On Discrete–Time Dynamic Programming in Insurance: Exponential Utility and Minimizing the Ruin Probability.

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This paper studies an insurance model where the risk process can be controlled by reinsurance and by investment in a financial market. The performance criterion is either the expected exponential utility of the terminal surplus or the ruin probability. It is shown that the problems can be imbedded in the framework of discrete–time stochastic dynamic programming but with some special features. A short introduction to control theory with infinite state space is provided which avoids the measure–theoretic apparatus by use of the so–called structure assumption. Moreover, in order to treat models without discount factor, a weak contraction property is derived. Explicit conditions are obtained for the optimality of employing no reinsurance.

Key words: reinsurance, investment, Markov decision processes, Howard improvement, verification theorem

1. INTRODUCTION

An introductory survey of the use of optimal control theory for treating Markovian control problems in non–life insurance was given by Martin–Löf (1994). There, models with finite state space and a discount factor are considered. In the present paper we concentrate on models with infinite state space without discounting in order to treat the control of the ruin probability in a variant of the Cramér–Lundberg model. The usual problems of measurability are here avoided by the use of so–called structure assumptions which were already explained by Porteus (1975). In sections 2 and 4 the theory of dynamic programming is explained for a finite and an infinite horizon, respectively. Applications to insurance, however, will lead to some new situations not yet considered in the literature.

In these applications, given in sections 3, 5, 6, an insurance model is studied which can be controlled by reinsurance and by investment in a financial market. The period lengths may be deterministic or random; e.g., a period may be the time between two successive claims. In section 3, we want to maximize the expected exponential utility of the terminal surplus. Since the general theory of dynamic programming is here explained for minimizing costs rather than for maximizing rewards, we will actually minimize an exponential disutility. This is also convenient from a mathematical point of view, since we can then restrict attention to nonnegative functions. By assuming that the decision maker (insurance company) enjoys an unbounded credit, the optimality of a very simple control is established which is given by a universal retention level for reinsurance and a universal portfolio vector for investment. Similar results are obtained by Browne (1995) for a diffusion model.

At first view, the ruin probability is not a classical performance criterion for control problems. In section 5, it will be shown that one can write the ruin probability as some total cost without discounting where one has to pay one unit of cost when entering a ruin state. After this simple observation, the results from discrete–time dynamic programming apply. However, the usual continuity conditions do not hold since the system function is discontinuous. In spite of the lack of discounting, the model enjoys a contraction property which is weaker than the usual ones considered in dynamic programming. This property was established by Schmidli (2001b) for a continuous–time insurance model. Here it is shown that the property is strong enough for the validity of the Howard improvement and a verification theorem. By use of the Howard improvement, one can look for a plan which is at least better than employing no reinsurance. As an application of the verification theorem, it can be shown for a model with exponentially distributed claims that it is optimal to have no reinsurance if the safety loading of the reinsurer is too high. Similar results are obtained by Schmidli (1999) for maximizing the adjustment coefficient and by Taksar & Markussen (2002) for a diffusion approximation.

The effect of reinsurance on the probability of ultimate ruin is also studied by Dickson & Waters (1996). Minimizing the ruin probability in continuous–time models is considered by Browne (1995), Hipp & Plum (2000), Hipp & Vogt (2001), Hϕjgaard & Taksar (1998a,b) Schmidli (2001a,b), Schäl (2002). In these papers, diffusion models, piecewise–deterministic models, and mixtures of these models are studied. There the decision maker can adjust the retention level and the portfolio at every time $t \in [0,\infty)$ whereas the control action is constant throughout one period in the present paper. We remark that also a continuously controlled piecewise–deterministic model as the Cramér–Lundberg model can be reduced to a discrete–time model. But then the control space has to be chosen as function space (see Schäl 1998, 2002) whereas in the present paper the control space is a subset of some Euclidean space.

We will consider an insurance model which can be controlled by reinsurance and investment in a financial market. The process $\{X_n, n\geq 0\}$ is the risk process where $X_n \in \mathbb{R}$ describes the *surplus* (size of the fund of reserves) of an insurance company after n periods. The *claim* (payment) in period n will be described by the random variable Y_n with values in $[0,\infty)$. The process can be controlled by reinsurance, i.e. by choosing the *retention level* (or risk exposure) $b \in [\underline{b},\bar{b}]$ of a reinsurance for one period. The (measurable) function h(b,y) specifies the part of the claim y paid by the insurer. Then h(b,y) depends on the retention level b (fixed in the risk sharing contract) at the beginning of the respective period where $0 \leq h(b,y) \leq y$. Hence $y - h(b,y)$ is the part paid by the reinsurer. It is natural to assume that h(b,y) is increasing in b. In the case of an *excess of loss reinsurance* we have:

$$h(b,y) = \min(b,y) \quad \text{with retention level} \quad 0 \leq \underline{b} \leq b \leq \bar{b} = \infty. \tag{1}$$

In case of a *proportional reinsurance* we have:

$$h(b,y) = b \cdot y \quad \text{with retention level} \quad 0 \leq \underline{b} \leq b \leq \bar{b} = 1. \tag{2}$$

We allow for the case that the length Z_n of period n is random. Thus we can cover a controlled

version of the Cramér–Lundberg model if we assume that periods are given by the intervals between the jump times of a Poisson process. Of course, we also can think of the case where $Z_n = 1$ is deterministic. We set $Y = Y_1$ for a typical claim and $Z := Z_1$ for a typical period length.
There is a *premium (income) rate* c which is fixed. For each retention level b, the insurer pays a premium rate to the reinsurer which has to be deducted from c. This leads to a *net income rate* c(b) where

$$0 \le c(b) \le c = c(\bar{b}) \quad \text{for } \underline{b} \le b \le \bar{b} \quad \text{and } c(b) \text{ is increasing.} \tag{3}$$

There, the retention level $\bar{b}$ stands for the control action "no reinsurance" which explains the property "$c = c(\bar{b})$". The smallest retention level $\underline{b}$ may be chosen in such a way that the condition (3) is satisfied. Then c(b) may be calculated according to the *expected value principle* with *safety loading* θ of the reinsurer:

$$c(b) = c - (1+\theta) \cdot E[Y - h(b,Y)]/E[Z]. \tag{4}$$

In addition, the insurance company can invest the capital (surplus) in a financial market where d assets can be traded which are called stocks and are described by the price process $\{S_n = (S_n^1,\dots,S_n^d),\ n \ge 0\}$ where S_n^k is the price of one share of stock k at the beginning of period n+1. We define the return process $\{R_n = (R_n^1,\dots,R_n^d),\ n \ge 1\}$ by $S_n^k =: S_{n-1}^k \cdot (1 + R_n^k)$, $1 \le k \le d$, where of course $1 + R_n^k > 0$ a.s. for all k. We assume that the $\{R_n\}$ are independent and identically distributed and set $R := R_1$ for a typical return.
A dynamic portfolio specifies at the beginning, i.e. at the beginning of period 1, a portfolio vector $\delta_0 \in \mathbb{R}^d$ and subsequently at the beginning of any period n+1 a portfolio vector $\delta_n \in \mathbb{R}^d$. There, the component δ_n^k of δ_n represents the amount invested in stock k during period n+1. We will allow for negative amounts δ_n^k, thus admitting short selling of stocks in section 3, but not in section 5.
Thus a control action $u = (b,\delta)$ will consist of two components where b specifies the retention level and $\delta = (\delta^1,\dots,\delta^d)$ specifies the portfolio vector. At the beginning of period n+1, the decision about the control action will depend on the present size X_n of the capital (surplus). Given the surplus X_n and the control action $u_n = (b_n,\delta_n)$, we now want to compute the surplus X_{n+1}. Then δ_n^k/S_n^k denotes the number of shares the investor holds during period n+1. Thus the value of these shares at the beginning of the next period is $\delta_n^k \cdot S_{n+1}^k/S_n^k$ and we have

$$\begin{aligned} X_{n+1} &= X_n + c(b_n)\cdot Z_{n+1} - h(b_n,Y_{n+1}) - \textstyle\sum_{k=1}^d \delta_n^k + \sum_{k=1}^d \delta_n^k \cdot S_{n+1}^k/S_n^k \\ &= X_n + c(b_n)\cdot Z_{n+1} - h(b_n,Y_{n+1}) - \textstyle\sum_{k=1}^d \delta_n^k + \sum_{k=1}^d \delta_n^k \cdot (1 + R_{n+1}^k) \\ &= X_n + c(b_n)\cdot Z_{n+1} - h(b_n,Y_{n+1}) + \textstyle\sum_{k=1}^d \delta_n^k \cdot R_{n+1}^k, \quad \text{and thus} \end{aligned}$$

$$X_{n+1} = X_n + c(b_n)\cdot Z_{n+1} - h(b_n,Y_{n+1}) + \langle \delta_n, R_{n+1} \rangle \tag{5}$$

where $<z,y>$ denotes the inner product in $\mathbb{R}^d$. It is convenient to set

$$X_{n+1} = f(X_n,b_n,\delta_n,R_n,Y_n,Z_n) \quad \text{where}$$

$$f(x,b,\delta,\rho,y,z) = x + c(b)\cdot z - h(b,y) + <\delta,\rho> . \tag{6}$$

We will call f the *system function* as in Bertsekas & Shreve (1978). Moreover, we will look on $W_n = (R_n,Y_n,Z_n)$ as the disturbance for period n. The sequence $\{W_n, n\geq 1\}$ forms the source of randomness of the model. We make the following assumption:

INDEPENDENCE ASSUMPTION. The $W_n = (R_n,Y_n,Z_n)$, $n \geq 1$, are iid (independent, identically distributed) random variables. In addition, it is assumed that (R_n,Z_n) and Y_n are independent.

As a consequence, $\{R_n\}$ as well as $\{Y_n\}$ and $\{Z_n\}$ are also iid random variables. However, it is reasonable to allow for a dependence of Z_n and R_n. In a combination of a Cramér–Lundberg model and a Black–Scholes model (with d=1), Z is exponentially distributed and $1 + R = \exp\{\sigma B_Z + (a - \frac{1}{2}\sigma^2) Z\}$ with a standard Brownian motion $\{B_t\}$, the volatility σ and the appreciation rate a.
In section 2, we will consider a more general control model which contains the present insurance model and also the model of section 5, 6 as special cases. The section 2 contains the notation and formal results of dynamic programming. But we will use the same notation as in this introduction.

