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The numeraire portfolio in discrete time: existence, related concepts and applications

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Abstract. We survey the literature on the numeraire portfolio, explain its relation to various other concepts in financial mathematics and present two applications in insurance mathematics and portfolio optimization.

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1 Introduction and summary

An important subject of financial mathematics is adequate pricing of financial derivatives, in particular options. In the modern theory (see e.g. Duffie 1992), the historical concept based on expectations of discounted quantities (the present value principle) is replaced by the concept of deflators, numeraires (inverse deflators) or the application of the present value principle after a change of measure. In this paper we focus on the concept of the numeraire portfolio, present its definition, its relation to various valuation concepts and its role in important applications. When the value process of a numeraire portfolio is used as a discount process, the relative value processes of all other portfolios with respect to it will be martingales or at least supermartingales (see Vasicek 1977, Long 1990, Artzner 1997, Bajeux-Besnainou and Portait 1997, Korn & Schäl 1999, Schäl 2000a, Becherer 2001, Platen 2001, 2006, Christensen and Larsen 2007, Karatzas & Kardaras 2007).

We will study a financial market with small investors which is free of arbitrage opportunities but incomplete (although we will see that much is valid under a weaker assumption than the no arbitrage assumption). Then in discrete time, one has several choices for an equivalent martingale measure (EMM) needed to value derivatives. In continuous time an EMM exists under more restrictive conditions. It is known (see Harrison & Kreps 1979) that each EMM corresponds to a consistent price system. Thus in incomplete markets, no preference-independent pricing of financial derivatives is possible. In the present paper, the unique martingale measure Q^* is studied which is defined by the concept of the numeraire portfolio (see Korn & Korn 2001, Section 3.7). The choice of Q^* can be justified by a change of numeraire in place of a change of measure. Uniqueness is obtained by the fact that the EMM after the change of numeraire should be the original real-world probability measure.

It is known that in many cases one can get a numeraire portfolio from the growth optimal portfolio (GOP) which maximises the expected utility when using the log-utility. Utility optimisation is now a classical subject. Recent papers with the log-utility are Goll & Kallsen 2000, Kallsen 2000, Goll and Kallsen 2003.

When looking for a numeraire portfolio (in the strict martingale sense), we are interested in optimal portfolios which can be chosen from the interior of the set of admissible portfolios. Also for more general utilities, optimal 'interior' portfolios can be used to define equivalent martingale measures (see Karatzas & Kou 1996, Schäl 2000a,b).

In order to get full equivalence of a numeraire portfolio and a GOP, one has to generalise the concept by defining a weak numeraire portfolio introduced by Becherer 2001 under the name 'numeraire portfolio'. Such a portfolio defines a supermartingale measure in the above sense.

The paper is laid out as follows. We consider a discrete-time market. It turns out that all the ideas can be explained in a simple one-period model starting in 0 and finishing at the time-horizon $T = 1$. In fact, for a log-utility investor, the optimal strategy is myopic even for market models where optimal power-utility strategies are not guaranteed to be myopic (see Hakansson & Ziemba, 1995). Given the solution to a one-period model, the form of the optimal strategy for a multi-period model is obvious. Therefore we will restrict to such a (0,1)-period. Then strategies ξ and portfolios π can be described by d-dimensional vectors. In fact when considering general semi-martingale models, it is sufficient (in most passages) to replace the inner products $\xi^\top \Delta S$ or $\pi^\top R$ by stochastic integrals $\xi \cdot S$ or $\pi \cdot R$, where S describes the prices and R the cumulative returns.

Except for the restriction to a $(0, 1)$-period, we try to choose the framework as general as possible where the recent paper by Karatzas & Kardaras 2007 will serve as a model. In particular, we accept the framework with general convex constraints.

We then consider various valuation and optimisation concepts that are directly related to the numeraire portfolio. Among them are the GOP, the benchmark portfolio, the value preserving portfolio and of course the valuation with the help of EMMs. This is followed by existence considerations for (weak) numeraire portfolios. Finally, we give two important applications of the numeraire portfolios in insurance mathematics and in portfolio optimisation.

2 The one-period market setting

On the market an investor can observe the prices of $1 + d$ assets at the dates $t = 0, 1$ which are described by S_t^0 and $S_t = (S_t^1, \ldots, S_t^d)^\top t = 0, 1$. [For any vector x we write $x^\top$ for the transposed vector and $x^\top y$ for the inner product of $x, y \in \mathbb{R}^d$ thought of as column vectors.]

Hence our time horizon will be $T = 1$. Then S_0^0 and S_0 are deterministic, S_1^0 is a random variable, S_1 is a random vector on a probability space (Ω, F, P) and S_t^0 is positive.

One of these assets will play a special role for which we will choose S^0. But any other component S^k can be chosen in place of S^0. An important situation will be the case where the asset with price S^0 describes the bank account (or money market) and the other d assets are stocks. This is a very useful interpretation and we will use it. The interpretation of S^0 as money market leads to further convenient interpretations. But remember that, mathematically, all price components will satisfy the same assumptions.

Given an initial capital $V_0 > 0$, one can invest in the assets described by S by choosing some $\xi \in \mathbb{R}^d$ which describes the *strategy* in the present simple case with $T = 1$. The number ξ^k represents the number of shares for stock k bought and held by the investor at time 0. The total amount invested in stocks is $\xi^\top S_0 = \sum_{k=1}^d \xi^k S_0^k$. For satisfying the self-financing condition, the remaining wealth of the initial value V_0, namely $\xi^0 := V_0 - \xi^\top S_0$ is invested in the bank account. Then $V_0 = V_0^\xi = \sum_{k=0}^d \xi^k S_0^k$. Upon defining $\Delta X := X_1 - X_0$ for X being defined for $t = 0, 1$, the value V_1^ξ of ξ at time 1 is described by

$$\Delta V^\xi = \xi^0 \Delta S^0 + \xi^\top \Delta S \ \left(= \sum_{k=0}^d \xi^k (S_1^k - S_0^k) \right). \tag{2.1}$$

Upon defining discounted quantities $\breve{S}_t = (\breve{S}_t^1, \ldots, \breve{S}_t^d)^\top$ and $\breve{V}_t$ by

$$\breve{S}_t^k := S_t^k / \, S_t^0, \ \breve{V}_t^\xi := V_t^\xi / \, S_t^0\,, \tag{2.2}$$

we easily obtain

$$\Delta \breve{V}^\xi = \xi^\top \Delta \breve{S}\,. \tag{2.3}$$

This simple relation is the mathematical reason for using "discounted" quantities. Since we might as well work in discounted terms, from now on we assume that $S_t^0 \equiv 1$ as is common in Mathematical Finance (see Harrison & Kreps 1979). Then $\Delta S^0 \equiv 0$ and one can dispense with ξ^0. Starting with capital $V_0 = x > 0$ and investing according to strategy ξ, the investor's value at time 1 is $V_1^\xi(x) := x + \xi^\top \Delta S$. For any $V_0 = x > 0$ and any strategy ξ, $V_1^\xi(x) = x + \xi^\top \Delta S$ is called *admissible* if $V_1^\xi(x) \geq 0$. The *return* R^k for stock k is defined by

$$S_1^k = S_0^k \cdot (1 + R^k)\,. \tag{2.4}$$

Then we can write $V_1^\xi(x) = x \cdot (1 + \sum_{k=1}^d (\xi^k S_0^k / V_0) R^k)$. Defining $\pi \in \mathbb{R}^d$ as the vector with components $\pi^k = \xi^k S_0^k / V_0$, π^k signifies the proportion of V_0 invested in stock k and we have

$$V_1^\xi(x) = x \cdot (1 + \pi^\top R) =: x \cdot V_1^\pi\,. \tag{2.5}$$

The equivalent of "$V_1^\xi(x) > 0$ (≥ 0)", for $x > 0$, is "$V_1^\pi = 1 + \pi^\top R > 0$ (≥ 0)". This simple representation is the reason for our *restriction* to the case $x = 1$ in the sequel where we write V_1^π in place of $V_1^\xi(1)$. By use of π, admissibility is independent of the initial wealth x and thus easier to handle.

We will now introduce constraints, where Karatzas & Kardaras 2007, Kardaras 2006 will serve as a model. For the sake of motivation, we will start with the following example.

Example A. The case where the investor is prevented from selling stock short or borrowing from the bank can be describe by $\xi^k \geq 0$, $1 \leq k \leq d$, and $\xi^0 := V_0 - \xi^\top S_0 \geq 0$.

This condition is equivalent to $\pi^k \geq 0$, $1 \leq k \leq d$, and $\pi^0 := 1 - \sum_{k=1}^d \pi^k \geq 0$. By setting $C := \{\pi \in \mathbb{R}^d : \pi^k \geq 0 \text{ and } \sum_{k=1}^d \pi^k \leq 1\}$, the prohibition of short sales and borrowing is translated into the requirement $\pi \in C$.

