is that, by putting these results together, we are able to determine an explicit value for the value of the informed agent's information, in terms of the change in price of a variance option; we give some numerical evidence to this extent.

4.2.1. *Hedging of variance options for insiders.* Suppose, following the discussion above, we are interested in the insiders problem of subhedging a variance option with payoff $F(V_T)$, where F is a non-negative, convex, increasing function, which we further assume that we can write in the form $F(t) = \int_0^t f(r)\,\mathrm{d}r$, for some non-negative, non-decreasing function (which is equivalent to assuming that $F(0) = 0$). Our arguments will be based on the construction in Section 3.2 of [COT15]. In particular, we consider the set $\mathcal{R}_2 := \mathcal{R}$ defined above, and $\mathcal{R}_1 := \mathbb{R}_+ \times \mathbb{R}$. Following [COT15], we define $\sigma_{\mathcal{R}_k} := \inf\{t : (t, B_t) \in \mathcal{R}_k\}$, and let $\mathbb{P}^{t,x}$ denote the probability measure where the space-time Brownian motion starts at x at time t, so in particular, $\mathbb{P}^{t,x}$-almost surely $\sigma_{\mathcal{R}_1} = t$. Then for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, we set

$$\varphi_3(t,x) := f(t), \quad \varphi_k(t,x) = \mathbb{E}^{t,x}[\varphi_{k+1}(\sigma_{\mathcal{R}_k}, B_{\sigma_{\mathcal{R}_k}})], \quad \text{and} \quad \phi_k(x) := \int_0^x \varphi_k(0,y)\,\mathrm{d}y,$$

for $k = 1, 2$, and introduce also $t_k(x) := \inf\{t : (t,x) \in \mathcal{R}_k\}$, and

$$h_k(t,x) := \int_0^t \varphi_k(s,x)\,\mathrm{d}s - 2\int_0^x \phi_k(y)\,\mathrm{d}y, \quad \text{and} \quad \lambda_k(x) := (h_{k+1} - h_k)(t_k(x), x).$$

We first observe that $\varphi_1(t,x) = \varphi_2(t,x) =: \varphi(t,x)$ since $\sigma_{\mathcal{R}_1} = t$ $\mathbb{P}^{t,x}$-almost surely. Then $\phi_1 \equiv \phi_2 := \phi$, $h_1 \equiv h_2 := h$, and hence $\lambda_1 \equiv 0$. We therefore also write $\lambda := \lambda_2$. By Theorem 3.2(i) of [COT15], we therefore have

$$F(s_2) \geq \lambda(x_2) + [h(s_2,x_2) - h(s_1,x_1)] + h(s_1,x_1), \tag{4.11}$$

whenever $0 \leq s_1 \leq s_2$. Moreover, equality is attained whenever $(s_2, x_2) \in \mathcal{R}_2$.

Finally, we define the martingale $M_t := \mathbb{E}[h(\underline{\tau}, B_{\underline{\tau}})|\mathcal{F}_t]$. Note that since we are in a Brownian filtration, $M_t = M_0 + \int_0^t \alpha_s\,\mathrm{d}B_s$, for some process α. Similarly, by Lemma 3.3 of [COT15], (and assuming that (3.8) therein holds), $h(t, B_t)$ is a submartingale, and hence there exists an increasing process A_t, and a process β such that $h(t,B_t) = h(0,B_0) + A_t + \int_0^t \beta_s\,\mathrm{d}B_s$.

Theorem 4.5. *Suppose the insider has feasibility set Λ given by* (4.8)*, and let $\mathcal{R}$ be given by Theorem 4.4. Then the hedging strategy consisting of:*

Hold payoff $\lambda(S_T)$ in call options with maturity T
Hold α_t units of the underlying up to time $\underline{\tau}$
Hold β_t units of the underlying after time $\underline{\tau}$
Hold M_0 in cash at time 0

is a sub-hedge (on Λ) of the variance option with payoff $F(V_T)$. Moreover, there exists a feasible model under which this is a perfect hedge.