2. DYNAMIC PROGRAMMING WITH FINITE HORIZON

A general discrete–time stochastic process $\{X_n, n \geq 0\}$ is considered which can be observed and controlled at the beginning of periods with numbers n = 1,2,... The stochastic development is determined by a sequence of iid random variables $\{W_n, n \geq 1\}$ on some probability space $(\Omega,\mathfrak{F},P)$ with values in some measurable space $(E,\mathfrak{E})$. There, $(E,\mathfrak{E})$ is the *disturbance space*. We write $W := W_1$ for a typical disturbance. The model is further specified by the following quantities:
$(S,\mathfrak{S})$ is the *state space* which is a measurable space;
$(U,\mathfrak{U})$ is the *control (action) space* which is a measurable space;
$f : S\times U\times E \mapsto S$ is the (measurable) *system function*;
$\alpha \in [0,1]$ is the *discount factor*;
$g : S\times U \mapsto (-\infty,\infty]$ is the *one-period cost function*, which is measurable and bounded from below;
$V_0 : S \mapsto (-\infty,\infty]$ is the *terminal cost function*, which is measurable and bounded from below;
$N \in \mathbb{N}$ is a *time horizon* (number of periods).

Of course, a negative cost can be interpreted as a reward. However for the present applications, it is sufficient to consider nonnegative cost functions. For simplicity, we will not restrict the set of control actions available in some state x. If the action u is not admissible at x one can set $g(x,u) = \infty$ or one can identify u with some other admissible action. In applications to insurance, we also consider the case where the period lengths are random. Therefore, it may be reasonable to assume that the discount factor is random. It may be a function of the length of the time period. Such models can be treated in the same way (see Schäl 1975); but for the present applications it would be even sufficient to consider the case $\alpha = 1$.

DEFINITION. A *decision function* is a measurable function $\varphi : S \mapsto U$.
A *plan* (policy, strategy) is a sequence $\pi = (\varphi_n)_{n\geq 0}$ of decision functions φ_n.

Then $\varphi_n(X_n)$ will represent the action chosen at the beginning of period n+1. Even in a model with finite horizon N, it is convenient to describe a plan by an infinite sequence (φ_n) where φ_n can be defined in an arbitrary way for $n \geq N$. Actually we only consider nonrandomized Markov plans. However, this restriction is justified (see Bertsekas & Shreve 1978 Proposition 8.4).

For an initial state $x \in S$ and a plan π, the state $X_n = X_n^{x,\pi}$ of the system at the beginning of period n+1 is defined as a random variable on $(\Omega,\mathfrak{F},P)$ according to:

$$X_0 = x,\ X_{n+1} := f(X_n,\varphi_n(X_n),W_{n+1}),\ n\geq 0. \tag{7}$$

In state $x_n \in S$ at the beginning of period n+1, a control action $u_n = \varphi_n(x_n) \in U$ is chosen which will result in a cost $g(x_n,u_n)$. Then the system is influenced by a disturbance $W_{n+1} = w_{n+1}$ in such a way that the state at the beginning of the next period is given by $x_{n+1} = f(x_n,u_n,w_{n+1})$. An example for a system function f is given in (5). The costs will be discounted by α.

REMARK. The processes studied in this paper are also called *Markov decision processes*. The underlying *Markov property* will now be formulated. If one defines the σ–algebra $\mathfrak{F}_n$ of the past at time n by $\mathfrak{F}_n := \sigma(W_1,\dots,W_n)$, $n\geq 1$, where $\mathfrak{F}_0 = \{\emptyset,\Omega\}$ is the trivial σ–algebra, then $\{X_n,\ n \geq 0\}$ is a discrete–time Markov process (w.r.t. the filtration $\{\mathfrak{F}_n\}$) and one has the following relation for measurable functions $v : S \mapsto (-\infty,\infty]$ bounded from below:

$$E\left[v(X_{n+1})\,|\,\mathfrak{F}_n\right] = \bar{v}_n(X_n) \quad \text{where} \quad \bar{v}_n(x) := E\left[\, v(f(x,\varphi_n(x),W))\, \right] . \tag{8}$$

DEFINITION. Given the initial state $X_0 = x \in S$, the *total discounted cost* $J_N^\pi(x)$ and the *value function* $V_N(x)$ in N periods are

$$J_N^\pi(x) = E\left[\sum_{0\leq n<N} \alpha^n \cdot g(X_n,\varphi_n(X_n)) + \alpha^N \cdot V_0(X_N)\right] \quad \text{with} \quad X_n = X_n^{x,\pi}. \tag{9}$$

$$V_N(x) := \inf_\pi J_N^\pi(x) . \tag{10}$$

The following lemma is easy to prove.

LEMMA 1. $J_N^\pi(x) = g(x,\varphi_0(x)) + \alpha \cdot E\left[J_{N-1}^{\overleftarrow{\pi}}(f(x,\varphi_0(x),W))\right]$ where

$$\overleftarrow{\pi} = (\varphi_1,\varphi_2,\dots) \quad \text{for} \quad \pi = (\varphi_0,\varphi_1,\dots). \tag{11}$$

It is convenient to describe the relation in Lemma 1 by the *one-step cost operator* T.

DEFINITION. For any $v : S \mapsto (-\infty,\infty]$, which is measurable and bounded from below, we set:

$Tv(x,u) := g(x,u) + \alpha \cdot E[v(f(x,u,W))]$, $x \in S$, $u \in U$;

$T_\varphi v(x) := Tv(x,\varphi(x))$ for any decision function φ;

$T^*v(x) := \inf_{u \in U} Tv(x,u)$.

If $v(z)$ represents the cost incurred at the beginning of the next period [or the expected cost from the next period onwards] given the state z at that time, then $Tv(x,u)$ tells us the expected discounted cost from the present period onwards given the present state x and the chosen action u.

LEMMA 2. For any functions v, v' as in the definition above we have:

(a) $T_\varphi v \geq T^*v$;

(b) T_φ and T^* are order preserving, i.e., $v \leq v' \Rightarrow T_\varphi v \leq T_\varphi v'$, $T^*v \leq T^*v'$;

(c) $J_N^\pi = T_{\varphi_0} J_{N-1}^{\overleftarrow{\pi}} = T_{\varphi_0} \ldots T_{\varphi_{N-1}} V_0$ with $\overleftarrow{\pi}$ as in (5).

(d) $T_{\varphi_0} \ldots T_{\varphi_{N-1}} v = J_N^\pi + \alpha^N \cdot E[v(X_N) - V_0(X_N)]$ if $v - V_0$ is bounded from below.

The proof of Lemma 2 is easy (see Bertsekas & Shreve 1978 Lemma 8.1). The measurability of T^*v is a problem, which we will avoid by the following assumption (see Porteus 1975).

STRUCTURE ASSUMPTION. There exist a set $\mathcal{V}$ of measurable functions $v : S \mapsto (-\infty,\infty]$ bounded from below and a set Φ of decision functions such that:

(i) $V_0 \in \mathcal{V}$;

(ii) $T^*v \in \mathcal{V}$ for all $v \in \mathcal{V}$;

(iii) for all $v \in \mathcal{V}$ there exists a decision function $\varphi \in \Phi$ with:

$T_\varphi v = T^*v$, i.e. $\varphi(x)$ is a measurable minimizer of the function $U \ni u \mapsto Tv(x,u)$.

One main problem in dynamic programming is to find such a class $\mathcal{V}$. In general semicontinuous models, candidates for $\mathcal{V}$ and Φ are the set of l.s.c. functions $v : S \mapsto (-\infty,\infty]$ bounded from below and the set of all decision functions, respectively, where we write l.s.c for lower semicontinuous (as usual). In a heavy measure–theoretic apparatus, one can choose $\mathcal{V}$ as the set of lower semianalytic function (see Bertsekas & Shreve 1978 § 7.7, § 8.3). But of course, one wants to choose $\mathcal{V}$ and Φ as small as possible. In inventory theory e.g., a famous result of Scarf (1960) says that one can choose $\mathcal{V}$ as the set of K–convex functions and Φ as the set of (s.S) decision functions.

LEMMA 3. Under the structure assumption one has:

(a) $T_{\varphi_0} \ldots T_{\varphi_{n-1}} V_0 \geq T^{*n} V_0 \in \mathcal{V}$ for all $\pi = (\varphi_0,\varphi_1,\ldots)$, $n > 0$;

(b) there exists a sequence of decision function $(\varphi_n^*, n\geq1)$ with $\varphi_n^* \in \Phi$ and

$T_{\varphi_n^*} \ldots T_{\varphi_1^*} V_0 = T^{*n} V_0$, $n\geq1$.

The proof immediately follows from the structure assumption.

DEFINITION. A plan is *optimal* (for horizon N), if $J_N^\pi(x) = V_N(x)$ for all $x \in S$.
The sequence of decision functions $(\varphi_n^*, n>0)$ is *value conserving* if $T_{\varphi_n^*} V_{n-1} = T^* V_{n-1}$, i.e., $\varphi_n^*(x)$ is a minimizer of the function $u \mapsto TV_{n-1}(x,u)$ for all $x \in S$, $n > 0$.

The notion of value conserving was introduced by Dubins & Savage (1965). From Lemmata 2 and 3, one obtains the following theorem in a straightforward manner.

THEOREM 1. Under the structure assumption one has:

(a) $V_n = T^{*n} V_0 \in \mathscr{V}$, $n > 0$, (*value iteration*);
(b) $V_n = T^* V_{n-1}$, $n > 0$, (*optimality equation*);
(c) there exists a value conserving sequence $(\varphi_n^*, n>0)$ such that $\varphi_n^* \in \Phi$ for $n > 0$;
(d) there exists an optimal plan for each finite horizon. More exactly, if $(\varphi_n^*, n>0)$ is value conserving, then $\pi^{(N)} := (\varphi_N^*, \dots, \varphi_1^*, \varphi_0, \varphi_0, \dots)$ is optimal for the horizon $N < \infty$ (where φ_0 can be chosen arbitrarily).

INTERPRETATION. *Principle of dynamic programming,* Bellman 1957.
Assume that you know the minimal total costs V_n if there are n periods ahead. [For $n = 0$, V_0 is indeed known and given.] Now you want to compute V_{n+1}. The way of solution is the following: Compute the quantity $TV_n(x,u)$ which can be interpreted as the (discounted) cost at time 0 (i) if you start in state $X_0 = x$, (ii) if you choose an arbitrary control action u, and (iii) if you choose an optimal control for the n periods lying ahead at the end of the first period, i.e., if you have to pay $V_n(X_1)$ then. Now minimize over $u \in U$; then you obtain $T^*V_n(x) = V_{n+1}(x)$. The minimizer $u^* = \varphi_{n+1}^*(x)$ is the optimal control action. Hence, the information which is important for the choice of the control action consists in the present state x and the number n+1 of periods ahead.