Definition 2.1. Consider an arbitrary convex closed set $C \subset \mathbb{R}^d$ with $0 \in C$. The admissible value V_1^π is called C-constrained, if $\pi \in C$. Here the following set

$$\check{C} := \cap_{a>0}\, aC \tag{2.6}$$

is called the *set of cone points* (or recession cone) of C.

Note in particular that the "safe" portfolio $\pi = 0$ is always admissible.

Example A (continuation). Here we have $aC = \{a\pi \in \mathbb{R}^d : \pi^k \geq 0 \text{ and } \sum_{k=1}^d \pi^k \leq 1\} = \{\vartheta \in \mathbb{R}^d : \vartheta^k \geq 0 \text{ and } \sum_{k=1}^d \vartheta^k \leq a\}$. This leads to the relation $\check{C} = \{0\} \subset \mathbb{R}^d$.

The following example describes the positivity constraints for admissibility.

Example B (Natural Constraints).

$$C := \Theta := \{\vartheta \in \mathbb{R}^d\,;\ 1 + \vartheta^\top R \geq 0 \ \text{ a.s.}\} = \{\vartheta \in \mathbb{R}^d\,;\ 1 + \vartheta^\top z \geq 0\, \forall z \in Z\},$$

where Z is the support of R, i.e. the smallest closed subset B of $\mathbb{R}^d$ such that $P[R \in B] = 1$.

The representation of Θ by means of Z is easily proved (see Korn and Schäl, 1999 Lemma 4.3a).

We use "$\geq$"in place of "$>$" in the definition of Θ to keep the set Θ closed. Then $aC = \{a\pi \in \mathbb{R}^d;\ 1 + \pi^\top R \geq 0 \ \text{ a.s.}\} = \{\vartheta \in \mathbb{R}^d;\ a + \vartheta^\top R \geq 0 \ \text{ a.s.}\}$ and $\check{C} = \cap_{a>0}\, aC = \{\vartheta \in \mathbb{R}^d;\ \vartheta^\top R \geq 0 \ \text{ a.s.}\}$.

The requirement of admissibility of V_1^π is exactly what corresponds to π being Θ-constrained.

Consider the special case $d = 1$ and the no-arbitrage condition: $-\alpha, \beta \in Z$ for some $\alpha, \beta > 0$. Then again $\check{C} = \{0\} \subset \mathbb{R}^1$. We shall always assume that C is enriched with the natural constraints, i.e. $C \subset \Theta$. Otherwise, we can replace C by $C \cap \Theta$.

Example C. The case where the investor is prevented from selling stock short but not from borrowing from the bank can be described by $\xi^k \geq 0$, $1 \leq k \leq d$. This condition is equivalent to $\pi^k \geq 0$, $1 \leq k \leq d$. By setting $C := \{\pi \in \mathbb{R}^d : \pi^k \geq 0\}$, the prohibition of short sales is translated into the requirement $\pi \in C$. Here C is a cone and thus we get $\check{C} = C = aC$ for $a > 0$.

In the sequel we will write

$$\Pi := \{\pi \in C\,;\ 1 + \pi^\top R > 0 \ \text{ a.s.}\}\,. \tag{2.7}$$

The elements of Π will be called *portfolios*; we make this distinction with the corresponding notion of strategy, denoted by ξ.

Lemma 2.2. *For $\rho \in C$ and $\vartheta \in \check{C}$ we have $\rho + \vartheta \in C$.*

Proof (See Karatzas & Kardaras 2007). We know that $a\vartheta \in C$ for all $a > 0$. Then $(1 - \frac{1}{a})\rho + \frac{1}{a} a\, \vartheta = (1 - \frac{1}{a})\rho + \vartheta \in C$ by the convexity of C. But C is also closed, and so $\rho + \vartheta \in C$. □

3 Weak numeraire portfolio

In general, by "numeraire" one understands any strictly positive random variable Y such that it acts as an "inverse deflator $D = Y^{-1}$", e.g. a stochastic discount factor, for the values V_1^π. Then we see our investment according to portfolio π relative to Y, giving us a wealth of V_1^π / Y. There Y may not even be generated by a portfolio.

Definition 3.1. A portfolio $\rho \in \Pi$ will be called *weak numeraire portfolio*, if for the relative value defined as V_1^π / V_1^ρ one has: $\mathbb{E}\,[V_1^\pi / V_1^\rho] \le 1$ $(= V_0^\pi / V_0^\rho)$ for every portfolio π.

The qualifier "weak" is used because we have "$\le$" in place of "$=$" in the definition above. Since $0 \in \Pi$, one has $\mathbb{E}\,[1/V_1^\rho] \le 1$ $(= 1/V_0^\rho)$. Thus V_t^π / V_t^ρ and $1/V_t^\rho$ are positive supermartingales. The definition in this form first appears in Becherer 2001.

Proposition 3.2. *If ρ_1 and ρ_2 are weak numeraire portfolios, then $V_1^{\rho_1} = V_1^{\rho_2}$* a.s.

Proof. We have both $\mathbb{E}[V_t^{\rho_1} / V_t^{\rho_2}] \le 1$ and $\mathbb{E}[V_t^{\rho_2} / V_t^{\rho_1}] \le 1$ which implies that $V_1^{\rho_1} = V_1^{\rho_2}$ a.s. □

Therefore the value generated by weak numeraire portfolios is unique. Moreover $\rho_1^\top R = \rho_2^\top R$ a.s. In this sense, the weak numeraire portfolio is unique, too.

Of course, if ρ satisfies the requirements of the definition above, V_1^ρ can act as a numeraire in the sense of this discussion. For a weak numeraire portfolio ρ, V_1^ρ is in a sense the best tradable benchmark: whatever anyone else is doing, it looks as a supermartingale (decreasing in the mean) through the lens of relative value to V_1^ρ.

An obvious example for a numeraire would be $Y_1 = S_1^0$ before assuming $S_t^0 \equiv 1$. Obviously the relative values do not depend on the discount factor since $\check{V}_1^\pi / \check{V}_1^\rho = V_1^\pi / V_1^\rho = Y^{-1} V_1^\pi / Y^{-1} V_1^\rho$. Now we again see that there was no loss of generality in considering discounted values. It will turn out that ρ satisfies certain optimality properties. Thus, when using V_t^ρ as inverse deflator in place of the classical S_t^0, we take into account that an investment in the bank account may be far from optimal.

The following relation will be used for sake of motivation (see Kardaras 2006):

$$V_1^\pi / V_1^\rho = (1 + \rho^\top R)^{-1} (1 + \pi^\top R) = 1 + (\pi - \rho)^\top R^\rho$$

where $R^\rho = (1 + \rho^\top R)^{-1} R$ is the return in an auxiliary market. Therefore the relative value can be seen as the usual value generated by investing in the auxiliary market. If

$\mathbb{E}[\pi^\top R] \in [-1, \infty]$ is called the *rate of return* or *drift rate*, then

$$\begin{aligned} r(\pi|\rho) &:= \mathbb{E}[(\pi-\rho)^\top R^\rho] \qquad (3.1) \\ &= \mathbb{E}[(1+\rho^\top R)^{-1} \cdot (\pi-\rho)^\top R] = (\pi-\rho)^\top \mathbb{E}[(1+\rho^\top R)^{-1} R] \end{aligned}$$

is the rate of return of the relative value process V_t^π / V_t^ρ. Since $\mathbb{E}[V_1^\pi / V_1^\rho] = 1 + r(\pi|\rho)$, we now obtain the following lemma.

Lemma 3.3. *ρ is a weak numeraire portfolio if and only if*

$$r(\pi|\rho) \leq 0 \textit{ for every } \pi \in \Pi \,. \qquad (3.2)$$

It is obvious that if (3.2) is to hold for C, then it must also hold for the closed convex hull of C, so it was natural to assume that C is closed and convex if we want to find the portfolio ρ.

The market may show some degeneracies. This has to do with linear dependence that some stocks might exhibit and which are not excluded. As a consequence, there may be seemingly different portfolios producing exactly the same value. Thus they should then be treated as equivalent. To formulate this notion, consider two different portfolios π_1 and π_2 producing exactly the same value, i.e. $\pi_1^\top R = \pi_2^\top R$ a.s. Now $(\pi_2 - \pi_1)^\top R = 0$ a.s. is equivalent to $(\pi_2 - \pi_1)^\top z$ for all $z \in Z$ where Z is again the support of R. Let $\mathcal{L}$ be the smallest linear space in $\mathbb{R}^d$ containing Z and $\mathcal{L}^\perp = \{\vartheta \in \mathbb{R}^d; \vartheta \perp \mathcal{L}\}$ its orthogonal complement.

