Control of ruin probabilities by discrete–time investments

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Dedicated to Ulrich Rieder on the occasion of his sixtieth birthday

Abstract. The control problem of controlling ruin probabilities by investments in a financial market is studied. The insurance business is described by the usual Cramer–Lundberg–type model and the risk driver of the financial market is a compound Poisson process. Conditions for investments to be profitable are derived by means of discrete–time dynamic programming. Moreover Lundberg bounds are established for the controlled model.

Key words. ruin probability, optimal investment, financial market, dynamic programming, Markov decision processes, optimal control
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1 Introduction

The control problem of minimizing ruin probabilities is studied in a Cramér– Lundberg model where the insurance company can invest the capital (surplus) in a financial market. The return process describing the financial market is driven by a (multi dimensional) Lévy process as in the Black–Scholes model. However, a compound Poisson process is chosen in place of a Wiener process. Indeed, a process with jumps seems to be more realistic than a process whose trajectories are continuous and have unbounded variation. The main advantage of the Black–Scholes model is the completeness of the financial market. But this property is not needed in the present control problem.

At first view, the ruin probability is not a classical performance criterion for control problems. However, one can write the ruin probability as some total cost in the embedded discrete–stage model where one has to pay one unit of cost when entering the ruin state. After this simple observation, results from discrete–time dynamic programming for minimizing costs apply but with some special features. In fact, applications to insurance will lead to some new situations not yet considered in the literature. These facts are summarized in section 3. In section 4, Lundberg inequalities are derived for the controlled model which extend the classical inequalities for the uncontrolled model (a recent paper is Cai & Dickson 2003).

In section 5 and 6 we study the problem when can the insurance company do better than keeping all the funds as risk reserve. More exactly, we look for a Howard improvement of the simple policy that prescribes not to invest at all. In section 5 the case of exponentially distributed claims is studied and it is shown that it is profitable to invest if and only if the expected return is positive.

The situation is different for Erlang distributed claims studied in section 6. Here, even in the case where the expected return is zero, i.e., where the price process is a martingale, investments do reduce the ruin probability in the situation where the insurer is poor.

In the appendix, the uncontrolled model is studied. The classical method for determining the ruin probability for exponentially distributed claims is extended to Erlang– (E_2–) distributed claims. In place of a first order ordinary differential equation we obtain a differential equation of second order.

The present paper is closely related to Gaier, Grandits & Schachermayer and Hipp & Schmidli (2003) studying models with continuous–time control and a Black–Scholes market model. Earlier papers are those of Hipp & Plum (2000) and Schmidli (2002). The papers Schäl (2003,2004) are similar to the present one, but study control by reinsurance.

2 The model

We consider a joint model of an insurance business and of a financial market. The real–valued discrete–time process $\{X_n, n\geq 0\}$ describes the *risk process* (surplus process, fund of reserves) immediately after time T_n. The time epochs $\{T_n\}$ result from the superposition of the claim times T'_n at the insurance company and the jump times T''_n at the financial market, $n \geq 1$. As in the Cramér–Lundberg model, the claim process is described by a compound Poisson process with claim size Y_n at time T'_n. We write N'_t for the number of claims in $(0,t]$ where $\{N'_t\}$ is a Poisson process with rate/intensity λ. There is a *premium (income) rate* c which is fixed. The insurance company can invest the capital (risk reserve) in a financial market. We consider a financial market where 1+d assets can be traded. One of them is called the bond and is described by the interest rate which here is assumed to be zero.

However, the case where the interest force of the bond is positive and equal to the inflation force can be treated in the same way since the ruin probability is the same if one replaces the surplus by the discounted surplus. From the theory of finance it is well known that considering the discounted wealth leads to a scenario which can be looked upon as market with interest rate zero. Moreover, the premium as well as the claim sizes then are interpreted as discounted quantities.

The other d assets are called stocks and are described by a d–dimensional price process $\{S_n = (S^1_n,\ldots,S^d_n), n \geq 1\}$ where S^k_n is the price of one share of stock k at time T''_n. More exactly, the return process will be driven by a d–dimensional compound Poisson process which can be defined by the sequence $\{T''_n, n\geq 1\}$ of jump times and the sequence of returns $R_n = (R^1_n,\ldots,R^d_n)$, $n\geq 1$, where

(2.1) $$S^k_n = S^k_{n-1} \cdot (1 + R^k_n).$$

For most of the present results we will assume d = 1.