EXAMPLE. Finite control spaces (see e.g. Martin–Löf 1994).
Assume that U is finite and w.l.o.g. $U \subset \mathbb{R}$. We can choose Φ as the set of all decision functions and $\mathscr{V}$ as the set of all functions $v : S \mapsto (-\infty.\infty]$ which are measurable and bounded from below . Then condition (i) of the structure assumption is obviously fulfilled. For (ii) and (iii), we have $T^*v(x) = \min_{u \in U} Tv(x,u)$ where $Tv(x,u)$ and thus $T^*v(x)$ is measurable in x. Because of the finiteness of U we can write 'min' in place of 'inf'. We obtain a measurable selection of minimizers by $\varphi(x) := \min \{u \in U; Tv(x,u) = T^*v(x)$.
The measurability of φ follows from $\{\varphi(x) \le a\} = \cup_{u \le a} \{x; Tv(x,u) = T^*v(x)\}$. $\square$

3. THE INSURANCE MODEL WITH MAXIMIZING AN EXPONENTIAL UTILITY

In this section we study the insurance model introduced in §1 where the investor (insurance company) is allowed to borrow an unlimited amount of money. We choose the state space and the control space as

$$S = \mathbb{R}, \quad U = [\underline{b},\bar{b}] \times \mathbb{R}^d . \tag{11}$$

Then a decision function φ consists of two components $\varphi = (\varphi',\varphi'')$ where φ' specifies the retention level b of reinsurance and φ'' specifies the portfolio vector $\delta = (\delta^1,...,\delta^d)$.

The cost structure is given by the idea that the insurance company is not ruined but only penalized if the size of the surplus is negative or small. The penalty cost for being in state x is of the form $\text{const} \cdot e^{-\beta \cdot x}$ for some $\beta > 0$. Therefore we define the cost functions as

$$g(x,u) := \gamma \cdot e^{-\beta x}, \; V_0(x) := \nu_0 \cdot e^{-\beta x} \quad \text{for some } \gamma, \nu_0 \geq 0. \tag{12}$$

Then the performance criterion is the expected total penalty paid. An important special case is defined by $\gamma = 0$ and (w.l.o.g.) $\nu_0 = 1$. Then the insurer has only to pay a penalty at the end and wants to minimize $E[\exp\{-\beta \cdot X_N^{x,\pi}\}]$. This is the same problem as maximizing the expected utility of terminal wealth if one chooses the exponential utility function $-e^{-\beta x}$. Thus, one can also speak of minimizing the expected exponential disutility of terminal surplus. This is an interesting problem, since exponential utility is also used in determining *fair premiums* by many property–liability insurance companies (see Goovaerts et al. 1990 III.6).

In the present situation, we claim that the structure assumption is satisfied if we choose $\mathcal{V}$ as the set of all functions $v : \mathbb{R} \mapsto [0,\infty)$ such that $v(x) = \nu \cdot e^{-\beta x}$ for some $\nu \geq 0$ and Φ as the set of all constant decision function $\varphi : \mathbb{R} \mapsto U$. Now we want to show the properties (i) – (iii) of the structure assumption. Obviously (i) holds by definition of V_0.

Moreover, we have for $v(x) = \nu \cdot e^{-\beta x}$, $u = (b,\delta)$, $W = (R,Y,Z)$:

$Tv(x,u) = g(x,u) + \alpha \cdot E[v(f(x,u,W))] = \gamma \cdot e^{-\beta x} + \alpha \cdot \nu \cdot E[\exp\{-\beta \cdot f(x,u,W)\}]$

$= \gamma \cdot e^{-\beta x} + \alpha \cdot \nu \cdot E[\exp\{-\beta \cdot [x + \langle\delta,R\rangle + c(b) \cdot Z - h(b,Y)]\}]$

$= e^{-\beta x} \cdot \left[\gamma + \alpha \cdot \nu \cdot E[\exp\{-\beta \cdot [\langle\delta,R\rangle + c(b) \cdot Z - h(b,Y)]\}]\right]$. Hence we have

$$T^*v(x) = \nu^* \cdot e^{-\beta x} \tag{13}$$
$$\text{with} \quad \nu^* := \gamma + \alpha \cdot \nu \cdot \inf_{(b,\delta)\in U} E[\exp\{-\beta \cdot [\langle\delta,R\rangle + c(b) \cdot Z - h(b,Y)]\}].$$

Thus, our model also enjoys property (ii) and we now have to concentrate on property (iii). It is sufficient to show that the infimum in (13) is attained by some (b^*,δ^*), say. Then we may define the decision function $\varphi \in \Phi$ as the constant function $\varphi(x) = (b^*,\delta^*)$. In fact, φ does not even depend on v. As a consequence, we will obtain a value conserving sequence (φ_n^*) by setting

$\varphi^*_n := \varphi$. Thus the optimal action will then be universal in the sense that it neither depends on the present state nor on the number of periods lying ahead.

We know that $U = [\underline{b},\bar{b}]\times\mathbb{R}^d$ where $[\underline{b},\bar{b}]$ is compact. In the examples, we have $U = [\underline{b},1]$ in the case of proportional reinsurance and $U = [\underline{b},\infty]$ in the case of an excess of loss reinsurance. Then we will need the property that

$$(b,\delta) \mapsto E[\exp\{-\beta\cdot[<\delta,R> + c(b)\cdot Z - h(b,Y)]\}] \text{ is continuous.} \tag{14}$$

For that purpose we make the following assumption

ASSUMPTION. The functions $c(b)$ and $h(b,y)$ are continuous in b (for each y) and

$$E[\exp\{\beta\cdot Y\}] < \infty, \quad E[\exp\{\varepsilon\cdot\|R\|\}] < \infty \text{ for all } \varepsilon > 0.$$

The latter assumption on R is satisfied if R is bounded or if R has a normal distribution. Since Y and R are assumed to be independent and since $0 \le h(b,y) \le y$, we know that
$E[\exp\{-\beta\cdot[<\delta,R> + c(b)\cdot Z - h(b,Y)]\}] \le E[\exp\{-\beta\cdot<\delta,R>\}]\cdot E[\exp\{\beta\cdot Y\}] < \infty$ for all b, δ. Now we may conclude from the dominated convergence theorem that property (14) holds. The set $\mathbb{R}^d$ is not compact; but we will show that the infimum is attained under the following well–known condition (NA) (see e.g. Jacod & Shiryaev 1998, Pliska 1997).

NO ARBITRAGE ASSUMPTION. For any portfolio vector $\delta \in \mathbb{R}^d$:

$$P[\,<\delta,R> \ge 0\,] = 1 \text{ implies } P[\,<\delta,R> = 0\,] = 1\,. \tag{NA}$$

In fact, the property $<\delta,R_1> \ge 0$ implies for $X_0 = x > 0$ that $X_1 = x + <\delta,R_1> + c(b)\cdot Z_1 - h(b,Y_1) \ge x\cdot + c(b)\cdot Z_1 - h(b,Y_1)$. Thus, using the portfolio $\delta \in \mathbb{R}^d$ is not worse than not investing in the stocks, i.e., there is no risk in using this portfolio. Then everybody would indeed like to use such an opportunity if there is a chance that $<\delta,R_1>$ is positive. Such a portfolio δ is called an *arbitrage opportunity* which is excluded by our assumption.

We write $\mathscr{L}$ for the smallest linear space L in $\mathbb{R}^d$ such that $P[R \in L] = 1$. Then it is easy to show that (NA) is equivalent to:

$$\text{for all } \delta \in \mathscr{L}, \delta \ne 0 : P[<\delta,R> < 0] > 0\,. \tag{NA)*}$$

LEMMA 4. The function $(b,\delta) \mapsto v(b,\delta) := E[\exp\{-\beta\cdot[<\delta,R> + c(b)\cdot Z - h(b,Y)]\}]$ attains the infimum over U at some (b^*,δ^*) where δ^* can be chosen in $\mathscr{L}$.

Proof. If χ denotes the orthogonal projection on $\mathscr{L}$, then $<\delta,R> = <\chi\delta,R>$ a.s. $\forall\ \delta \in \mathbb{R}^d$. Now we can restrict attention to $\delta \in \mathscr{L}$. In view of (3), we have

$$\lim_{\lambda\uparrow\infty} v(b,\lambda\cdot\delta) \ge \lim_{\lambda\uparrow\infty} E[\exp\{-\beta\cdot[\lambda\cdot<\delta,R> + c\cdot Z]\}\cdot\mathbf{1}_{\{<\delta,R> < 0\}}] = \infty \tag{15}$$

for $\delta \in \mathscr{L}\setminus\{0\}$ by (NA)*. Now define $F_\lambda := \{(b,\delta) \in [\underline{b},\bar{b}]\times\mathscr{L};\ \|\delta\| = 1,\ v(b,\lambda\cdot\delta) \le v(b,0) + 1\}$. Then F_λ is compact and it is easy to show that the convexity of $v(b,\delta)$ in δ implies that $F_\lambda \subset F_{\lambda'}$

for $0 < \lambda' < \lambda$. Moreover, we conclude from (15) that $\cap_{n\in\mathbb{N}} F_n = \emptyset$. Hence we know that there exists some $n_o \in \mathbb{N}$ such that $F_\lambda = \emptyset$ for all $\lambda \geq n_o$, i.e. $v(b,\delta) \geq v(b,0) + 1$ for all δ with $\|\delta\| \geq n_o$. Thus $\inf_{(b,\delta)\in U} v(b,\delta) = \min_{(b,\delta)\in[\underline{b},\bar{b}],\ \delta\in\mathscr{L},\ \|\delta\|\leq n_o} v(b,\delta)$, i.e., the infimum over all $(b,\delta) \in U$ is attained on the compact set $[\underline{b},\bar{b}]\times\{\delta \in \mathscr{L};\ \|\delta\| \leq n_o\}$. $\square$

Therefore the model indeed enjoys property (iii) and the structure assumption is satisfied. The use of convexity in the present minimization problem is well–known (see Rockafellar 1970 Theorems 27.1, 27.3, Bertsekas 1974 Proposition 1, Rogers 1994 Proposition 2.2). Also the use of the no–arbitrage condition for such problems is known (see Rogers 1994, Schäl 2000a,b, 20001).

The situation is further simplified under the assumption that the random variables R, Z, Y are independent which is the case if the period length Z is deterministic. Then

$$E\left[\exp\{-\beta\cdot[<\delta,R>+c(b)\cdot Z-h(b,Y)]\}\right] = E\left[\exp\{-\beta\cdot<\delta,R>\}\right]\cdot E\left[\exp\{-\beta\cdot[<c(b)\cdot Z-h(b,Y)]\}\right]$$

and we can get the minimizer (b^*,δ^*) in such a way that

$$b^* \text{ is a minimizer of the function } b \mapsto E\left[\exp\{-\beta\cdot[c(b)\cdot Z - h(b,Y)]\}\right], \tag{16}$$

$$\delta^* \text{ is a minimizer of } \delta \mapsto E\left[\exp\{-\beta\cdot<\delta,R>\}\right]. \tag{17}$$

It is remarkable that under this additional independence assumption the control of the reinsurance and the control of the investments can be chosen independently of each other. The investments are controlled in such a way that the expected utility is maximized for an exponential utility function $-e^{-\beta x}$. In fact, a pure investment problem with an exponential utility is a special case of the present model if one chooses $c(b) = 0$, $h = 0$. Moreover, the optimal plan invests a fixed amount, regardless of the surplus of the company, in accordance with Ferguson (1965) and Merton (1990). Similar results for a diffusion model are obtained by Browne (1995) who also explains that the well–known constant proportional schemes may be inappropriate in the present scenario.