Lemma 3.4. *(a)* $\pi_1^\top R = \pi_2^\top R$ a.s. *is equivalent to* $\pi_2 - \pi_1 \in \mathcal{L}^\perp$.

(b) $\vartheta \in \mathbb{R}^d \setminus \mathcal{L}^\perp$ *if and only if* $P[\vartheta^\top R \neq 0] > 0$.

Two portfolios π_1 and π_2 satisfy $\pi_2 - \pi_1 \in \mathcal{L}^\perp$ if and only if $V_1^{\pi_1} = V_1^{\pi_2}$ a.s. It is convenient to assume that $\mathcal{L}^\perp \subset C$. So the investor should have at least the freedom to do nothing; that is, if an investment leads to absolutely no profit or loss, one should be free to make it. In the non-degenerate case $\mathcal{L} = \mathbb{R}^d$ this just becomes $0 \in C$. The natural constraints Θ can easily be seen to satisfy this requirement as well as the requirements of closedness and convexity.

Definition 3.5. Let us define the set I of *arbitrage opportunities* to be the set of portfolios ϑ such that

$$P[\vartheta^\top R > 0] > 0 \text{ and } P[\vartheta^\top R \geq 0] = 1\,,$$

i.e., the set of portfolios $\vartheta \in \mathbb{R}^d \setminus \mathcal{L}^\perp$ such that $\vartheta^\top R \geq 0$ a.s.

4 The NUIP condition $I \cap \check{C} = \emptyset$

The condition $I \cap \check{C} = \emptyset$ will play an important role and will be called *No Unbounded Increasing Profit* (NUIP) condition as by Karatzas and Kardaras 2007. The qualifier "increasing" stems from the fact that $\vartheta^\top R \geq 0$ a.s. for $\vartheta \in I$. The qualifier "unbounded" reflects the following fact:

Suppose that $\vartheta \in I \cap \check{C}$ and $V_1^\vartheta = 1 + \vartheta^\top R$ where $\vartheta^\top R \geq 0$ a.s. and $P[\vartheta^\top R > 0] > 0$. Since $\vartheta \in \check{C}$, we know that $a\vartheta \in C$ for all $a > 0$. Moreover $a\vartheta^\top R \geq 0$ a.s. and $\{a\vartheta^\top R, a > 0\}$ is unbounded on the set $\vartheta^\top R > 0$ with positive measure.

Now suppose that $\vartheta \in I \cap \check{C}$ and ρ is a weak numeraire portfolio, then

$$\mathbb{E}[V_1^{a\vartheta}/V_1^\rho] = \mathbb{E}[1/V_1^\rho] + a\mathbb{E}[\vartheta^\top R/V_1^\rho] \text{ where } \mathbb{E}[\vartheta^\top R/V_1^\rho] > 0\,.$$

Thus $\mathbb{E}[V_1^{a\vartheta}/V_1^\rho]$ is unbounded in a, in particular $\mathbb{E}[V_1^{a\vartheta}/V_1^\rho] > 1$ for large a which is a contradiction. Therefore we can obtain the following result:

Proposition 4.1. *The NUIP condition $I \cap \check{C} = \emptyset$ is necessary for the existence of a weak numeraire portfolio.*

Note in particular that the NUIP condition is far weaker than the no arbitrage condition.

Example A (continuation). In the case $C := \{\pi \in \mathbb{R}^d : \pi^k \geq 0 \text{ and } \sum_k \pi^k \leq 1\}$, we know that $\check{C} = \{0\} \subset \mathbb{R}^d$.

Since $I \subset \mathbb{R}^d \setminus \mathcal{L}^\perp$, in particular $0 \notin I$, the NUIP condition $I \cap \check{C} = \emptyset$ is always satisfied.

Example B (Natural Constraints continuation). In the case $C := \Theta := \{\vartheta \in \mathbb{R}^d; 1 + \vartheta^\top R \geq 0 \text{ a.s.}\}$ we have $\check{C} = \{\vartheta \in \mathbb{R}^d; \vartheta^\top R \geq 0 \text{ a.s.}\} \supset I$. Here the NUIP condition $I \cap \check{C} = \emptyset$ amounts to the no arbitrage condition $I = \emptyset$.

Example C (continuation). In the case $C := \{\pi \in \mathbb{R}^d : \pi^k \geq 0\}$ we have $\check{C} = C$. Here the NUIP condition $I \cap \check{C} = \emptyset$ amounts to the no arbitrage condition $I \cap C = \emptyset$.

We now present an example where $\mathbb{E}[\log\ V_1^\pi] = \infty$ for nearly all π, but where V_1^π/V_1^ϑ is bounded in π for nearly all ϑ and where a unique numeraire portfolio exists.

Example D (see Kardaras 2006). Consider the case where $d = 1$ and

$$P[R \in dx] \propto \left(1_{(-1,1]} + x^{-1}(\log\{1+x\})^{-2} \cdot 1_{(1,\infty)}\right) dx\,.$$

Since the support Z of R is $[-1, \infty)$, we have $\Theta = [0,1] =: C$. Now the expected log-utility is

$$\mathbb{E}[\log\ V_1^\pi] = \mathbb{E}[\log(1 + \pi R)] = \infty \text{ for } \pi \in (0,1]$$

since $\int_1^\infty \log(1+\pi x)x^{-1}(\log(1+x))^{-2}dx = \infty$ which easily follows by use of the substitution $y = \log(1+x)$. Obviously $\mathbb{E}[\log\ V_1^\pi] = 0$ for $\pi = 0$.

However if we consider relative values $V_1^\pi/V_1^\vartheta = \frac{1+\pi R}{1+\vartheta R}$, then V_1^π/V_1^ϑ is bounded since

$$\min\left(\frac{\pi}{\vartheta}, \frac{1-\pi}{1-\vartheta}\right) \leq V_1^\pi/V_1^\vartheta \leq \max\left(\frac{\pi}{\vartheta}, \frac{1-\pi}{1-\vartheta}\right)\,.$$

Moreover, if we fix $\vartheta \in (0,1)$ and define

$$g(\pi) = \mathbb{E}[\log(V_1^\pi / V_1^\vartheta)] = \mathbb{E}\left[\log\left(\frac{1+\pi R}{1+\vartheta R}\right)\right],$$

then we obtain for $\pi \in (0,1)$

$$g'(\pi) = \mathbb{E}\left[\frac{R}{1+\pi R}\right]$$

where $g'(0+) = \infty$ and $g'(1-) = -\infty$. Therefore there exists a unique $\rho \in (0,1)$ such that $g'(\rho) = 0$. As a consequence we obtain the relation

$$\mathbb{E}[V_1^\pi / V_1^\rho] = \mathbb{E}\left[\frac{1+\pi R}{1+\rho R}\right] = 1 + (\pi - \rho)\mathbb{E}\left[\frac{R}{1+\rho R}\right] = 1$$

for any $\pi \in \Theta$. Then ρ will be called a numeraire portfolio (in the strict sense). The portfolio ρ is computed by Kardaras as $\rho \cong .916$.

Although the expected log-utility is infinite, the numeraire portfolio does not put all the weight on the stock. Finally we know that ρ is the unique portfolio such that

$$\mathbb{E}[\log(V_1^\rho / V_1^\rho)] = \sup_{\pi \in \Pi} \mathbb{E}[\log(V_1^\pi / V_1^\rho)] = 0\,.$$

5 The weak numeraire portfolio and the growth-optimal portfolio

Definition 5.1. (a) A portfolio $\rho \in \Pi$ is *log-optimal* if $\mathbb{E}[\log\ V_1^\pi] \leq \mathbb{E}[\log\ V_1^\rho]$ for every $\pi \in \Pi$.

(b) A portfolio $\rho \in \Pi$ will be called *growth optimal portfolio* (GOP) [or *relatively log-optimal*] if $\mathbb{E}[\log(V_1^\pi / V_1^\rho)] \leq 0$ for every $\pi \in \Pi$.

The present concept of GOP is used e.g. by Christensen & Larsen 2007, the name (relatively) log-optimal is used e.g. by Karatzas and Kardaras 2007. Of course, if the portfolio ρ is log-optimal with $\mathbb{E}[\log\ V_1^\rho] < \infty$, then ρ is also a GOP and we will prefer the notation GOP in that case. The two notions coincide if $\sup_{\pi\in\Pi} \mathbb{E}[\log\ V_1^\pi] < \infty$. In the Example D above, this condition fails and almost every portfolio is log-optimal. But we have existence of a unique numeraire portfolio which is the unique GOP.

Theorem 5.2. *A portfolio is a weak numeraire portfolio if and only if it is a GOP.*

Note that this result shows in particular that the existence of a weak numeraire portfolio implies the existence of a GOP and vice versa.

Proof of Theorem 5.2. (See Becherer 1999, Christensen and Larsen 2007, Bühlmann and Platen 2003.)