Proof. The fact that this is a sub-hedge for any model follows immediately from the discussion above. To show it is a hedge, we consider the model implied by Theorem 4.4. Then under this model, at the corresponding stopping time, we have equality in (4.11). Moreover since M_t is a martingale, at time t the dynamic strategy α_t plus the cash holding delivers a portfolio worth $h(\underline{\tau}, B_{\underline{\tau}})$ at time $\underline{\tau}$. Finally, by Lemma 3.3 of [COT15] (and in particular (iv) of the proof), $h(t, B_t)$ is a martingale between $\underline{\tau}$ and $\sigma_{\mathcal{R}} \equiv \tau$. (Note that (3.8) of Lemma 3.3 of [COT15] can be verified in the same manner as in the proof of Theorem 3.2 therein). Hence $A_{\underline{\tau}} = A_\tau$, and so $h(\tau, B_\tau) = h(\underline{\tau}, B_{\underline{\tau}}) + \int_{\underline{\tau}}^{\tau} \beta_s \, \mathrm{d}B_s$. The claim follows. □

4.2.2. *Numerical results.* In this section we illustrate the previous example with some numerical evidence. In particular, we are interested in illustrating how the insider's price and hedging strategies change as the information set changes.

Our basic setup is as follows: we suppose that the insider's information set Λ is determined by the following stopping time:

$$\underline{\tau} := \inf\{t \geq 0 : B_t \notin [-1, 1]\} \wedge \mathfrak{e}_\rho, \tag{4.12}$$

where $\mathfrak{e}_\rho$ is an independent exponential random variable with rate ρ. Strictly speaking, this independent randomisation depends on additional randomisation, and so is outside our current framework, however it is easy to approximate such times by considering

$$\bar{\mathfrak{e}}_\rho := N^{-1} \inf\{n \geq 1 : (B_{s+(n-1)/N} - B_{(n-1)/N})_{s \in [0, N^{-1}]} \in \mathcal{V}\}, \tag{4.13}$$

where $\mathcal{V}$ is a subset of $\mathcal{C}([0, N^{-1}])$ which is given mass ρN^{-1} under Wiener measure. Then $\mathfrak{e}_\rho$ and $\bar{\mathfrak{e}}_\rho$ are approximately equal in law, for large N. The smaller the value of ρ, the larger $\underline{\tau}$, and hence the insider has 'more' information.

To compare different amounts of information, we consider the case where the insider wishes to subhedge a Variance option with payoff function $F(v) = v^3/3$, and we suppose that the measure μ implied from call prices is a mix of truncated Gaussians. Specifically, we suppose

$$\begin{aligned} \mu(\mathrm{d}x) = {} & \frac{n(2, [1, \infty)) + n(3, [1, \infty))}{2} \delta_1(\mathrm{d}x) + \frac{n(2, \mathrm{d}x) + n(3, \mathrm{d}x)}{2} \mathbf{1}_{\{x \in (-1,1)\}} \\ & + \frac{n(2, (-\infty, -1]) + n(3, (-\infty, -1])}{2} \delta_{-1}(\mathrm{d}x), \end{aligned} \tag{4.14}$$

where $n(\sigma, \mathrm{d}x)$ is the Gaussian measure with mean 0 and variance σ^2.

Our numerical algorithm proceeds as follows: to find $v^{\underline{\tau}}$, we solve $v^{\underline{\tau}}_t = v^{\underline{\tau}}_{xx}/2 - \rho v^{\underline{\tau}}$, using an implicit numerical scheme. Once we have $v^{\underline{\tau}}$, we can compute v by (4.10), again using an implicit scheme to solve the corresponding Bellman equation, together with a solution of Complementary Pivot Algorithm to determine the stopping rule (see [CW13b] for details). Once we have determined v, we can immediately find $\mathcal{R}$. Once $\mathcal{R}$ has been determined, φ_2 can be found by solving $(\varphi_2)_t + (\varphi_2)_{xx}/2 = 0$ in $\mathcal{R}^\complement$, with appropriate boundary conditions; once again, this is done using an implicit numerical scheme. From φ_2, it is immediate to compute the functions λ and h. The insider's sub-hedging price is then determined by integrating λ and h against the relevant stopping distributions, which can be computed from $v^{\underline{\tau}}$ and u^μ.