THEOREM 2. Under the assumption of this section, the structure assumption is satisfied with $\mathscr{V}$ and Φ defined as above. Moreover, there exists a value conserving sequence (φ_n^*) such that $\varphi_n^*(x) = (b^*,\delta^*)$ is independent of x and n where (b^*,δ^*) is the minimizer of Lemma 4. If R and (Z,Y) are independent, then b^* and δ^* can be obtained independently of each other by (16), (17).

The present results only hold if the interest rate r for the capital of the company not invested in stocks is zero. However, the case $r \geq 0$ can be treated in the same way if one considers the problem of minimizing the expected disutility of the *discounted* terminal capital. From the theory of finance it is well known that considering the discounted wealth leads to a scenario which can be looked upon as market with interest rate zero. Then one has to replace the price process by the discounted price process and a portfolio vector δ_n then describes the discounted invested amounts (see Harrison & Kreps 1979, Schäl 2000a,b). Moreover, the premium as well the claim sizes are interpreted as discounted quantities.

EXAMPLE. We want of solve (16) in the situation of a Cramér–Lundberg model where we consider constant period lengths, Z = 1, say, and a proportional reinsurance (2). If the single claims have expectation μ and moment generating function m(s), then Y has a compound distribution with Expectation $\lambda \cdot \mu$ and moment generation function $\exp\{\lambda \cdot (m(s) - 1)\}$ (see Grandell 1991 Theorem 14). We obtain from (4): $c(b) = c - (1+\theta) \cdot (1-b) \cdot \lambda\mu$. Then $E\left[\exp\{-\beta \cdot [c(b) \cdot Z - h(b,Y)]\}\right] = \exp\{-\beta \cdot [c - (1+\theta) \cdot (1-b) \cdot \lambda\mu]\} \cdot \exp\{\lambda \cdot (m(\beta b) - 1)\}$.
Then it is easy to see that b* = 1, (i.e., it is optimal to have no reinsurance) if and only if θ is too high in the sense that one has (with m' denoting the derivative of m):

$$1 + \theta \geq \frac{1}{\mu} \cdot m'(\beta). \tag{18}$$

4. DYNAMIC PROGRAMMING WITH INFINITE HORIZON

We will look on a model with infinite horizon as an approximation of a model with a finite but large horizon N. Therefore we will define the performance criterion J^π as the limit $\lim_N J_N^\pi$ of those with finite horizon as $N \to \infty$. Then J^π is also affected by the terminal cost function V_0. In most applications, one sets $V_0 = 0$ in an infinite horizon model. But in some cases it is convenient to allow for more general terminal cost functions. Examples are models with optimal stopping. In the next section, we do not want to have the function 0 in the set $\mathcal{V}$. Therefore, we will assume that $V_0 = g$ and we will then look on the one–period cost incurred in the last period as terminal cost. We make the following assumption which is called *Uniform Increase Assumption* (see Bertsekas & Shreve 1978 p. 70).

ASSUMPTION. $TV_0(x,u) \geq V_0(x)$ for all $u \in U$, i.e. $T^*V_0(x) \geq V_0(x)$ for all $x \in S$.

We will discuss the assumption below.

Lemma 4. (a) $V_0 \leq J_n^\pi \leq J_{n+1}^\pi$ for $n \geq 0$;
(b) $V_0 \leq V_n \leq V_{n+1}$ for $n \geq 0$.

Proof. We have $T_\varphi V_0 \geq V_0$ for all φ and thus
$J_{n+1}^\pi = T_{\varphi_0} \ldots T_{\varphi_{n-1}} (T_{\varphi_n} V_0) \geq T_{\varphi_0} \ldots T_{\varphi_{n-1}} V_0 = J_n^\pi$. Now part (a) and (b) are obvious. $\square$

Now we can define the performance criterion which is justified by the preceding lemma.

DEFINITION. $J^\pi(x) := \lim_{n\to\infty} J_n^\pi(x)$ is the *total discounted cost* for any plan π,
$V(x) := \inf_\pi J^\pi(x)$ is the *value function*.

In the *positive case* (in the sense of Bertsekas & Shreve 1978, which is the negative case in the sense of Strauch 1966 where rewards are maximized) we have

$$g(x,u) \geq 0,\ V_0 = 0. \tag{19a}$$

Then the preceding assumption is satisfied. This is also true in the *discounted case* where:

$$0 \leq \alpha < 1,\ \text{ g is bounded and } V_0 = g_o/(1-\alpha) \ \text{ where } g_o \text{ is a lower bound of g.} \tag{19b}$$

There $J^{\pi}(x)$ does not depend on the special form of V_0 in view of Lemma 2d provided only that V_0 is bounded. Thus there is no loss of generality in choosing V_0 as in (19b).
From the monotone convergence theorem we obtain the following lemma.

LEMMA 5. (a) $J_n^{\pi} = T_{\varphi_0} ... T_{\varphi_{n-1}} V_0 \uparrow J^{\pi} = T_{\varphi_0} J^{\overleftarrow{\pi}}$ with $\overleftarrow{\pi}$ as in (11);

(b) $V_0 \leq V^{\infty}(x) := \lim_{n\to\infty} V_n(x) \leq V(x)$.

(c) $\lim_{n\to\infty} TV_n(x,u) = TV^{\infty}(x,u)$.

We saw in §2 that the information which is important for the choice of the control action consists in the present state x and the number of periods ahead. But in an infinite horizon model the number of periods ahead is always the same and that is ∞. This motivates the following:

DEFINITION. A plan $\pi = (\varphi_n)$ is *stationary* if $\varphi_n = \varphi$ for a decision function φ and we will write

$$\pi = \varphi^{\infty}.$$

From Lemma 5 we obtain for every stationary plan $\pi = \varphi^{\infty}$:

$$T_{\varphi} J^{\varphi^{\infty}} = J^{\varphi^{\infty}} = \lim_{n\to\infty} (T_{\varphi})^n V_0. \tag{20}$$

For this section we will use the following general assumption.

STRUCTURE ASSUMPTION. S is a Borel subset of some Euclidian space (or more generally of some Polish space). There exists a set $\mathscr{V}$ of measurable functions $v : S \mapsto (-\infty,\infty]$ bounded from below such that:

(i) $V_0 \in \mathscr{V}$;

(ii) $T^*v \in \mathscr{V}$ for all $v \in \mathscr{V}$;

(iii) U is a compact metric space;

(iv) $Tv(x,u)$ is l.s.c. in u for all $x \in S$, $v \in \mathscr{V}$.

By a selection theorem (Brown & Purves 1973) this assumption implies the following property:

$$\text{for any } v \in \mathscr{V}, \text{ there exists a decision function } \varphi \text{ with } T_{\varphi} v = T^*v . \tag{21}$$

Thus the structure assumption for the infinite horizon implies that for the finite horizon if one defines Φ as the set of all decision functions. *Optimality* of a plan is defined as for the finite horizon model with J^{π} in place of J_N^{π}.

THEOREM 3. (a) $V(x) = \lim_{n\to\infty} T^{*n}V_0$ (*value iteration*);

(b) $V = T^*V$ (*optimality equation*);

(c) there exists a stationary optimal plan;

(d) If $T_\varphi V = T^*V$, i.e. $\varphi(x)$ is a minimizer of the function $u \mapsto TV(x,u)$, then the stationary plan φ^∞ is optimal (*optimality criterion*).

Proof. As in the finite horizon case, we conclude that $V_n \in \mathcal{V}$ for all n. Now fix any $x \in S$ and set $t_n(u) := TV_{n-1}(x,u)$. By assumption we know that $t_n(u)$ is increasing in n and l.s.c. in u. Therefore it follows that $\lim_{n\to\infty} \inf_{u\in U} t_n(u) = \inf_{u\in U} \lim_{n\to\infty} t_n(u)$ from a variant of Dini's theorem (see Schäl 1975 Proposition 10.1). By use of Lemma 5c, this equation now means:

$V^\infty(x) = \lim_{n\to\infty} V_n(x) = \lim_{n\to\infty} T^*V_{n-1}(x) = \inf_{u\in U} T(\lim_{n\to\infty} V_{n-1})(x,u) = T^*V^\infty(x)$.

As the limit of an increasing sequence of l.s.c. functions, $\lim_{n\to\infty} t_n(u) = TV^\infty(x,u)$ is l.s.c.

As for (21), we conclude that there exists a decision function φ with $T_\varphi V^\infty(x) = T^*V^\infty(x)$, $x \in S$. Thus $V^\infty(x) = T_\varphi V^\infty(x)$, $x \in S$, which implies that $V^\infty(x) = (T_\varphi)^n V^\infty(x) \geq (T_\varphi)^n V_0(x)$ and hence for $n\to\infty$: $V^\infty(x) \geq J^{\varphi^\infty}(x)$ for all x . But in view of Lemma 5b, $V^\infty(x) \leq V(x) \leq J^{\varphi^\infty}(x)$ for all x. Thus (a), (b), and (c) are proved. From $T_\varphi V = T^*V$, we finally get:

$V = T_\varphi V = (T_\varphi)^n V \geq (T_\varphi)^n V_0 \to J^{\varphi^\infty}$ and (d) is also proved. $\square$

For some of the statements of theorem 3, the structure assumption is too strong (see Bertsekas & Shreve 1978). Now we want to extend the *Howard improvement procedure*, well–known for the discounted case, to a more general situation.

THEOREM 4. (*Howard Improvement*). Let φ, ψ be any decision functions and set $J := J^{\varphi^\infty}$ and:

$$U(x,\varphi) := \{ u \in U;\ TJ(x,u) < J(x) \},\ x \in S.$$

If, for some subset S^* of S, $\psi(x) \in U(x,\varphi)$, $x \in S^*$, and $\psi(x) = \varphi(x)$, $x \notin S^*$, then one has:

$J^{\psi^\infty} \leq J = J^{\varphi^\infty}$ and $J^{\psi^\infty}(x) < J(x)$, $x \in S^*$.

Proof. We certainly have $T_\psi J \leq J$; in fact we have by (20)

$T_\psi J(x) < J(x)$ if $x \in S^*$ and $T_\psi J(x) = T_\varphi J(x) = J(x)$ if $x \notin S^*$.