(i) Suppose ρ is numeraire portfolio. Then we have by Jensen's inequality

$$\mathbb{E}[\log(V_1^\pi / V_1^\rho)] \leq \log\left(\mathbb{E}[V_1^\pi / V_1^\rho]\right) \leq \log\ 1 = 0\,.$$

(ii) Suppose that ρ is GOP and π is an arbitrary portfolio. Then $V_1^\varepsilon := (1-\varepsilon)V_1^\rho + \varepsilon V_1^\pi$ is the value of some portfolio where $V_1^\varepsilon - V_1^\rho = \varepsilon(V_1^\pi - V_1^\rho)$. From $1 - t^{-1} \leq \log\ t$ for $t > 0$ we obtain

$$0 \geq \varepsilon^{-1} \cdot \mathbb{E}[\log(V_1^\varepsilon / V_1^\rho)] \geq \varepsilon^{-1} \cdot \mathbb{E}[(V_1^\varepsilon - V_1^\rho)/V_1^\varepsilon] = \mathbb{E}[(V_1^\pi - V_1^\rho)/V_1^\varepsilon].$$

From

$$-2 \leq 2\frac{x-y}{x+y} \leq \frac{x-y}{(1-\varepsilon)y + \varepsilon x} \uparrow \frac{x}{y} - 1 \text{ for } \frac{1}{2} \geq \varepsilon \downarrow 0 \text{ (where } x, y > 0)$$

we finally get $\mathbb{E}[V_1^\pi / V_1^\rho] \leq 1$ from the monotone convergence theorem.

□

Proposition 5.3. *The NUIP condition $I \cap \check{C} = \emptyset$ is necessary for the existence of a GOP.*

Proof. Suppose that ρ is a GOP and suppose that $\vartheta \in I \cap \check{C}$. Since $\vartheta \in I$, we know that $\vartheta^\top R \geq 0$ a.s. and $P[\vartheta^\top R > 0] > 0$. Now we conclude from Lemma 2.2 that $\rho + \vartheta \in C$ where $\mathbb{E}[\vartheta^\top R / V_1^{\rho+\vartheta}] > 0$ and thus $\mathbb{E}[\log(V_1^\rho / V_1^{\rho+\vartheta})] \leq \log\ \mathbb{E}[V_1^\rho / V_1^{\rho+\vartheta}] = \log\{1 - \mathbb{E}[\vartheta^\top R / V_1^{\rho+\vartheta}]\} < 0$. Now we have a contradiction to the optimality of ρ. □

However, one can also directly derive Proposition 5.3 from Theorem 5.2 and Proposition 4.1 without a proof.

6 Existence of weak numeraire portfolios

In this section we will show that the NUIP condition is also sufficient for the existence of a weak numeraire portfolio. This in particular shows that valuation via discounting by the wealth process of a weak numeraire can even be performed in situations where the no arbitrage condition is not satisfied. This was already emphasised by Platen 2006. For getting the existence result we need some technical notations and results.

Definition 6.1. For $f : \mathbb{R}^d \mapsto (0,1]$ we write $f \in \mathcal{F}$ if $\mathbb{E}[f(R) \cdot \log(1 + \|R\|)] < \infty$.

Example E. (see Kardaras 2006) We have $f_k \in \mathcal{F}$ and $f_k \uparrow 1$ for

$$f_k(x) := 1_{\{\|x\| \leq 1\}} + 1_{\{\|x\| > 1\}} \cdot \|x\|^{-1/k}\,.$$

Under the no-arbitrage condition $I \cap \Theta = \emptyset$, one knows that $\Theta \cap \mathcal{L}$ is compact (see Korn and Schäl 1999) where Θ is defined in Example B. Under the weaker NUIP condition we need the following more technical lemma.

Lemma 6.2. *Assume $I \cap \check{C} = \emptyset$. Let $\mathcal{F}^*$ be some subset of $\mathcal{F}$ which is bounded from below in the following sense: there is some $f^* \in \mathcal{F}$ such that $f \geq f^*$ for all $f \in \mathcal{F}^*$.*

Let $R \subset C$ be a set of portfolios which are "not too bad" in the following sense: for every $\rho \in R$, $\rho \in \mathcal{L} \setminus \{0\}$, there exists some $f \in F^*$ such that the function $[0,1] \ni u \mapsto g_f(u\rho)$ is increasing where

$$g_f(\pi) := \mathbb{E}[\log(1 + \pi^\top R) \cdot f(R)] \; [\leq (\log \|\pi\|)^+ + \mathbb{E}[\log(1 + \|R\|) f(R)] < \infty] \,.$$

Then R is bounded.

The lemma is hidden in the proof of Theorem 3.15 in Karatzas and Kardaras 2007.

Proof by contradiction. Suppose there exists some sequence $(\rho_m, f_m) \subset R \times \mathcal{F}^*$ such that $\rho_m \in \mathcal{L} \cap C$ and $[0,1] \ni u \mapsto g_{f_m}(u\rho_m)$ is increasing where $\|\rho_m\| \to \infty$. Define $\xi_m := \|\rho_m\|^{-1}\rho_m$. We can assume that $\xi_m \to \xi$ for some $\xi \in \mathcal{L}$. We want to show that $\xi \in \check{C}$.

Choose any $a > 0$ and m_a such that $0 < u = a/\|\rho_m\| < 1$ for $m \geq m_a$. Then $a\xi_m = u\rho_m = u\rho_m + (1-u)0 \in C$ since C is convex. Moreover, since C is closed, we also have $a\xi \in C$. This proves $\xi \in \check{C} \cap \mathcal{L}$ with $\|\xi\| = 1$. Now for $u \in (0,1]$ we have

$$\begin{aligned} 0 &\leq \varepsilon^{-1}[g_{f_m}(u\rho_m) - g_{f_m}((1-\varepsilon)u\rho_m)] \\ &= \mathbb{E}\left[\varepsilon^{-1}\left(\log\left\{1 + u\,\rho_m^\top R\right\} - \log\left\{1 + (1-\varepsilon)u\,\rho_m^\top R\right\}\right) \cdot f_m(R)\right]. \end{aligned}$$

From the concavity of log we conclude that the integrand is decreasing for $\varepsilon \downarrow 0$. Since the expectation if finite for $\varepsilon = 1$, we apply the monotone convergence theorem and obtain

$$0 \leq \mathbb{E}\left[\frac{d}{du}\log\{1 + u\rho_m^\top R\} \cdot f_m(R)\right] = \mathbb{E}\left[(1 + u\rho_m^\top R)^{-1}\rho_m^\top R f_m(R)\right].$$

Again choose any $a > 0$ and m_a such that $0 < u = a/\|\rho_m\| < 1$ for $m \geq m_a$. Then

$$0 \leq \mathbb{E}[(1 + a\,\xi_m^\top R)^{-1}\xi_m^\top R\, f_m(R)] \text{ where } (1 + a\,\xi_m^\top R)^{-1}\xi_m^\top R\, f_m(R) \leq a^{-1}.$$

From Fatou's lemma we now obtain

$$\begin{aligned} a^{-1} &\geq \mathbb{E}\Big[(1 + a\xi^\top R)^{-1}\xi^\top R\, \overline{\lim_m}\, f_m(R)\Big] \\ &\geq \overline{\lim_m}\, \mathbb{E}\Big[(1 + a\xi_m^\top R)^{-1}\xi_m^\top R f_m(R)\Big] \geq 0 \end{aligned}$$

Since $\overline{\lim}\, f_m(R) \geq f^*(R) > 0$, we conclude from the first inequality that $1 + a\,\xi^\top R > 0$ a.s.

Now $a > 0$ was arbitrary, so we conclude that $\xi^\top R \geq 0$ a.s. where $\|\xi\| = 1$ and $\xi \in \mathcal{L}$. Therefore $P[\xi^\top R = 0] > 0$, otherwise $\xi \in \mathcal{L}^\perp$. Thus we finally have $\xi \in I$ and hence $\xi \in I \cap \check{C}$ which is a contradiction to our assumption. □

Theorem 6.3. *Under the NUIP assumption $I \cap \check{C} = \emptyset$, there exists a weak numeraire portfolio ρ.*

If $\mathbb{E}[\log(1+\|R\|)] < \infty$*, then* ρ *is obtained as the unique solution of the following concave optimisation problem and thus the only GOP in* $C \cap \mathcal{L}$:

$$\rho = \arg \max_{\pi \in C \cap \mathcal{L}} g(\pi) \text{ where } g(\pi) := \mathbb{E}[\log(1+\pi^\top R)].$$

Remark 6.4. In the general case, where the condition $\mathbb{E}[\log(1+\|R\|)] < \infty$ does not hold, one can solve a sequence of optimisation problems and show that the corresponding solutions converge to the solution of the original problem, see below and Theorem 3.15 in Karatzas and Kardaras 2007.