We display the results graphically. We typically plot values of ρ in the range $[0.5, 2]$, and in Figure 5 we show the effect of changing ρ on the barrier given by Theorem 4.4.

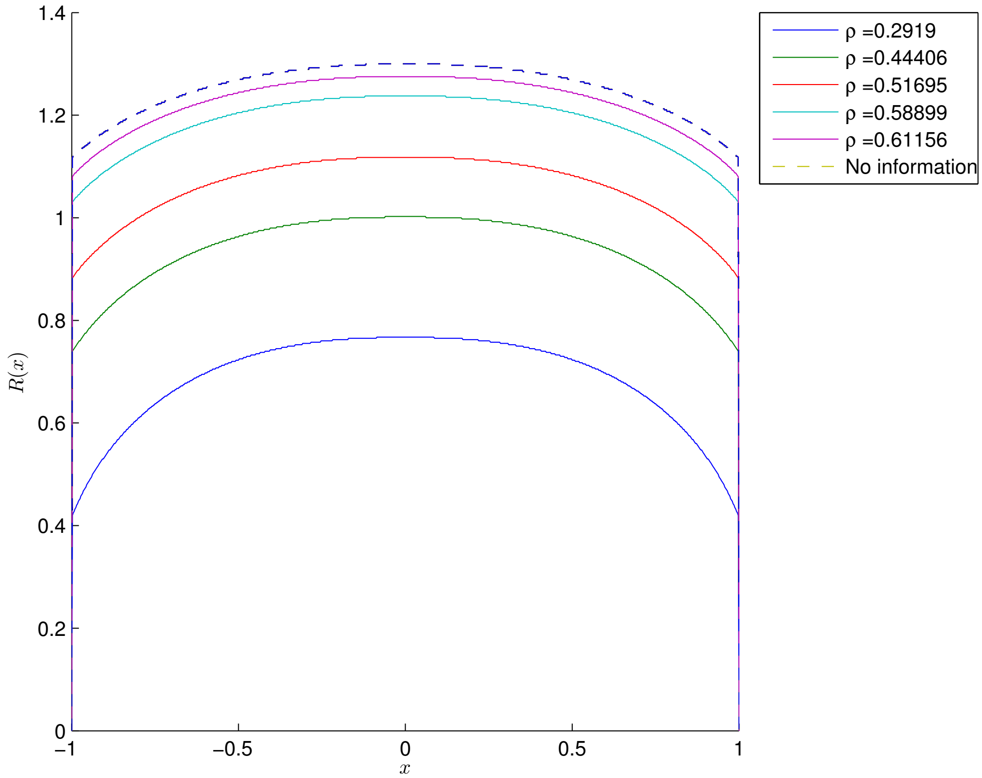

FIGURE 5. Effect of changing ρ on barrier.

Recall that $\mathcal{R}(x)$ is the barrier function. In Figure 6 we show the effect of changing ρ on the resulting function λ, and finally in Figure 7 we show the effect of changing ρ on the price of the sub-hedge for the insider. In each of these figures, the case of no information $\Lambda = \mathcal{S}$ is given by the dotted lines.