Then we obtain $T_\psi^2 J \leq T_\psi J \leq J$ and by induction $T_\psi^n J \leq T_\psi J \leq J$ for all n. Now we obtain

$J^{\psi^\infty} = \lim_{n\to\infty} T_\psi^n V_0 \leq \lim_{n\to\infty} T_\psi^n J \leq T_\psi J \leq J$, in particular $J^{\psi^\infty}(x) < J(x)$ if $x \in S^*$. $\square$

Now one can ask what happens in the situation where $U(x,\varphi) := \{ u \in U;\ TJ(x,u) < J(x) \} = \emptyset$ for all x, i.e. $T^*J \geq J$. Since we always have $T^*J \leq T_\varphi J = J$, this means that $T^*J = T_\varphi J = J \geq V_0$. The next theorem gives an answer.

Theorem 5. *Verification theorem.* Let $v : S \mapsto (-\infty,\infty]$ be a measurable function with $v \geq V_0$ and φ some decision function with $v = T^*v = T_\varphi v$. Then we have: The function v is the value function V and φ defines a stationary optimal plan φ^∞ provided that $\alpha^n \cdot E[(v-V_0)(X_n)] \to 0$ as $n\to\infty$ with $X_n = X_n^{x,\psi^\infty}$ for all decision functions ψ and for all $x \in S$.

Proof. On the one hand we obtain as above: $J^{\varphi^\infty} = \lim_n T_\varphi^n V_0 \leq \lim_n T_\varphi^n v = v$. On the other hand we have for any ψ: $T_\psi v \geq T^*v = v$ and by induction $T_\psi^n v \geq v$. Now we get from Lemma 2d

$$v(x) \leq T_\psi^n v(x) = T_\psi^n V_0(x) + \alpha^n \cdot E[v(X_n)-V_0(X_n)] \quad .$$

From our condition we obtain $J^{\psi^\infty} = \lim_n T_\psi^n V_0 \geq v$. Thus $V \leq J^{\varphi^\infty} \leq v \leq \inf_\psi J^{\psi^\infty} = V$.
The last identity follows from Theorem 3c. $\square$

Theorem 3(b) contains a fixed–point equation. It is remarkable that it can be solved by an iteration scheme without any contraction property in view of Theorem 3(a). In the discounted case Banach's fixed–point theorem applies. The Howard Improvement is another tool to approach the value function V. It works well in finite–state discounted models (see Martin–Löf 1994).

5. MINIMIZING THE RUIN PROBABILITY

We again consider the insurance model which can be controlled by reinsurance and investment in a financial market. Now we want to minimize the ruin probability. Therefore, we now assume that the company is ruined if the surplus is negative. Therefore we must modify the system function of section 1 for negative values. Moreover, we add a cemetery state $-\infty$ to the state space $\mathbb{R}$ of sections 1, 3. Again the insurance company can invest the capital (surplus) in a financial market where d assets can be traded and the random variables $W_n = (R_n,Y_n,Z_n)$ are given as in sections 1, 3 and satisfy the independence assumption. A dynamic portfolio will again be specified by a portfolio vector $\vartheta_n \in \mathbb{R}^d$ at the beginning of period n+1. But in this section, the component ϑ_n^k of ϑ_n represents the proportion of the capital x which is invested in the k^{th} stock, $k = 1,...,d$, i.e. $x\,\vartheta = \delta$. We write Θ for the set of all admissible portfolio vectors which is the simplex

$$\Theta := \{\vartheta = (\vartheta^1,...,\vartheta^d) \in \mathbb{R}^d;\ \vartheta^k \geq 0,\ 1 \leq k \leq d,\ \textstyle\sum_1^d \vartheta^k \leq 1\} \ . \tag{22}$$

For $\vartheta \in \Theta$, $\vartheta^0 := 1 - \sum_1^d \vartheta^k \geq 0$ represents the proportion of the capital which is not invested in some stock. In this model we do not allow for negative amounts ϑ_n^k, thus excluding short selling, in order to get a compact action space for the structure assumption.

In addition, the risk process may be controlled by reinsurance as in section 3. Thus a decision function φ again consists of two components $\varphi = (\varphi',\varphi'')$ where φ' specifies the retention level b of reinsurance as above and φ'' specifies the portfolio $\vartheta = (\vartheta^1,...,\vartheta^d)$. Thus state space S and control space U are given according to $S = [-\infty,\infty)$, $U = [\underline{b},\bar{b}]\times\Theta$.

We now want to compute the system function f. From (5) we obtain with $X_n \cdot \vartheta_n = \delta_n$:

$$X_{n+1} = X_n \cdot \left[1 + \langle \vartheta_n, R_{n+1} \rangle\right] + c(b_n) \cdot Z_{n+1} - h(b_n, Y_{n+1}) \quad \text{for } X_n \geq 0. \tag{23}$$

Now we choose with $u = (b,\vartheta)$, $w = (\rho,y,z)$:

$$f(x,u,w) = x \cdot \left[1 + \langle \vartheta, \rho \rangle\right] + c(b) \cdot z - h(b,y) \quad \text{for } x \geq 0. \tag{24}$$

The cost will be defined by

$$g(x,u) = V_0(x) := \mathbf{1}_{(-\infty,0)}(x), \quad \alpha = 1. \tag{25}$$

Once the system is in state $x \in (-\infty,0)$, then it shall move to $-\infty$ in the next step, i.e. we set $f(x,u,w) = -\infty$ for $x \in [-\infty,0)$. Thus the cost of 1 unit has to be paid at most once. Then

$$\begin{aligned} J_n^{\pi}(x) &= E\left[\sum_{0 \leq m < n} g(X_m, \varphi_m(X_m)) + V_0(X_N)\right] \\ &= P\left[X_m^{x,\pi} \in (-\infty,0) \text{ for some } 0 \leq m \leq n\right], \end{aligned} \tag{26}$$

in particular $J_n^{\pi}(x) = 1$ for $x \in (-\infty,0)$, $J_n^{\pi}(-\infty) = 0$,

which is just the probability of being ruined after n periods.

As in §3 we only consider the case where the interest rate r for the surplus not invested in stocks is zero. However, the case $r \geq 0$ can be treated in the same way since the ruin probability is the same if one replaces the surplus X_n by the discounted surplus. As explained in §3, this leads to a scenario which can be looked upon as market with interest rate zero. As in section 3 we assume:

CONTINUITY ASSUMPTION. The functions c(b) and h(b,y) are continuous in b (for each y).

However, there is a discontinuity of the system function f(x,u,w) at x=0. Thus the usual continuity assumption (see Bertsekas & Shreve, 1978, pp 46, 209) is not satisfied. We will overcome that difficulty by choosing a suitable class $\mathscr{V}$ in the structure assumption. In fact we set:

$$\mathscr{V} := \{\, v : [-\infty,\infty) \mapsto [0,1];\ v \text{ is l.s.c. on } [0,\infty),\ v(x) = 1 \text{ for } x \in (-\infty,0),\ v(-\infty) = 0 \,\}. \tag{27}$$

Obviously we have $V_0 \in \mathscr{V}$. Now choose some $v \in \mathscr{V}$. Then v is obviously l.s.c. also on the whole of $\mathbb{R}$. Since f(x,u,w) is continuous in (x,u) on $[0,\infty)\times U$, it follows that $v(f(x,u,w))$ is l.s.c. in (x,u) on $[0,\infty)\times U$. From Fatou's lemma we now conclude that $Tv(x,u) = E[v(f(x,u,W))]$ is l.s.c. in (x,u) on $[0,\infty)\times U$. Since U is compact, it is known that $T^*v(x) = \min_{u\in U} Tv(x,u)$ is l.s.c. in x on $[0,\infty)$ (see Bertsekas & Shreve, 1978, Proposition 7.33). For $x \in (-\infty,0)$ we have $Tv(x,u) = 1$ and finally $Tv(-\infty,u) = 0$. Thus we have the following result:

PROPOSITION 1. The structure assumption of section 4 (and of section 2) is satisfied under the continuity assumption of this section.

As a consequence, Theorems 3 – 5 hold for the present insurance model.

For the sequel, we make the following natural assumption, also used by Waters (1983).

ASSUMPTION. $P[c(b)\cdot Z < h(b,Y)] > 0$ for all b.

LEMMA 6. Let $\xi : (0,\infty) \mapsto (0,1]$ be any measurable function.
(a) For any $x \in \mathbb{R}$, the following function d(b,x) is l.s.c in b:

$$d(b,x) := E[\xi(Z)\cdot \mathbf{1}_{\{c(b)\cdot Z - h(b,Y) < x\}}] = \int \xi(z)\, P[c(b)\cdot z - h(b,Y) < x]\; P[Z \in dz]\ .$$

(b) There is some $\varepsilon > 0$ such that $\inf_b d(b,-\varepsilon) > 0$.

Proof. a) The two representations of the function d follow from the independence of Z and Y. We set $H(b,y,z) := c(b)\cdot z - h(b,y)$. Then H(b,y,z) is continuous in b and thus $\mathbf{1}_{\{H(b,y,z) < x\}}$ is l.s.c. in b for all y,z. Now the result follows from Fatou's lemma.

b) By use of (a) and the compactness of $[\underline{b},\bar{b}]$, we obtain from a variant of Dini's theorem (see Schäl 1975, Proposition 10.1) for $\varepsilon \downarrow 0$: $\inf_b d(b,-\varepsilon) \uparrow \inf_b d(b,0) = d(b_o,0)$ for some b_o. The last expression is positive for all b_o by our assumption from above and since ξ is positive. []
We will use a slightly stronger version of the no-arbitrage condition already considered in §3.

NO ARBITRAGE ASSUMPTION. For all z in the range (or support) of Z and for any portfolio vector $\delta \in \mathbb{R}^n$ we have: $P[\, <\delta,R> \geq 0 \mid Z = z] = 1$ implies $P[\, <\delta,R> = 0 \mid Z = z] = 1$.

LEMMA 7. (a) For all z in the range (or support) of Z we have: $\inf_{\delta\in\mathbb{R}^n} P[\, <\vartheta,R> \leq 0 \mid Z = z] > 0$.
(b) There is some $\varepsilon > 0$ such that $\inf_{b\in [\underline{b},\bar{b}],\delta\in\mathbb{R}^n} P[\, <\delta,R> + c(b)\cdot Z - h(b,Y) < -\varepsilon] > 0$.