Proof. We start with a sequence $(f_k) \subset \mathcal{F}$ where $f_k \uparrow 1$. The sequence can be chosen as in Example E above. Now define $g_k(\pi) = g_{f_k}(\pi) := \mathbb{E}[\log(1+\pi^\top R) \cdot f_k(R)]$.

Then g_k is strictly concave on $C \cap \mathcal{L}$ and $-\infty \le g_k(\pi) < +\infty$. Further set

$$0 \le g_k^* := \sup_{\pi \in C} g_k(\pi) = \lim_{n \to \infty} g_k(\rho_{kn})$$

for some sequence $(\rho_{kn}) \subset C$. Since $g_k(\pi+\zeta) = g_k(\pi)$ for $\zeta \perp \mathcal{L}$ we can choose $\rho_{kn} \in \mathcal{L} \cap C$. Moreover, we may choose ρ_{kn} such that $g_k(\rho_{kn}) = \max_{0 \le u \le 1} g_k(u\rho_{kn}) \le \sup_{\pi \in C} g_k(\pi)$.

Then by concavity, $u \mapsto g_k(u\rho_{kn})$ is increasing. From the preceding lemma we know that $R = (\rho_{kn})$ is bounded, in particular $g_k^* \in [0,\infty)$ and $g_k^* = g_k(\rho_k^*)$.

Now fix some k and assume $w.l.o.g.$ that $\rho_{kn} \to \rho_k^*$ for some $\rho_k^* \in C$ where $g_k^* = g_k(\rho_k^*)$ since C is closed. Choose $\pi \in C$, then $[0,1] \ni u \mapsto g_k(\rho_k^* + u(\pi - \rho_k^*))$ is real valued, concave. Since ρ_k^* is a maximum point, we conclude from the concavity that

$$0 \le G(u) := \frac{1}{u}[g_k(\rho_k^*) - g_k(\rho_k^* + u(\pi - \rho_k^*))] \le g_k(\rho_k^*) - g_k(\pi)$$

is increasing in $0 < u \le 1$. From the monotone convergence theorem, we obtain for $u \downarrow 0$, again by concavity of log,

$$G(u) \downarrow E\left[-\frac{d}{du} \log\left\{1 + [\rho_k^* + u(\pi - \rho_k^*)]^\top R\right\}\Big|_{u=0} \cdot f_k(R)\right].$$

Thus we get

$$\mathbb{E}[(\pi - \rho_k^*)^\top R \,/(1 + \rho_k^{*\top} R)] \le 0.$$

Since we know that (ρ_k^*) is also bounded, we may assume that $\rho_k^* \to \rho$ for some $\rho \in C \cap \mathcal{L}$.

Now $(\pi - \rho_k^*)^\top R \,/(1 + \rho_k^{*\top} R) = (1+\pi^\top R)/(1+\rho_k^{*\top} R) - 1 \ge 1$.

Then we obtain from Fatou's lemma $r(\pi|\rho) \le 0$, since

$$r(\pi|\rho) = \mathbb{E}[(\pi-\rho)^\top R \,/(1+\rho^\top R)] = \mathbb{E}[\lim_k (\pi-\rho_k^*)^\top R \,/(1+\rho_k^{*\top} R)] \le \overline{\lim}\, \mathbb{E}[\cdots] \le 0.$$

From Lemma 3.3 we finally conclude that ρ is a weak numeraire portfolio.

In the case where $\mathbb{E}[\log(1+\|R\|) < \infty] < \infty$, we can choose $f_k \equiv 1$ for all k and thus $\rho_k = \rho$.

$$\text{Then } g^* := \sup_{\pi \in C} g(\pi) = g(\rho_k^*) = \max_{\pi \in C \cap \mathcal{L}} \text{ where } g(\pi) := \mathbb{E}[\log(1+\pi^\top R)].$$

Since g is strictly concave on $C \cap \mathcal{L}$, the maximum point is unique. □

7 Deflators and value preserving portfolios

The concept of a deflator is important for the valuation of uncertain payment streams and is more general than that of a numeraire portfolio.

Definition 7.1. The class $\mathcal{D}$ of *supermartingale deflators* is defined as

$\mathcal{D} := \{D \geq 0;\ D$ is a random variable with $\mathbb{E}[DV_1^\pi] \leq 1 (= V_0^\pi)\}$ for all portfolios $\pi\}$.

Since $0 \in \Pi$, we know that $\mathbb{E}[D] \leq 1$ for $D \in \mathcal{D}$.

Corollary 7.2. *(a) A portfolio $\rho \in \Pi$ is a weak numeraire portfolio if and only if $(V_1^\rho)^{-1}$ is a supermartingale deflator.*

(b) $\mathbb{E}[\log\ V_1^\rho] = \inf_{D \in \mathcal{D}} \mathbb{E}[\log(D^{-1})]$.

The second property in (a) is introduced by Korn 1997 and called "ρ is *interest-oriented*". The property (b) of ρ can be seen as an optimal property dual to log-optimality.

Proof. (a) is clear by definition. (b) See Becherer 2001. $\mathbb{E}[\log(D^{-1})]$ makes sense since $\mathbb{E}[\log^-(D^{-1})] \leq \mathbb{E}[D] \leq 1$. Assume *w.l.o.g.* that the right hand in (b) is finite and $\mathbb{E}[\log(D^{-1})] \in \mathbb{R}$. Then $\mathbb{E}[\log\ V_1^\rho - \log(D^{-1})] = \mathbb{E}[\log(DV_1^\rho)] \leq \log\ \mathbb{E}[DV_1^\rho] = 0$. □

Definition 7.3 (Hellwig 1996). For $\pi \in \Pi$ and $D \in \mathcal{D}$, $V_D^\pi := D \cdot (1 + \pi^\top R)$ is called *present economic value* of π (at time 0) associated with $D \in \mathcal{D}$.

Since D is a supermartingale deflator, we always have $\mathbb{E}[V_D^\pi] \leq 1$ where 1 is here the initial value. Therefore the following definition is interesting:

Definition 7.4 (Hellwig 1996). A portfolio $\pi \in \Pi$ is called *value preserving* if $V_D^\pi \equiv 1$ a.s. for some $D \in \mathcal{D}$.

Theorem 7.5. *The following properties are equivalent:*

(1) π is value preserving w.r.t. the supermartingale deflator D;

(2) π is a weak numeraire portfolio and $D = (1 + \pi^\top R)^{-1}$.

Thus, by Theorem 7.5 existence of a value preserving portfolio is also related to the existence of a GOP (see Korn and Schäl 1998).

Proof. "(1) $\Rightarrow$ (2)" From $D \cdot (1 + \pi^\top R) = 1$ we get $D = (1 + \pi^\top R)^{-1}$ where $D \in \mathcal{D}$. Now Corollary 7.2 (a) applies. "(2) $\Rightarrow$ (1)" Again from Corollary 7.2 we know that $D = (1 + \pi^\top R)^{-1}$ is a deflator and $D \cdot (1 + \pi^\top R) = 1$. □

8 Fair portfolios and applications in actuarial valuation

Benchmarked portfolios and fair valuation is a concept that is suggested for use in actuarial valuation by Bühlman and Platen 2003. As ibidem we call V_1^π/V_1^ρ the *benchmarked value of portfolio* π if ρ is a weak numeraire portfolio and hence V_1^ρ is uniquely determined according to Proposition 3.2. Then we know that: $\mathbb{E}[V_1^\pi/V_1^\rho] \leq 1$ for every portfolio π.

In financial valuations in competitive markets, a price is typically chosen such that seller and buyer have no systematic advantage or disadvantage. Let the random variable H be a contingent claim which is a possibly negative random payoff. Candidates for prices of H are $\mathbb{E}[DH]$ for some deflator $D \in \mathcal{D}$. For $H = V_1^\pi$ we thus have $\mathbb{E}[DH] \leq 1$. For the case $\mathbb{E}[DH] < 1$, this could give an advantage to the seller of the portfolio π; its expected future benchmarked payoff is less than its present value. The only situation when buyers and sellers are equally treated is when the benchmarked price process V_t^π/V_t^ρ is a martingale, that means in our situation: $\mathbb{E}[V_1^\pi/V_1^\rho] = 1$.

Definition 8.1 (see Bühlmann and Platen 2003). A value process V_t, $t = 0, 1$, is called *fair* if its benchmarked value V_t/V_t^ρ is a martingale, i.e. if $\mathbb{E}[V_1/V_1^\rho] = V_0$ (since $V_0^\rho = 1$).