We write N''_t for the number of jumps in $(0,t]$ of the return process where $\{N''_t\}$ is a Poisson process with rate ν and $1 + R^k_n > 0$ a.s. In general, ν will be much larger than λ. We define the Poisson process $\{N_t\}$ by superposition:

(2.2) $\quad N_t := N'_t + N''_t$ is the Poisson process with parameter $\lambda + \nu$ and jump times T_n, $n \geq 1$.

We write $K_n = 1$ if the jump at T_n is caused by the financial market and $K_n = 0$ if the jump is caused by a claim. Then we make the following asumption:

Model Assumption:
All random variables $Z_n := T_n - T_{n-1}$, Y_n, R_n, K_n, $n\geq 1$, *are independent. The* (Z_n) *are iid and have an exponential distribution with parameter* $\lambda+\nu$; *the* (Y_n) *are iid and positive; the* (R_n) *are iid with* $P[R_n < 0] > 0$ *and* $E[\|R_n\|^2] < \infty$; *the* (K_n) *are iid with*

(2.3) $$P[K_n = 1] = \frac{\nu}{\lambda+\nu} = 1 - P[K_n = 0].$$

We set $(Z_1,R_1,Y_1,K_1) = (Z,R,Y,K)$ for the generic elements.

Example. Assume $d = 1$ and that there is an underlying **Black–Scholes price process** $(\tilde{S}_t)$, i.e. S_n is the price $\tilde{S}_{T''_n}$ at time T''_n and $1 + R_n = \tilde{S}_{T''_n}/\tilde{S}_{T''_{n-1}}$ where

$$\tilde{S}_t = S_0 \cdot \exp\{\sigma \cdot W_t + a \cdot t\} \quad \text{and} \quad (W_t) \text{ is a standard Wiener process.}$$

Then $E[(1 + R)^m] = \nu \cdot \{\nu - m \cdot a - \frac{1}{2} \cdot m^2 \cdot \sigma^2\}^{-1}$, $m \geq 0$. $\square$

A dynamic portfolio specifies a portfolio vector $\theta_n \in \mathbb{R}^d$ at any time T_n. There the component θ^k_n of θ_n represents the amount of capital (value) which is invested in the k^{th} stock.
We have the following law of motion:

(2.4) $$X_{n+1} = X_n + c \cdot Z_{n+1} + \langle\theta_n, R_{n+1}\rangle \cdot K_{n+1} - Y_{n+1} \cdot (1 - K_{n+1}) \quad \text{for } X_n \geq 0,$$
$$X_{n+1} = -\infty \quad \text{for } X_n < 0,$$

where $\langle z,y\rangle$ denotes the inner product in $\mathbb{R}^d$.

We write $\Theta(x)$ for the set of all portfolio vectors θ *admissible* at x which is assumed to be

$$\Theta(x) = \{\theta = (\theta^1,\ldots,\theta^d) \in \mathbb{R}^d;\ \theta^k \geq 0,\ 1 \leq k \leq d,\ \textstyle\sum_1^d \theta^k \leq \alpha \cdot x + A\}, \quad x \geq 0.$$

There $\alpha \cdot x$ denotes a constant fraction of capital and A some extra amount where in most cases $0 \leq \alpha \leq 1$, $A = 0$. In the case $A > 0$ it is possible to borrow money. This case will be interesting for Lundberg bounds. We defined $\Theta(x) := \{0\}$ for $x < 0$.

Remark: One can replace $\alpha \cdot x + A$ by $\max(\alpha \cdot x, A)$ in the definition of $\Theta(x)$. $\square$

For $\theta \in \Theta(x)$, $\theta^0 := x - \Sigma_1^d \theta^k$ represents the amount of the capital which is invested in the bond, i.e., which is not invested in the stocks. In this model we do not allow for negative amounts θ_n^k, thus excluding short selling of the stock. We have

(2.5) $$X_n + <\theta_n, R_{n+1}> \;\geq\; X_n - \Sigma_1^d \theta_n^k \;\geq\; (1-\alpha)\cdot X_n - A \;\geq\; -A \quad \text{for } X_n \geq 0.$$

We choose the *state space* as $\mathbb{R}$ enlarged by the cemetery state $-\infty$. There $x < 0$ represents a state of ruin. A *plan* π is a sequence $\pi = (\varphi_n, n\geq 0)$ where φ_n is a measurable (decision) function such that $\varphi_n(x) \subset \Theta(x)$ for all x. Then $\varphi_n(X_n)$ specifies the portfolio vector $\theta_n \in \Theta(X_n)$ for the period $(T_n, T_{n+1}]$. A plan is stationary and we write $\pi = \varphi^\infty$ if $\varphi_n = \varphi$ for all n.