Proof. a) We write $\mathscr{L}_z$ for the smallest linear space L in $\mathbb{R}^d$ such that $P[R \in L \mid Z = z] = 1$. Then it follows as in §3 that the no arbitrage condition from above is equivalent to the following condition (NA)':

For all z in the range (or support) of Z and $\delta \in \mathscr{L}_z$, $\delta \neq 0$: $P[<\delta,R> < 0 \mid Z = z] > 0$. (NA)'

For $\delta \perp \mathscr{L}_z$ we have $P[\, <\delta,R> = 0 \mid Z = z] = 1$.
For $\delta \in \mathscr{L}_z$, $\delta \neq 0$, we have $P[\, <\delta,R> \leq 0 \mid Z = z] \geq P[\, <\frac{1}{\|\delta\|}\delta,R> < 0 \mid Z = z] > 0$ by (NA)'.
Now as in the proof of Lemma 6, we can show that the function $\delta \mapsto P[\, <\delta,R> < 0 \mid Z = z]$ is l.s.c. and thus its infimum is attained on the compact sphere $\{\delta \in \mathscr{L}_z ; \|\delta\| = 1\}$.
b) For all z in the range (or support) of Z we have $\xi(z) := \inf_\delta P[\, <\delta,R> \leq 0 \mid Z = z] > 0$ by (a).
Now $P[<\delta,R> + c(b)\cdot Z - h(b,Y) < -\varepsilon] \geq P[\, <\delta,R> \leq 0,\ c(b)\cdot Z - h(b,Y) < -\varepsilon]$
$= \int P[\, <\delta,R> \leq 0 \mid Z = z]\cdot P[c(b)\cdot z - h(b,Y) < -\varepsilon]\; P[Z \in dz]$
$\geq \int \xi(z)\cdot P[c(b)\cdot z - h(b,Y) < -\varepsilon]\; P[Z \in dz]$.
From Lemma 6b we now know that the last expression is positive for some $\varepsilon > 0$. []

PROPOSITION 2. Let $M > 0$ be arbitrary.

(a) There exists some $n \in \mathbb{N}$ such that $\eta_{n,M} := \sup_{x\leq M,\pi} P[X_n^{x,\pi} \geq 0] < 1$.

(b) $P\left[0 \le X_m^{x,\pi} \le M \text{ for infinitely many } m\right] = 0$ for all x,π.

Proof. a) Choose $\varepsilon > 0$ as in Lemma 7b and set $H(b,y,z) := c(b)\cdot z - h(b,y)$. Then we have by (23):

$X_n = X_0 + \Sigma_{m=0}^{n-1}\ [\langle X_m\ \varphi''_m, R_{m+1}\rangle\ +\ H(\varphi'_m, Y_{m+1})]$ on the set $\{X_m \geq 0$ for $0 \leq m < n\}$.

We know that for the event $A_m := \{\langle X_m^+\ \varphi''_m, R_{m+1}\rangle\ +\ H(\varphi'_m, Y_{m+1}) < -\varepsilon\}$ we have:

$P[A_m | W_1,...,W_m] \geq \inf_{b,\delta} P[\ \langle\delta,R\rangle + H(b,Y) < -\varepsilon] =: \nu$ where $\nu > 0$ by Lemma 7b.

By use of $\cap_{m=0}^{n-1} A_m \in \sigma(W_1,...,W_n)$, we now can prove by induction that

$P[\Sigma_{m=0}^{n-1}\ \{\langle X_m^+\ \varphi''_m, R_{m+1}\rangle + H(\varphi'_m, Y_{m+1})\} < -n\cdot\varepsilon] \geq\ P[\cap_{m=0}^{n-1}\ A_m] \geq \nu^n.$

Now we obtain for $X_0 = x$: $P[X_n < 0] \geq P[\Sigma_{m=0}^{n-1}\ \{\langle X_m^+\ \varphi''_m, R_{m+1}\rangle\ +\ H(\varphi'_m, Y_{m+1})\} < -x]$

$\geq P[\Sigma_{m=0}^{n-1}\ \{\langle X_m^+\ \varphi''_m, R_{m+1}\rangle\ +\ H(\varphi'_m, Y_{m+1})\} < -n\cdot\varepsilon] \geq \nu^n$ for $x \leq M \leq n\cdot\varepsilon$,

and thus $P[X_n \geq 0] \leq 1 - \nu^n$ for $x \leq M$ if $n \geq M/\varepsilon$.

b) It is easy to see that we obtain from (a): $P[X_{m+n}^{x,\pi} \geq 0 | W_1,...,W_m] \leq \eta_{n,M} < 1$ on the set $\{X_m^{x,\pi} \leq M\}$. We define the stopping times $\tau_0 := 0$, $\tau_{k+1} := \inf\ \{m \geq \tau_k + n;\ X_\ell^{x,\pi} > 0,\ \ell \leq m,$ $X_m^{x,\pi} \leq M\} = \inf\ \{m \geq \tau_k + n;\ 0 \leq X_m^{x,\pi} \leq M\}$. Then we have

$P[\tau_1 < \infty, \ldots, \tau_{k+1} < \infty] \leq P[\tau_1 < \infty, \ldots, \tau_k < \infty]\cdot P[X_{\tau_k+n}^{x,\pi} > 0 | \tau_1 < \infty, \ldots, \tau_k < \infty]$

$\leq P[\tau_1 < \infty, \ldots, \tau_k < \infty]\cdot\eta_{n,M}$ and thus

$P[0 \leq X_m^{x,\pi} \leq M$ for infinitely many $m] = \lim_k P[\tau_1 < \infty, \ldots, \tau_k < \infty]$

$\leq P[0 \leq X_m^{x,\pi} \leq M$ for infinitely many $m]\cdot\eta_{n,M}$ and the result follows. []

From Proposition 2b we conclude that

$$\lim_{n\to\infty} P[0 \leq X_n^{x,\pi} \leq M\ ] = 0 \quad \forall\ x,\pi,\ M > 0. \tag{28}$$

Now we are able to prove a kind of contraction property.

LEMMA 8. If $\xi : [-\infty,\infty) \mapsto [0,\infty)$ is any bounded measurable function such that $\xi(\infty) := \lim_{x\to\infty} \xi(x) = 0$ and $\xi(x) = 0$ for $x \leq 0$, then $\lim_{n\to\infty} E[\xi(X_n^{x,\pi})] = 0 \quad \forall\ x,\pi.$

Proof. For any $\delta > 0$ we can choose any M such that $\xi(x) \leq \delta$ for $x > M$. Thus if $\|\xi\|$ is the upper bound of ξ we obtain: $E[\xi(X_n)] = E[\mathbf{1}_{\{0 \leq X_n \leq M\}}\cdot\xi(X_n)] + E[\mathbf{1}_{\{X_n > M\}}\cdot\xi(X_n)]$ $\leq \|\xi\|\cdot P[0 < X_n \leq M] + \delta$. Now the result follows from (28). []

We can and will use Lemma 8 for an application of Theorems 4 and 5. There we choose $\xi = J - V_0$ and $J = v = J^{\varphi^\infty}$ as the ruin probability under the stationary plan φ^∞ which describes the situation where the decision maker employs no reinsurance and does not invest in stocks, i.e.,

$$\varphi(x) = (\varphi'(x),\varphi''(x)) = (\bar{b},0) \text{ where } \bar{b} \text{ stands for "no reinsurance" as before.} \tag{29}$$

Thus we want to know how to improve φ^∞ and under what conditions φ^∞ is the optimal policy. The following first two properties are well–known. In fact, the condition $\xi(\infty) = J(\infty) = 0$ just means that the ruin probability tends to zero as the initial surplus tends to infinity.

LEMMA 9. Assume $c \cdot E[Z] > E[Y]$. In the situation of (29) we then have for $J = J^{\varphi^\infty}$:

(a) $J(x)$ is decreasing;

(b) $J(\infty) := \lim_{x\to\infty} J(x) = 0$;

(c) $J(x) - V_0(x) = 0$ for $x \le 0$ and $\lim_{x\to\infty} J(x) - V_0(x) = 0$.

Proof. Part (a) is obvious. Part(b) follows from the law of large numbers (see Grandell 1991 p.5). Part (c) follows from (b) since $J(x) = V_0(x)$ for $x \le 0$ and $V_0(x) = 0$ for $x > 0$. $\square$

COROLLARY 1. Assume $c \cdot E[Z] > E[Y]$, let φ^∞ be as in (29), set $J := J^{\varphi^\infty}$ and

$$U(x,\varphi) := \{ u \in U;\ TJ(x,u) < J(x) \},\ x \in S.$$

(a) *Howard Improvement.* For each decision function ψ with $\psi(x) \in U(x,\varphi)$ for some states x and $\psi(x) = \varphi(x)$ for the other states x, one has: $J^{\psi^\infty} \le J$ and $J^{\psi^\infty}(x) < J(x)$ if $\psi(x) \in U(x,\varphi)$.

(b) *Verification theorem.* If $J = T^*J$, i.e., $T_\varphi J = T^*J$, then φ defines a stationary optimal plan φ^∞.

Recall that $T_\varphi J = J$ by (20).

6. EXPONENTIALLY DISTRIBUTED CLAIMS

In this section we assume that there is no financial market, i.e., we can choose $R = 0$ for the return in one period. Thus the decision maker can only control by reinsurance. Then one can restrict attention to control actions of the form $u = (b,0)$. Therefore we will identify u with b. Moreover we will concentrate on the situation where

$$Y \sim \mathrm{Exp}(1/\mu) \text{ and } Z \sim \mathrm{Exp}(\lambda) \text{ for some } 0 < \lambda, \mu < \infty \text{ with } c > \lambda\mu. \tag{30}$$

There $\mathrm{Exp}(\lambda)$ stands for the exponential distribution with parameter λ. Then the property $Z \sim \mathrm{Exp}(\lambda)$ is always fulfilled in the Cramér–Lundberg model. The following identity is well–known (see e.g. Grandell 1991 (II)).