Let us consider a contingent claim H, which has to be paid at the maturity date 1. Let ρ be the weak numeraire portfolio. We choose the following pricing formula

$$pr(H) := \mathbb{E}[H/V_1^\rho] \tag{8.1}$$

which by definition is fair. In contrast to classical actuarial valuation principles no loading factor enters the valuation formula. For premium calculations in insurance business the use of a change of measure is explained in Delbaen & Haezendonck 1989. An important case arises when H is independent of the value V_1^ρ. Then we obtain

$$pr(H) = \mathbb{E}[H] \cdot \mathbb{E}[1/V_1^\rho]\,. \tag{8.2}$$

Here $P(0,1) = \mathbb{E}[1/V_1^\rho]$ is the fair price of the contingent claim $H' \equiv 1$ to be paid at the maturity date $T = 1$ and thus the *zero coupon bond* with maturity 1. Thus (8.2) is the classical actuarial pricing formula in the case of stochastic interest rates and $pr(H) := \mathbb{E}[H/V_1^\rho]$ is an extension to the more general case where dependence may occur.

For *equity-linked* or *unit-linked insurance contracts* we look again at a claim H payed at $T = 1$ where H has the following form: $H = U \cdot V_1^\pi$. Intuitively, H stands now for unit linked benefit and premium. Then H can be of either sign. The benefit at maturity is linked to some strictly positive reference portfolio V_1^π with given portfolio π. The insurance contract specifies the reference portfolio π and the random variable U depending on the occurrence of insured events during the period $(0, 1]$, for instance, death, disablement or accident.

These products offer the insurance company as well as the insurance customer advantages compared to traditional products. The insurance industry may benefit from offering more competitive products and the customer may benefit from higher yields in financial markets. Compared to classical insurance products, one distinguishing feature of unit-linked products is the random amount of benefit. But the traditional basis for pricing life insurance policies, the principle of equivalence, based on the idea that premiums and expenses should balance in the long run, does not deal with random benefits. Therefore, we have to use financial valuation theories together with elements of actuarial theory to price such products.

The *standard actuarial value* $pr_o(H)$ of the contingent claim $H = U \cdot V_1^\pi$ is determined by the properly defined liability of prospective reserve as

$$pr_o(H) = V_0^\pi \cdot \mathbb{E}[H/V_1^\pi] = \mathbb{E}[H \cdot V_0^\pi / V_1^\pi].$$

The standard actuarial methodology assumes that the insurer invests all payments in the reference portfolio π. Then one obtains for $pr_o(H)$, when expressed in units of the domestic currency, the expression

$$pr_o(H) = pr_o(U \cdot V_1^\pi) = V_0^\pi \cdot \mathbb{E}[U].$$

We observe the difference between $pr_o(H)$ and $pr(H)$. Hence the standard actuarial pricing and fair pricing will, in general, lead to different results. As one could see this is to be expected when the endowments depend on the numeraire portfolio. Indeed, let us assume that ρ is a numeraire portfolio (in the strict sense), then $\mathbb{E}[V_1^\pi / V_1^\rho] = V_0^\pi / V_0^\rho = V_0^\pi$ and we obtain

$$pr(U \cdot V_1^\pi) - pr_o(U \cdot V_1^\pi) = \mathrm{Cov}(U, V_1^\pi / V_1^\rho). \tag{8.3}$$

A similar formula is derived by Dijkstra 1998. Hence, the two prices coincide if and only if U and V_1^π / V_1^ρ are uncorrelated. Moreover, the sign of the difference is the sign of the covariance. This condition differs from the one given by Bühlmann and Platen 2003.

In many cases, the endowment H of the insurance contract will include a guaranteed (non-stochastic) amount $g(K)$ where K is the premium paid by the insured. Then the benefit at maturity is composed of the guaranteed amount plus a call option with exercise price $g(K)$ and with the reference portfolio as underlying assets. Then the fair premium is the solution to an equation in K and $g(K)$ (see Nielsen and Sandmann 1995).

9 Existence of numeraire portfolios

It seems to be a general agreement that S_t^k should be fair, since S_0^k is a fair price for $H = S_1^k$ for every $k \in \{0, 1, \dots, d\}$. This leads to the requirement

$$\mathbb{E}[S_1^k / V_1^\rho] = S_0^k,\ 0 \le k \le d. \tag{9.1}$$

Definition 9.1. A portfolio $\rho \in \Pi$ will be called *numeraire portfolio* (in the strict sense), if the above condition (9.1) holds.

Proposition 9.2. *(a) If ρ is a numeraire portfolio, then we have for any strategy ξ and*

$$V_1^\xi(x) := x + \xi^\top \Delta S : \quad \mathbb{E}[V_1^\xi(x)/V_1^\rho] = x = V_0^\xi .$$

(b) A numeraire portfolio is a weak numeraire portfolio.

Proof. Set $V_t^\xi(x) = \sum_{k=0}^d \xi^k S_t^k$. Then we obtain $\mathbb{E}[V_1^\xi/V_1^\rho] = \sum_{k=0}^d \xi^k \mathbb{E}[S_1^k/V_1^\rho] = \sum_{k=0}^d \xi^k S_0^k = V_0$. In the present simple situation, where the horizon is $T = 1$ we do not have any integrability problems and we even get the martingale property. In the more general case we obtain the supermartingale property from the fact that each non-negative local martingale is a supermartingale. □

As in Lemma 3.3 we know that ρ is a numeraire portfolio if and only if $r(\pi|\rho) = 0$, where π is a unit vector or the zero vector in $\mathbb{R}^d$. There $r(\pi|\rho)$ is the directional derivative of $g(\pi) := \mathbb{E}[\log(1 + \pi^\top R)]$ at the point ρ in the direction of $\pi - \rho$ (if g is finite).

In general, we cannot expect to be able to compute the numeraire portfolio just by naively trying to solve the first-order condition $\nabla g(\rho) = r(0|\rho) = 0$, because sometimes this equation simply fails to have a solution. In this section, we make the following assumptions.

Assumption 9.3. $C = \Theta$ describes the natural constraints, $I = \emptyset$ which here is the NUIP condition, and integration of the log exists in the following sense: $\mathbb{E}[\log(1 + \|R\|)] < \infty$.

We now introduce another condition given in the following theorem proved in Schäl (1999, Theorem 4.15):

Theorem 9.4. *Let ρ be the only GOP in $\Theta \cap \mathcal{L}$ according to Theorem 6.3. Then, the condition*

$$\mathbb{E}[\, \vartheta^\top \cdot R/(1 + \vartheta^\top \cdot R)] < 0 \textit{ for all } \vartheta \in \partial\Theta \cap \mathcal{L}\,, \tag{9.2}$$

implies the first order condition: $\mathbb{E}[R^k/(1 + \rho^\top \cdot R)] = 0$ for $k = 1, \dots, d$.

Corollary 9.5. *Let ρ be defined as in the preceding theorem. Then, under condition (9.2), ρ is a numeraire portfolio (in the strict sense) and*

$$\mathbb{E}[\, 1/V_1^\rho] = 1. \tag{9.3}$$

Proof. We obtain from Theorem 9.4

$$1 = 1 - \sum_{k=1}^d \rho^k \mathbb{E}[R^k/(1 + \rho^\top \cdot R)] = 1 - \mathbb{E}[\rho^\top \cdot R/(1 + \rho^\top \cdot R)]\,,$$

which implies (9.3). Now we get for $0 \le k \le d$ with $R^0 \equiv 0$:

$$\mathbb{E}[S_1^k/V_1^\rho] = \mathbb{E}[S_0^k(1 + R^k)/(1 + \rho^\top \cdot R)] = S_0^k \mathbb{E}[1/(1 + \rho^\top \cdot R)] = S_0^k\,.$$

Thus ρ is a numeraire portfolio. □

Example F. *The one-dimensional case.*

Consider the case where $d = 1$ and R is bounded, then the support Z is a compact subset of $\mathbb{R}$. Set $-\alpha = \min\ Z$, $\beta = \max\ Z$. Then $\mathrm{conv}(Z) = [-\alpha, \beta]$. For the no-arbitrage condition we need $\alpha > 0$, $\beta > 0$.

Then condition (9.3) is satisfied if and only if

$$\mathbb{E}[\ R/(1+\frac{1}{\alpha}R)] < 0 < \mathbb{E}[\ R/(1-\frac{1}{\beta}R)]\,. \tag{9.4}$$

For a proof we have

$$\min_{z\in Z} 1+\vartheta z = \min_{-\alpha\le z\le\beta} 1+\vartheta z = 1-\vartheta\alpha \text{ for } \vartheta > 0 \text{ and } \ = 1+\vartheta\beta \text{ for } \vartheta < 0\,.$$

Hence, we know that $\Theta = [-\frac{1}{\beta},\ \frac{1}{\alpha}]$ and $\partial\Theta = \{-\frac{1}{\beta},\ \frac{1}{\alpha}\}$.