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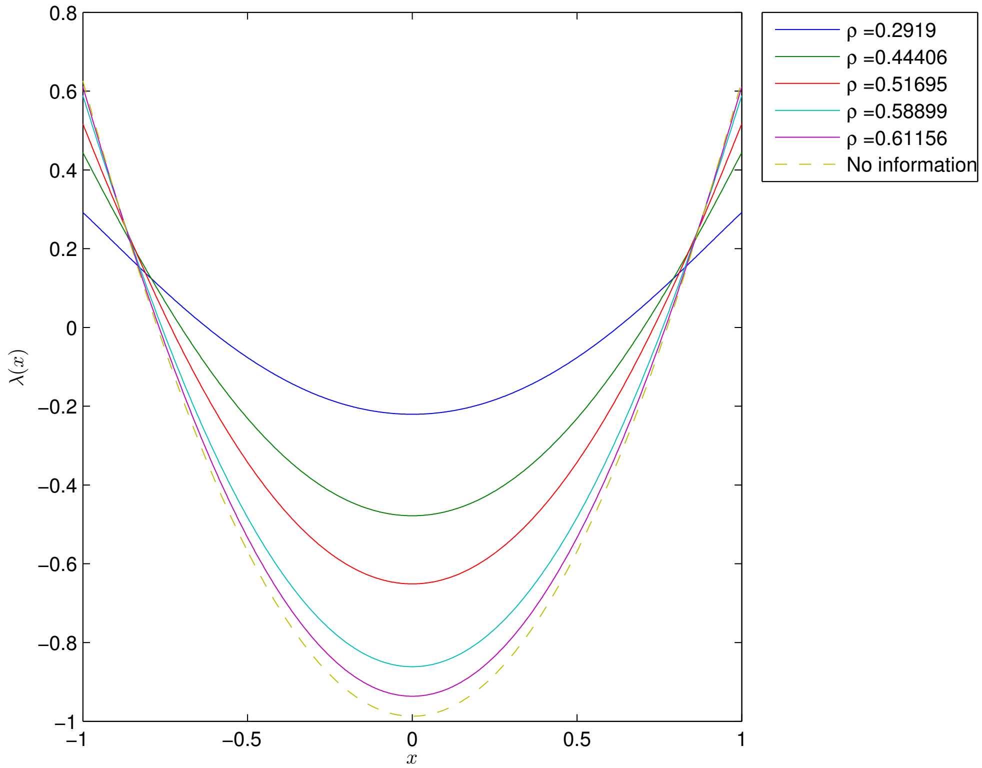

FIGURE 6. Effect of changing ρ on $\lambda(x)$.

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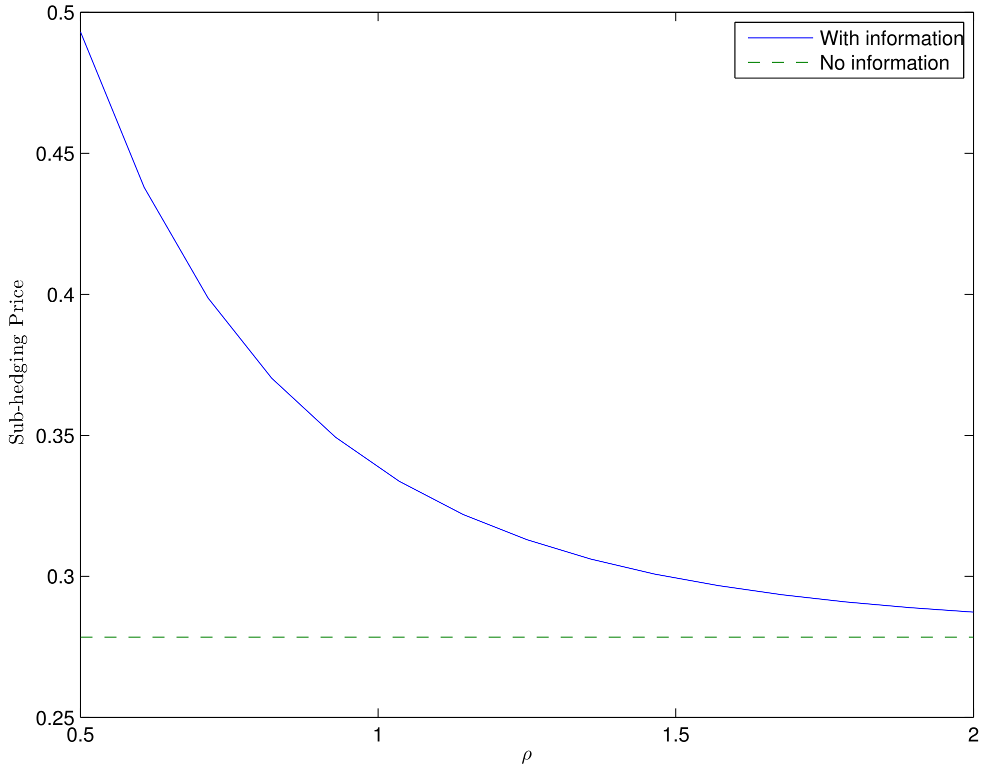

FIGURE 7. Effect of changing ρ on the sub-hedging price.

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