PROPOSITION 3. If $J(x)$ is defined as in Corollary 1, i.e., $J(x)$ is the ruin probability in the Cramér–Lundberg model, then in the situation of (30) one has for $x \ge 0$:

$$J(x) = (1-\kappa) \cdot e^{-\kappa \cdot x/\mu} \text{ with } \kappa := 1 - \frac{\lambda\mu}{c} > 0.$$

Obviously, the assumptions of §5 are satisfied in the case of proportional reinsurance. In order to improve the plan φ^∞ as defined by (29) which recommends to do nothing, one has to study the following quantity according to Corollary 1:

$$TJ(x,b) = E\left[J(f(x,b,Y,Z))\right] = E\left[J(x + c(b)\cdot Z - h(b,Y))\right] \quad \text{for } x \geq 0$$
$$TJ(x,b) = J(x) = 1 \quad \text{for } -\infty < x < 0. \tag{31}$$

LEMMA 10. Set $1 + \eta(b) := c(b)/\lambda\mu$ and $q(b) := \frac{1}{1 - \kappa\cdot b} \cdot \frac{1}{1 + \kappa\cdot\left[1+\eta(b)\right]}$.
Then one has in the case of proportional reinsurance (7):

$$TJ(x,b) = q(b)\cdot J(x) + (1-b) \cdot \frac{\kappa}{1 - \kappa\cdot b} \cdot \frac{b}{b + 1 + \eta(b)} \cdot \exp\{-\frac{x}{b\mu}\} \quad \text{for } x \geq 0.$$

The proof follows by a straightforward but lengthy computation. Now we want to apply the verification theorem to φ by showing that $b = 1$ is a minimizer of $TJ(x,\cdot)$. The second term of $TJ(x,b)$ is nonnegative, vanishes for $b = 1$, and thus attains its minimum for $b = 1$. Now we will consider the first term. According to (4) we have: $1 + \eta(b) = 1 + \eta - (1+\theta)\cdot(1-b)$ and $\eta'(b) = 1+\theta$. Now we get for the first derivative q' of q by a straightforward computation:

$$q'(b) = \kappa\cdot q(b)^2\cdot \left[\frac{1}{1-\kappa} - \left[1+\kappa\right]\cdot(1+\theta) + 2(1+\theta)\cdot\kappa\cdot b\right].$$

Hence, we know that $q'(b) < 0$ for all b if only $q'(1) < 0$. The latter condition obviously holds if and only if $\frac{1}{1-\kappa} - \left[1+\kappa\right]\cdot(1+\theta) + 2(1+\theta)\cdot\kappa < 0$. Now this leads to the condition: $q'(1) < 0 \Leftrightarrow 1+\theta > (1-\kappa)^{-2}$. If we denote the *safety loading of the insurer* by $\eta = \eta(1) > 0$, i.e.
$c\cdot E\left[Z\right] = (1 + \eta)\cdot E\left[Y\right]$ or $\frac{1}{1-\kappa} = 1 + \eta$, we obtain the condition:

$$1+\theta > (1 + \eta)^2, \text{ i.e., } \theta > 2\eta + \eta^2. \tag{32}$$

This condition is also obtained when maximizing the adjustment coefficient (see Schmidli 1999). The adjustment coefficient R(b) is in the present situation (see Dickson & Waters 1996)

$$R(b) = \frac{1}{\mu}\left[\frac{1}{b} - \frac{1}{b(1+\theta)-(\theta-\eta)}\right] = \frac{\eta - \theta(1-b)}{\mu\cdot b\cdot\left[1+\eta - (1+\theta)(1-b)\right]}.$$

Since η^2 is small, the condition (32) comes close to the condition $\theta > 2\eta$. The latter condition is obtained by Taksar & Markussen (2002, § 3.1) for a diffusion approximation and also appears in many different optimization problems in insurance (see Hϕjgaard & Taksar1998a).

THEOREM 6. In the model where (30) holds and where the decision maker can only control by proportional reinsurance, it is optimal to have no reinsurance under the condition (32).

On the other hand, we know from the validity of the Howard Improvement that it is optimal to have some reinsurance if $TJ(x,b) < J(x)$ for some $b < 1$ and some x.

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Appendix.

Positive interest rate. Now we consider the following extended model: The insurance company can invest the capital (surplus) in a financial market where 1+d assets can be traded. An investor can invest in a bank account (bond) with an interest rate $r \geq 0$. In addition as before, there are d stocks which can be described by the stock price process $\{S_n\}$ as in section 3. The random variables R_n, Y_n, Z_n will have the same meaning as before. We want to show that one gets the same system functions as in sections 3 and 5 if one considers discounted quantities.

The *discounted stock price process* $\{\breve{S}_n = (\breve{S}_n^1,...,\breve{S}_n^d), n \geq 0\}$ is defined by

(A.1) $$\breve{S}_n^k := \Pi_{m=1}^n \exp\{-r \cdot Z_m\} \cdot S_n^k.$$

The return process $\{\breve{R}_n = (\breve{R}_n^1,...,\breve{R}_n^d), n \geq 0\}$ for the discounted stock price process is defined by

(A.2) $$\breve{S}_n^k = \breve{S}_{n-1}^k \cdot (1 + \breve{R}_n^k) \text{ , i.e. } (1 + \breve{R}_n^k) = \exp\{-r \cdot Z_n\} \cdot (1 + R_n^k),\ 1 \leq k \leq d.$$

Again, it is reasonable to allow for a dependence of Z_n and R_n or for a dependence of Z_n and $\breve{R}_n$. In addition we define the discounted capital and the discounted claims by

(A.3) $$\breve{X}_n := \Pi_{m=1}^n \exp\{-r \cdot Z_m\} \cdot X_n, \quad \breve{Y}_n := \Pi_{m=1}^n \exp\{-r \cdot Z_m\} \cdot Y_n,$$

For the part of the claim paid by the insurer we make here the following weak assumption which is fulfilled for the two cases (5) and (7):

Assumption: $h(b,\gamma \cdot y) = \gamma \cdot h(b,y)$ for all $b, y, \gamma \geq 0$.

For the premium we assume that c and c(b) now are the discounted income rates. Then one has the following undiscounted net income:

(A.4a) $$\Pi_{m=1}^n \exp\{r \cdot Z_m\} \cdot c(b) \cdot Z_n$$

in period n with a retention level b. If one takes into account a continuous–time discounting in period n one would get an undiscounted net income of

(A.4b) $$\Pi_{m=1}^n \exp\{r \cdot Z_m\} \cdot \frac{1}{r} \cdot (1 - \exp\{-r \cdot Z_n\}) \cdot c(b).$$

A dynamic portfolio will again be specified at the beginning of any period n+1 by portfolio vectors δ_n or $\vartheta_n \in \mathbb{R}^d$ as in section 3 an 5, respectively. But in this section, the component δ_n^k of δ_n will represent the discounted amount invested in the k^{th} stock. We now want to compute the system functions f if one uses discounted quantities.

If x is the capital at the beginning of period n+1, then $\vartheta_n^k \cdot x/S_n^k$ denotes the number of shares the investor holds in period n+1; thus the value of these shares at the end of period n+1 is

$$\vartheta_n^k \cdot x \cdot S_{n+1}^k/S_n^k = \vartheta_n^k \cdot x \cdot \breve{S}_{n+1}^k \cdot \exp\{r \cdot Z_{n+1}\}/\breve{S}_n^k = \vartheta_n^k \cdot x \cdot \exp\{r \cdot Z_{n+1}\} \cdot (1 + \breve{R}_{n+1}^k).$$

If b_n is the retention level in period n+1, we have in situation A4a:

$$X_{n+1} = X_n \cdot [1 - \Sigma_{k=1}^{d} \vartheta_n^k] \cdot \exp\{r \cdot Z_{n+1}\} + X_n \cdot \exp\{r \cdot Z_{n+1}\} \cdot \Sigma_{k=1}^{d} \vartheta_n^k \cdot (1 + \breve{R}_{n+1}^k)$$
$$+ \Pi_{m=1}^{n+1} \exp\{r \cdot Z_m\} \cdot c(b_n) \cdot Z_{n+1} - h(b_n, Y_{n+1}) \quad \text{and hence}$$
$$\breve{X}_{n+1} = \breve{X}_n \cdot [1 - \Sigma_{k=1}^{d} \vartheta_n^k + \Sigma_{k=1}^{d} \vartheta_n^k \cdot (1 + \breve{R}_{n+1}^k)] + c(b_n) \cdot Z_{n+1} - h(b_n, \breve{Y}_{n+1})$$
$$= \breve{X}_n \cdot [1 + \Sigma_{k=1}^{d} \vartheta_n^k \cdot \breve{R}_{n+1}^k] + c(b_n) \cdot Z_{n+1} - h(b_n, \breve{Y}_{n+1})$$
$$= \breve{X}_n \cdot [1 + <\vartheta_n, \breve{R}_{n+1}>] + c(b_n) \cdot Z_{n+1} - h(b_n, \breve{Y}_{n+1})$$
$$= \breve{X}_n + <\delta_n, \breve{R}_{n+1}> + c(b_n) \cdot Z_{n+1} - h(b_n, \breve{Y}_{n+1}) \quad \text{with } \delta_n = \breve{X}_n \, \vartheta_n .$$

Thus we have the same system functions for sections 3 and 5 (see (13) and (24)). If one decides for (A.4b) in place of (A.4b) one should replace $c(b_n) \cdot Z_{n+1}$ with $\frac{1}{r} \cdot (1 - \exp\{-r \cdot Z_{n+1}\}) \cdot c(b_n)$. In the discounted case we will choose

(A.5) $\qquad W_n = (\breve{R}_n, \breve{Y}_n, Z_n).$

Thus we have to replace the quantities in sections 3 and 5 with the respective quantities discounted by the interest rate which may be close to the quantities discounted by inflation.

Proof of Lemma 10. We obtain from (31) and Proposition 3:

$$TJ(x,b) = P[x + c(b) \cdot Z - h(b,Y) \le 0] +$$
$$+ (1-\kappa) \cdot E[\mathbf{1}_{\{x + c(b) \cdot Z - h(b,Y) > 0\}} \cdot \exp\{-\tfrac{1}{\mu}\kappa \cdot (x + c(b) \cdot Z - h(b,Y))\}]$$
$$= P[x + c(b) \cdot Z - h(b,Y) \le 0] + (1-\kappa) \cdot \exp\{-\tfrac{1}{\mu}\kappa \cdot x\} \cdot$$
$$E[\mathbf{1}_{\{x + c(b) \cdot Z - h(b,Y) > 0\}} \cdot \exp\{-\tfrac{1}{\mu}\kappa \cdot (c(b) \cdot Z - h(b,Y))\}].$$

For the first term we have

$$P[x + c(b) \cdot Z \le h(b,Y)] = {}_0\!\int^\infty \lambda \cdot e^{-\lambda \cdot z} \cdot P[x + c(b) \cdot z \le h(b,Y)] \, dz \quad \text{where}$$
$$P[x + c(b) \cdot z \le h(b,Y)] = P[x + c(b) \cdot z \le b \cdot Y]$$
$$= P[\tfrac{1}{b}[x + c(b) \cdot z] \le Y] = \exp\{-\tfrac{1}{\mu} \cdot \left(\tfrac{1}{b}[x + c(b) \cdot z]\right)\} = \exp\{-\tfrac{1}{b\mu}[x + c(b) \cdot z]\}.$$

These computations lead to

$$P[x + c(b) \cdot Z \le h(b,Y)] = {}_0\!\int^\infty \lambda \cdot e^{-\lambda \cdot z} \cdot \exp\{-\tfrac{1}{b\mu}[x + c(b) \cdot z]\} \, dz$$
$$= \exp\{-\tfrac{x}{b\mu}\} \cdot {}_0\!\int^\infty \lambda \cdot e^{-\lambda \cdot z} \cdot \exp\{-\tfrac{1}{b\mu} c(b) \cdot z\} \, dz$$
$$= \lambda \cdot \exp\{-\tfrac{x}{b\mu}\} \cdot {}_0\!\int^\infty \exp\{-[\lambda + \tfrac{1}{b\mu} c(b)] \cdot z\} \, dz$$
$$= \lambda \cdot \exp\{-\tfrac{x}{b\mu}\} \cdot [\lambda + \tfrac{1}{b\mu} c(b)]^{-1}$$
$$= \lambda \cdot \exp\{-\tfrac{x}{b\mu}\} \cdot [\tfrac{\lambda \cdot b\mu + c(b)}{b\mu}]^{-1} = \exp\{-\tfrac{x}{b\mu}\} \cdot \frac{b \cdot \lambda\mu}{b \cdot \lambda\mu + c(b)}$$