Then $E\left[\frac{\vartheta\cdot R}{1+\vartheta\cdot R}\right] = \vartheta\cdot E\left[\frac{R}{1+\vartheta\cdot R}\right] < 0$ for $\vartheta \in \partial\Theta$ if and only if (9.4) holds. In fact, the condition (9.4) is weak. It can be looked upon as a kind of no-arbitrage condition. The martingale case $\mathbb{E}[R] = 0$ is not interesting as we can choose $\vartheta = 0$ then. Let us suppose that $\mathbb{E}[R] > 0$. Then $\mathbb{E}[R/(1-R/\beta)] \ge \mathbb{E}[R] > 0$ and the condition $\mathbb{E}[R/(1+R/\alpha)] < 0$ requires that there should not be too little probability for negative values of R. The condition (9.4) can easily be proved to be also necessary for the first order condition.

We will give a sufficient condition for (9.2) which is far from being necessary, however.

Theorem 9.6. *If Ω or Z is finite, then the condition (9.2) is always satisfied and thus the statements of Corollary 9.5 hold true.*

Proof (See also Long 1990). If Ω is finite, then Z is finite. Choose $\vartheta \in \partial\Theta \cap \mathcal{L}$, then one obtains the following relation :

$$0 = \min_{z\in Z}(1+\vartheta^\top\cdot z) = 1+\vartheta^\top\cdot z_o \text{ for some } z_o \in Z\,.$$

Further, $\{R = z_o\} \subset \{1+\vartheta^\top R = 0\} = \{\vartheta^\top\cdot R = -1\}$. Now

$$\begin{aligned}\mathbb{E}[\vartheta^\top\cdot R/(1+\vartheta^\top\cdot R)] &\le \mathbb{E}[1_{\{R=z_o\}}\cdot\vartheta^\top\cdot R/(1+\vartheta^\top\cdot R)]\\ &\quad +\mathbb{E}[1_{\{\vartheta^\top\cdot R>0\}}\cdot\vartheta^\top\cdot R/(1+\vartheta^\top\cdot R)]\\ &\le -\mathbb{E}[1_{\{R=z_o\}}/(1+\vartheta^\top\cdot R)]+1\\ &= -\infty,\ \text{since } P[R=z_o]>0\,.\end{aligned}$$

□

For the theorem one can also use a result by Hakansson 1971 that the GOP can be chosen as an interior point. The theorem is generalised in (Korn and Schäl 1999, Theorem 4.22).

It is known that the existence of a growth-optimal portfolio will not imply the existence of a numeraire portfolio (see Becherer 2001). We will give an example.

Example G. We may restrict attention to the case $d = 1$ (see Example F). Let the distribution of R on $Z := [-1, 1]$ be given by

$$\mathbb{E}[g(R)] := \lambda \cdot \int_{-1}^{0} \frac{3}{2}(1 - z^2)g(z)dz + (1 - \lambda) \cdot \int_{0}^{1} \frac{3}{2}(1 - z^2)g(z)dz\,,$$

where we choose $\lambda > 0$ sufficient small, e.g. $\lambda = 1/12$. Then

$$\begin{aligned} \mathbb{E}[R] \quad &:= \quad \lambda \cdot \int_{-1}^{0} \frac{3}{2}(1 - z^2)z\,dz + (1 - \lambda) \cdot \int_{0}^{1} \frac{3}{2}(1 - z^2)z\,dz \\ &= \quad (1 - 2\lambda) \int_{0}^{1} \frac{3}{2}(1 - z^2)z\,dz = \frac{3}{8}(1 - 2\lambda) > 0\,. \end{aligned}$$

[Obviously, by the choice of $\lambda = \lambda* = \frac{1}{2}$, one obtains an equivalent martingale measure (see below)]. Now set

$$f(\vartheta) := E\left[\frac{R}{1 + \vartheta \cdot R}\right]\,,$$

then f is strictly decreasing on $\Theta := [-1, 1]$, where $f(-1) \geq f(\vartheta) \geq f(1)$ for $\vartheta \in \Theta$. Now

$$\begin{aligned} f(1) \quad &= \quad E\left[\frac{R}{1 + R}\right] \\ &= \quad \lambda \int_{-1}^{0} \frac{3}{2}(1 - z)z\,dz + (1 - \lambda) \cdot \int_{0}^{1} \frac{3}{2}(1 - z)z\,dz \\ &= \quad \frac{1}{4} - \frac{3}{2}\lambda > 0\,, \\ f(-1) \quad &= \quad E\left[\frac{R}{1 - R}\right] \\ &= \quad \lambda \int_{-1}^{0} \frac{3}{2}(1 + z)z\,dz + (1 - \lambda) \cdot \int_{0}^{1} \frac{3}{2}(1 + z)z\,dz \\ &= \quad \frac{5}{4} - \lambda > 0\,. \end{aligned}$$

Hence there is no $\vartheta \in \Theta$ such that $f(\vartheta) = 0$ and ϑ is hence a numeraire portfolio. On the other hand, we have $\infty > f(-1) \geq f(\vartheta) = \frac{d}{d\vartheta} E\left[\ln(1 + \vartheta \cdot R)\right] \geq f(1) > 0$ for $-1 < \vartheta < 1$.

Thus, we know that $\max_{\vartheta \in \Theta} \mathbb{E}[\ln(1 + \vartheta \cdot R)] = \mathbb{E}[\ln(1 + R)]$ and $\vartheta* = 1$ defines the GOP.

10 Equivalent martingale measures and the numeraire portfolio

A well-known candidate for a fair price of a financial derivative described by the contingent claim H is given by an EMM Q (defined below) with positive density dQ/dP according to

$$E_Q\left[H/S_1^0\right] = \mathbb{E}\left[\frac{dQ}{dP}H/S_1^0\right]$$

where $\frac{dQ}{dP}/S_1^0$ is a deflator (see Duffie 1992, p. 23).

Definition 10.1. A probability measure Q is an equivalent martingale measure (EMM), if Q has a (a.s.) positive density dQ/dP such that

$$E_Q[S_1^k/S_1^0] = \mathbb{E}\left[\frac{dQ}{dP}S_1^k/S_1^0\right] = S_0^k/S_0^0,\ 0 \leq k \leq d. \tag{10.1}$$

Here, we present the general property though we decided to consider only the case $S_t^0 \equiv 1$.

Proposition 10.2. *(a) A portfolio $\rho \in \Pi$ is a numeraire portfolio if and only if $1/V_1^\rho = dQ^*/dP$ for some EMM Q^*.*

(b) In the case of existence, an EMM Q^* implied by a numeraire portfolio in the sense of (a) is unique.

Proof. (a) We make use of (9.1). For the 'only if'-direction we get $\mathbb{E}[dQ^*/dP] = 1$ from $dQ^*/dP = 1/V_1^\rho > 0$ and $\mathbb{E}[S_1^k/V_1^\rho] = S_0^k$ for $k = 0$.

Part (b) follows from the uniqueness of V_1^ρ according to Proposition 3.2. □

From the "Fundamental Theorem of Asset Pricing" (see Back and Pliska 1990, Dalang et al. 1990, Schachermayer 1992, Rogers 1994, Jacod & Shiryaev 1998) we know that there exists an EMM if and only if the no arbitrage condition $I = \emptyset$ holds. If in addition the market is complete, then the EMM Q is known to be unique and we may consider $L^{-1} := (dQ/dP)^{-1}$ as a contingent claim. Upon making use of the definition of completeness, we obtain $L^{-1} = V^\xi(x)$ for some strategy ξ and some initial capital x. Then we obtain $x = \mathbb{E}[L\ V_1^\xi(x)] = \mathbb{E}[L\ L^{-1}] = 1$. Therefore we conclude that $V_1^\xi(x) = 1 + \rho^\top R$ where $\rho^k = \xi^k S_0^k$. From the preceding proposition we obtain the following result:

Corollary 10.3. *Let $C = \Theta$ describe the natural constraints. If the market is complete and free of arbitrage opportunities, then a numeraire portfolio (in the strict sense) exists.*

For the remainder of this section, we consider the case where $d = 1$ and (as in Example F):

$$\text{conv}(Z) = [-\alpha, \beta] \text{ for some } \alpha,\ \beta > 0 \text{ with } -\alpha,\ \beta \in Z. \tag{10.2}$$

The minimal martingale measure was introduced by Föllmer and Schweizer 1991 in the context of option hedging and pricing in incomplete financial markets. By the discrete-time Girsanov transformation one obtains the minimal martingale Q^o according to $\frac{dQ^o}{dP} = b + a \cdot R$ (see Schweizer 1995). From the two conditions $\mathbb{E}[\frac{dQ^o}{dP}] = 1$ and $\mathbb{E}[\frac{dQ^o}{dP} R] = 0$, one can compute that

$$b = 1 + \{\mu/\sigma\}^2, \; a \; = -\mu/\sigma^2 \text{ where } \mu := \mathbb{E}[R] \text{ and } \sigma^2 := \text{Var}[R]\,. \tag{10.3}$$

One difficulty with the Girsanav transformation in discrete time is that it may lead to a density with negative values. The resulting martingale measure is then called a *signed martingale measure*. However, in the case where $Z \subset \{d-1, 0, u-1\}$ for some $0 < d < 1 < u$, it is easy to see that $\frac{dQ^o}{dP} > 0$. On the other hand, we know from Theorem 9.6 and Corollary 9.5 that $\frac{dQ^*}{dP} = \{1 + \rho R\}^{-1} > 0$ always defines a (positive) martingale measure if Z is finite. Thus we know that the minimal martingale measure cannot coincide with the martingale measure Q^* induced by the numeraire portfolio if Q^o is not a positive measure but a signed measure. It can be shown that the two measures only coincide in a binomial model that means only for a complete market (according to Harrison & Pliska 1981 and Jacod & Shiryaev 1998).