Given a plan π, the initial value x, and the sequence $Z_n(\omega), Y_n(\omega), R_n(\omega), K_n(\omega)$, $n \geq 1$, we can construct the state (risk) process $X_n(\omega)$ and we will sometimes write

(2.6) $$X_n = X_n^{x,\pi}.$$

Our performance criterion is the *ruin probability*:

(2.7) $$\psi^\pi(x) := P[X_n^{x,\pi} < 0 \text{ for some } n].$$

A policy π^* is called *optimal* if $\psi^{\pi^*}(x) = \inf_\pi \psi^\pi(x)$ for $x \geq 0$.

3 Dynamic programming

The one–period cost function g will be defined by $g(x) := \mathbf{1}_{(-\infty,0)}(x)$. Once in state $x \in (-\infty,0)$, the system moves to the absorbing state $-\infty$ in the next step. Thus the cost of 1 unit has to be paid at most once. Now we define

(3.1) $$\psi_n^\pi(x) := P[X_m^{x,\pi} \in (-\infty,0) \text{ for some } 0 \leq m < n], \; n \leq \infty,$$

in particular $\psi_n^\pi(x) = 1$ for $x \in (-\infty,0)$, $\psi_n^\pi(-\infty) = 0$,

which is just the probability of being ruined after n periods. Then we have

(3.2) $$\psi_n^\pi(x) := E[\Sigma_{0\leq m\leq n} g(X_m, \varphi_m(X_m))], \; n \leq \infty.$$

(3.3) $$\psi^\pi(x) := \psi_\infty^\pi(x).$$

Definition 3.4. *For any function* $v : \mathbb{R} \mapsto \mathbb{R}$ *and* $v(-\infty) := 0$ [*such the following expressions are well defined*] *set*:

$$Tv(x,\theta) = \frac{\nu}{\lambda+\nu} E[v(x + c\cdot Z + \theta\cdot R)] + \frac{\lambda}{\lambda+\nu} E[v(x + c\cdot Z - Y)], \; x \geq 0,$$
$$Tv(x,\theta) = g(x), \; x < 0,$$
$$T_\varphi v(x) := Tv(x,\varphi(x)),$$
$$T^*v(x) := \inf_{\theta\in\Theta(x)} Tv(x,\theta).$$

The following relation is obvious:

(3.5) $$(T_{\varphi_0}\ldots T_{\varphi_{n-1}}v)(x) = \psi_n^{\pi}(x) + E\lceil v(X_n^{x,\pi})\rceil \qquad \text{for } \pi = (\varphi_0,\varphi_1,\ldots).$$

Lemma 3.6. *For* $\pi = (\varphi_0,\varphi_1,\ldots)$ *one has*:

(a) $\psi_{n+1}^{\pi} = T_{\varphi_0}\ldots T_{\varphi_{n-1}} g = T_{\varphi_0}\ldots T_{\varphi_n} 0$,

(b) $0 \le g = \psi_0^{\pi} \le \psi_n^{\pi} \le \psi_{n+1}^{\pi} \uparrow \psi^{\pi}$.

*Proof***.** Part (a) is obvious where $g = T_{\varphi}0$.
(b) We have $T_{\varphi}g \ge g = T_{\varphi}0 \ge 0$ for all φ and thus

$\psi_{n+1}^{\pi} = T_{\varphi_0}\ldots T_{\varphi_{n-1}}(T_{\varphi_n}0) \ge T_{\varphi_0}\ldots T_{\varphi_{n-1}}0 = \psi_n^{\pi}$. []

From Lemma 3.6 and the monotone convergence theorem we obtain for stationary plans $\pi = \varphi^{\infty}$:

(3.7) $$T_{\varphi}\psi^{\varphi^\infty} = \psi^{\varphi^\infty} = \lim_{n\to\infty} T_{\varphi}^{n}0 .$$

Proposition 3.8. *Let* $v : \lceil-\infty,\infty) \mapsto \lceil 0,\infty)$ *be some measurable function*.