For the second term we obtain

$$E[\mathbf{1}_{\{x + c(b)\cdot Z - h(b,Y) > 0\}} \cdot \exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot Z - h(b,Y))\}]$$

$$= {}_0\int^\infty \lambda\cdot e^{-\lambda z}\, E[\mathbf{1}_{\{x + c(b)\cdot z > b\cdot Y\}} \cdot \exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot z - b\cdot Y)\}]\ dz \quad \text{where}$$

$$E[\mathbf{1}_{\{x + c(b)\cdot z > b\cdot Y\}} \cdot \exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot z - b\cdot Y)\}]$$

$$= {}_0\int^\infty \frac{1}{\mu} e^{-y/\mu}\, \mathbf{1}_{\{x + c(b)\cdot z > b\cdot y\}} \cdot \exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot z - b\cdot y)\}\ dy$$

$$= {}_0\int^{(x+c(b)\cdot z)/b} \frac{1}{\mu}\exp\{-\frac{1}{\mu} y\}\ \cdot\exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot z - b\cdot y)\}\ dy$$

$$= {}_0\int^{(x+c(b)\cdot z)/b} \frac{1}{\mu}\exp\{-\frac{1}{\mu} y - \frac{1}{\mu}\kappa\cdot(c(b)\cdot z - b\cdot y)\}\ dy$$

$$= \exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\ {}_0\int^{(x+c(b)\cdot z)/b} \frac{1}{\mu}\exp\{-\frac{1}{\mu} y + \frac{1}{\mu}\kappa\cdot b\cdot y\}\ dy$$

$$= \frac{1}{\mu}\cdot\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\ {}_0\int^{(x+c(b)\cdot z)/b}\ \exp\{-\frac{1}{\mu}(1-\kappa\cdot b)\cdot y\}\ dy$$

$$= \frac{1}{\mu}\cdot\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\cdot[\frac{1}{\mu}(1-\kappa\cdot b)]^{-1}\cdot\left[1 - \exp\{-\frac{1}{\mu}(1-\kappa\cdot b)\cdot\frac{x + c(b)\cdot z}{b}\ \}\right]$$

$$= \frac{1}{\mu}\cdot\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\cdot\frac{\mu}{1 - \kappa\cdot b}\cdot\left[1 - \exp\{-\frac{1}{\mu}(\frac{x + c(b)\cdot z}{b} - \kappa\cdot\frac{x + c(b)\cdot z}{1})\ \}\right]$$

$$= \frac{1}{1 - \kappa\cdot b}\left[\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\} - \exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z - \frac{x + c(b)\cdot z}{b\mu} + \kappa\cdot\frac{x + c(b)\cdot z}{\mu}\}\right]$$

$$= \frac{1}{1 - \kappa\cdot b}\left[\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \} - \exp\{-\frac{x + c(b)\cdot z}{b\mu} + \kappa\cdot\frac{x}{\mu}\}\right]$$

$$= \frac{1}{1 - \kappa\cdot b}\left[\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \} - \exp\{-\frac{c(b)\cdot z}{b\mu}\}\cdot\exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\right]$$

Thus we obtain

$$E[\mathbf{1}_{\{x + c(b)\cdot Z - h(b,Y) > 0\}} \cdot \exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot Z - h(b,Y))\}]$$

$$= {}_0\int^\infty \lambda\cdot e^{-\lambda z}\frac{1}{1 - \kappa\cdot b}\left[\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \} - \exp\{-\frac{c(b)\cdot z}{b\mu}\}\cdot\exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\right] dz$$

$$= \frac{1}{1 - \kappa\cdot b}\left[{}_0\int^\infty \lambda\cdot e^{-\lambda z}\ \exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\ dz - {}_0\int^\infty \lambda\cdot e^{-\lambda z}\cdot\exp\{-\frac{c(b)\cdot z}{b\mu}\}\cdot\exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\ dz\right]$$

$$= \frac{\lambda}{1 - \kappa\cdot b}\left[{}_0\int^\infty e^{-\lambda z}\exp\{-\frac{1}{\mu}\kappa\cdot c(b)\cdot z\ \}\ dz - \exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\cdot{}_0\int^\infty e^{-\lambda z}\cdot\exp\{-\frac{c(b)\cdot z}{b\mu}\}dz\right]$$

$$= \frac{\lambda}{1 - \kappa\cdot b}\left[{}_0\int^\infty \exp\{-(\lambda + \frac{1}{\mu}\kappa\cdot c(b))\cdot z\ \}\ dz - \exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\cdot\ {}_0\int^\infty \exp\{-(\lambda + \frac{c(b)}{b\mu})\cdot z\}\ dz\right]$$

$$= \frac{\lambda}{1 - \kappa\cdot b}\left[(\lambda + \frac{1}{\mu}\kappa\cdot c(b))^{-1} - \exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\cdot\ (\lambda + \frac{c(b)}{b\mu})^{-1}\right]$$

$$= \frac{\lambda}{1 - \kappa\cdot b}\left[\frac{\mu}{\lambda\mu + \kappa\cdot c(b)} - \exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\cdot\ \frac{b\cdot\mu}{b\cdot\lambda\mu + c(b)}\right]$$

For the whole second term we now have:

$$(1-\kappa)\cdot\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot E[\mathbf{1}_{\{x+c(b)\cdot Z-h(b,Y)>0\}}\cdot\exp\{-\frac{1}{\mu}\kappa\cdot(c(b)\cdot Z-h(b,Y))\}]$$

$$=(1-\kappa)\cdot\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{\lambda}{1-\kappa\cdot b}\left[\frac{\mu}{\lambda\mu+\kappa\cdot c(b)}-\exp\{-(1-\kappa\cdot b)\cdot\frac{x}{b\mu}\}\cdot\frac{b\cdot\mu}{b\cdot\lambda\mu+c(b)}\right]$$

$$=\frac{\lambda\cdot(1-\kappa)}{1-\kappa\cdot b}\cdot\left[\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{\mu}{\lambda\mu+\kappa\cdot c(b)}-\exp\{-\frac{x}{b\mu}\}\cdot\frac{b\cdot\mu}{b\cdot\lambda\mu+c(b)}\right]$$

Altogether we finally obtain:

$$TJ(x,b)=\exp\{-\frac{x}{b\mu}\}\cdot\frac{b\cdot\lambda\mu}{b\cdot\lambda\mu+c(b)}+\frac{\lambda\cdot(1-\kappa)}{1-\kappa\cdot b}\cdot\left[\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{\mu}{\lambda\mu+\kappa\cdot c(b)}-\exp\{-\frac{x}{b\mu}\}\cdot\frac{b\cdot\mu}{b\cdot\lambda\mu+c(b)}\right]$$

$$=\exp\{-\frac{x}{b\mu}\}\cdot\left\{\frac{b\cdot\lambda\mu}{b\cdot\lambda\mu+c(b)}-\frac{\lambda\cdot(1-\kappa)}{1-\kappa\cdot b}\cdot\frac{b\cdot\mu}{b\cdot\lambda\mu+c(b)}\right\}+\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{\lambda\cdot(1-\kappa)}{1-\kappa\cdot b}\cdot\frac{\mu}{\lambda\mu+\kappa\cdot c(b)}$$

$$=\exp\{-\frac{x}{b\mu}\}\cdot\frac{b\cdot\lambda\mu}{b\cdot\lambda\mu+c(b)}\cdot\left\{1-\frac{1-\kappa}{1-\kappa\cdot b}\right\}+\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{1-\kappa}{1-\kappa\cdot b}\cdot\frac{\lambda\cdot\mu}{\lambda\mu+\kappa\cdot c(b)}$$

$$=\exp\{-\frac{x}{b\mu}\}\cdot\frac{b\cdot\lambda\mu}{b\cdot\lambda\mu+c(b)}\cdot\frac{1}{1-\kappa\cdot b}\left\{1-\kappa\cdot b-(1-\kappa)\right\}+\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{1-\kappa}{1-\kappa\cdot b}\cdot\frac{\lambda\cdot\mu}{\lambda\mu+\kappa\cdot c(b)}$$

$$=\exp\{-\frac{x}{b\mu}\}\cdot\frac{b}{b+1+\eta(b)}\cdot\frac{1}{1-\kappa\cdot b}\,\kappa\cdot(1-b)+(1-\kappa)\cdot\exp\{-\frac{1}{\mu}\kappa\cdot x\}\cdot\frac{1}{1-\kappa\cdot b}\cdot\frac{1}{1+\kappa\cdot(1+\eta(b))}\cdot\;\square$$

Optimal reinsurance under certain other principles is recently studied e.g. in :

Gajek, L. & Zagrodny, D. (2000). Insurer's optimal reinsurance strategies. Insurance: Mathematics and Economics **27**, 105 – 112.

Kaluszka, M. (2001). Optimal reinsurance under mean–variance premium principles. Insurance: Mathematics and Economics **28**, 61 –67.

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26. September 2002

Professor Angus Macdonald
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Dear colleague,

Thank you very much for the report on my paper

"On discrete–time dynamic programming in insurance:
exponential utility and minimizing the ruin probability"

which I submitted for possible publication in the Scandinavian Actuarial Journal.
Now I revised my paper along the lines suggested by the referee.
I enclose the revised paper and an answer to the referee.
Would you please be so kind to acknowledge receipt of the paper.
My Email address is: schael@uni–bonn.de

Thank you very much and best regards

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26. September 2002

Answer to the referee.

Thank you very much for your prompt and careful report and for your excellent advice. I rewrote the paper along the lines you suggested.

I explained the insurance model before I introduced the formal notation.
I added further remarks just behind the introduction of the structure assumption.
I added some remarks on the computation of the fixed point of T* at the end of §4 on p 14.
I assumed that c(b) and h(b,y) are increasing in b.
I mentioned that the R_n are iid after the Independence Assumption on p 4.
I explained the meaning of V_0 at the beginning of §4.
I also made the corrections of my misprints and some further explaining remarks.

$$E\left[\Phi(t - h(b,Y))\right] = \int_0^\infty \frac{1}{\mu} e^{-y/\mu} \Phi(t - h(b,y))\, dy$$
$$= \int_0^\infty \frac{1}{\mu} e^{-y/\mu} \mathbf{1}_{(0,\infty)}(t - h(b,y)) \cdot \left[1 - \Psi(0) \cdot \exp\{-\frac{1}{\mu}\left[1-\Psi(0)\right] \cdot (t - h(b,y))\}\right] dy$$