A binomial model is characterised by the fact

$$R \in \{-\alpha, \; \beta\} \text{ a.s.} \tag{10.4}$$

Theorem 10.4. *Let Q^* be the measure defined by Proposition 10.2 and let Q^o be the minimal martingale measure. Then $Q^* = Q^o$ if and only if (10.4) holds.*

The proof is given in Korn and Schäl (1999, Theorem 5.18). The theorem is surprising because one always has $Q^* = Q^o$ in the important case of financial markets modeled by diffusion processes (see Becherer 2001, Korn 1998).

11 Portfolio optimisation and the numeraire portfolio

So far we mainly highlighted the role of the numeraire portfolio in valuation of uncertain payment streams. However, we already saw that the numeraire portfolio is closely related to the growth optimal portfolio. In this section, we generalise this and show that, for a wide class of portfolio optimisation problems, the numeraire portfolio is the main ingredient of their solution.

Definition 11.1. (a) A strictly concave function U on $(0, \infty)$ which is increasing, twice continuously differentiable and satisfies

$$U'(0+) = \infty, \quad U'(\infty) = 0 \tag{11.1}$$

is called a *utility function*.

(b) We call the optimisation problem

$$u(x) := \sup_{\pi \in \Pi} \mathbb{E}[U(V_1^\pi(x))], \text{ where } V_1^\pi(x) = x \cdot V_1^\pi, \tag{11.2}$$

the *portfolio problem* of an investor with initial value x.

Popular utility functions are $U(x) = \log\ x$ or $U(x) = \frac{1}{\gamma}x^\gamma$ for $\gamma < 1$. The portfolio problem can now be explicitly solved in a complete market setting:

Theorem 11.2. *Let ρ be the weak numeraire portfolio; define*

$$\begin{aligned} I(y) &= (U')^{-1}(y),\ X(y) = \mathbb{E}[I(y/V_1^\rho)/V_1^\rho],\ Y(x) = X^{-1}(y)\,, & (11.3)\\ B &= I(Y(x)/V_1^\rho) \ \textit{and assume} & (11.4)\\ X(y) &< \infty\,. & (11.5) \end{aligned}$$

(a) Then $\mathbb{E}[U(V_1^\pi(x))] \le \mathbb{E}[U(B)]$ and $\mathbb{E}[B/V_1^\rho] = x$ for all admissible portfolios π.

(b) If the market is complete and ρ is chosen as the numeraire portfolio, then B is the optimal final value for the portfolio problem of an investor with initial wealth x.

Proof. Under the assumption (11.5) it can easily be shown (by dominated and/or monotone convergence) that $X(y)$ is strictly decreasing with $X(0) = \infty$, $X(\infty) = 0$. Thus, an inverse $Y(x)$ exists and one can define B as in (11.4). Further, by construction, B satisfies

$$\mathbb{E}[B/V_1^\rho] = x\,, \tag{11.6}$$

while for all other admissible portfolios π we have

$$\mathbb{E}[V_1^\pi(x)/V_1^\rho] \le x\,. \tag{11.7}$$

The following property of a concave function

$$U(x) \le U(y) + U'(y)(x-y),\ y, x > 0\,, \tag{11.8}$$

implies that

$$U(x) \le U(I(y)) + y(x - I(y)),\ y, x > 0\,. \tag{11.9}$$

From (11.9), (11.4)–(11.7) we then obtain

$$\mathbb{E}[U(V_1^\pi(x))] \le \mathbb{E}[U(B) + Y(x)(\mathbb{E}[V_1^\pi(x)/V_1^\rho] - \mathbb{E}[B/V_1^\rho]) \le \mathbb{E}[U(B)]\,. \tag{11.10}$$

If the market is complete, then there exists a portfolio π^B and an initial value x^B generating the final payment of B, i.e. $V_1^{\pi^B}(x^B) = B$. Now, $x^B = \mathbb{E}[B/V_1^\rho] = x$ by (11.6), since ρ is a numeraire portfolio (see proof of Proposition 9.2). Thus π^B is a solution to the portfolio problem. □

Example H. (a) Note that for $U(x) = \log\ x$, we recover from Theorem 11.2 that we have

$$B = xV_1^\rho\,, \tag{11.11}$$

which restates the relation between the growth optimal and the numeraire portfolio.

(b) For $U(x) = \frac{1}{\gamma}x^\gamma$, Theorem 11.2 yields the optimal final wealth of

$$B = x(V_1^\rho)^{\gamma'}/\mathbb{E}[(V_1^\rho)^{\gamma\cdot\gamma'}] \text{ where } \gamma' = \frac{1}{1-\gamma}. \tag{11.12}$$

Remark 11.3. Portfolio optimisation problems in incomplete markets can be solved by duality methods in a similar way as in Kramkov and Schachermayer 1999. There, the problem is transformed to auxiliary markets which are complete. The portfolio problem in these markets is again solved with the help of the numeraire portfolio.

The optimisation problem (11.2) makes sense only if its value function u is finite. Due to the concavity of U, if $u(x) < +\infty$ for some $x > 0$, then $u(x) < +\infty$ for all $x > 0$ and u is continuous, concave and increasing. When we have $u(x) = \infty$ for some (equivalently, all) $x > 0$, there are two cases. Either the supremum in (11.2) is not attained, so there is no solution; or, in case there exists a portfolio with infinite expected utility, the concavity of U will imply that there will be infinitely many of them. We will show that one cannot do utility optimisation if the NUIP condition fails. We can use the same arguments as at the beginning of Section 4. Then $a\vartheta^\top R$ is unbounded for $a \to \infty$ on the set $\vartheta^\top R > 0$ where $P[\vartheta^\top R > 0] > 0$. Thus

$$u(x) \geq \lim_{a\to\infty} \mathbb{E}[U(xV_1^{a\vartheta})] = U(1)\cdot P[\vartheta^\top R = 0] + U(\infty)\cdot P[\vartheta^\top R > 0]$$

and we proved the following result (see Karatzas and Kardaras 2007 Prop. 4.19):

Proposition 11.4. *Assume that the NUIP condition fails. If $U(\infty) = \infty$ then $u(x) = \infty$ for all $x > 0$. If $U(\infty) < \infty$, then there is no solution.*

12 Additional remarks

1 Vasicek 1977 was perhaps the first who used the concept of a numeraire portfolio for an equilibrium characterisation of the term structure. In the language of Long 1990 and of the present paper, Vasicek constructed a numeraire portfolio investing in two assets: the short rate and a long rate.

2 By the use of the numeraire portfolio we can replace the change of measure $P\to Q$ where Q is an EMM by changing the numeraire $\{S_t^0\}\to\{V_t^\rho\}$ and sticking to the original probability measure P. There P models the 'true world'-probability which can be investigated by statistical methods. Long 1990, for example, studied the application of measuring abnormal stock returns by discounting NYSE-stock returns by empirical proxys of the numeraire portfolio.

3 Further properties and applications in the diffusion case, where the numeraire portfolio is mean-variance efficient and therefore related to the CAPM-theory, can also be found in Bajeux-Besnainou & Portait (97) and Johnson (96).

4 De Santis, Gerard and Ortu 2000 are interested in the case where no self-financing trading strategy has strictly positive value and introduce the concept of a generalised numeraire portfolio based on non-self-financing strategies.

5 A further advantage of the present discrete time market is the fact that there exists only one concept of no-arbitrage under the natural constraints (Example B). In particular, it cannot happen then as in continuous-time models that the weak numeraire portfolio exists but no equivalent martingale measure does. As mentioned above, a numeraire portfolio can be used for the purpose of pricing derivative securities. Platen 2002 argues that this can be done even in models where an equivalent martingale measure is absent and has developed a benchmark framework to do so (see Platen 2006).

6 Theorem 9.6 is generalised by Korn, Oertel and Schäl 2003 to a market modeled by a jump-diffusion process where the state space of the jumps is finite.

7 How to apply the results for the one-period model to the multi-period model is explained in Schäl 2000.

8 The concept of a numeraire portfolio (in the strict sense) is extended to financial markets with proportional transaction cost by Sass and Schäl 2009.

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