(a) *If* φ *is some decision function with* $T_{\varphi}v \le v$, *then we have*: $\psi^{\varphi^\infty} \le T_{\varphi}v \le v$.

(b) *If* $T^*v(x) \ge v(x)$ *then* $\psi^{\pi} \ge v$ *for all plans* π *provided that*

$$E\lceil v(X_n^{x,\pi}) - g(X_n^{x,\pi})\rceil \to 0\ (n\to\infty) \quad \textit{for all } x \ge 0 \textit{ and all plans } \pi.$$

In order to understand the condition in 3.8b it is useful to look on g also as a terminal cost function (see 3.6a)

Proof. a) By induction we obtain $T_{\varphi}^{n}v \le T_{\varphi}v \le v$ for all n. Thus we get:

$$\psi^{\varphi^\infty} = \lim_{n\to\infty} T_{\varphi}^{n}0 \le \lim_{n\to\infty} T_{\varphi}^{n}v \le T_{\varphi}v \le v.$$

b) We obtain $v \ge T_{\varphi}v$ for all decision functions φ and thus by Lemma 3.6

$$v \le T_{\varphi_0}\ldots T_{\varphi_{n-1}}v = T_{\varphi_0}\ldots T_{\varphi_{n-1}}g + E\lceil v(X_n) - g(X_n)\rceil \to \psi^{\pi}. \ []$$

Theorem 3.9 (*Howard Improvement*). *Let* φ, $\tilde\varphi$ *be any decision functions,*

set $\psi := \psi^{\varphi^\infty}$, $\tilde\psi := \psi^{\tilde\varphi^\infty}$ *and* $\Theta(x,\varphi) := \{\ \theta \in \Theta(x);\ T\psi(x,\theta) < \psi(x)\ \}$.

If, for some subset S *of* $\lceil 0,\infty)$, $\tilde\varphi(x) \in \Theta(x,\varphi)$, $x \in S$, *and* $\tilde\varphi(x) = \varphi(x)$, $x \notin S$, *then one has*:

$$\tilde\psi \le \psi \quad \text{and} \quad \tilde\psi(x) \le T_{\tilde\varphi}\psi(x) < \psi(x) \ \text{ for } x \in S.$$

In the situation of Theorem 3.9 with $S \neq \emptyset$, $\tilde\varphi$ is called a **Howard improvement** of φ, and φ^* is an **optimal Howard improvement** of φ if in addition the upper bound $T_{\varphi^*}\psi$ is minimal, that is if

(3.10) $\quad T_{\varphi^*}\psi(x) = T^*\psi(x)$, i.e., $\quad T\psi(x,\varphi^*(x)) = \inf_{\theta\in\Theta(x)} T\psi(x,\theta)$.

Proof of 3.9. We certainly have $T_{\tilde\varphi}\psi \leq \psi$; in fact we have by (3.7)

$$T_{\tilde\varphi}\psi(x) < \psi(x) \text{ if } x \in S \text{ and } \quad T_{\tilde\varphi}\psi(x) = T_{\varphi}\psi(x) = \psi(x) \text{ if } x \notin S.$$

From Proposition 3.8a we then obtain $\tilde\psi \leq T_{\tilde\varphi}\psi \leq \psi$, in particular $\tilde\psi(x) \leq T_{\tilde\varphi}\psi < \psi$ if $x \in S$. []

Now one can ask what happens in the situation where $\Theta(x,\varphi) := \{\, \theta \in \Theta(x);\ T\psi(x,u) < \psi(x) \,\}$ is empty for all x, i.e. $T^*\psi \geq \psi$. Since we always have $T^*\psi \leq T_{\varphi}\psi = \psi$, this means that $T^*\psi = T_{\varphi}\psi = \psi$. The next theorem gives an answer.

Theorem 3.11 (*Verification theorem*). *Let* $v : S \mapsto [0,\infty]$ *be a measurable function and* φ *some decision function with* $v = T^*v = T_{\varphi}v$. *Then we have*

$v(x) = \inf_{\pi} \psi^{\pi}(x) \ \forall\, x$ *and* φ *defines a stationary optimal plan* φ^{∞} *provided that*

$$E\left[v(X_n^{x,\pi}) - g(X_n^{x,\pi})\right] \to 0 \quad \textit{as } n\to\infty \quad \textit{for all plans } \pi \textit{ and for all } x \geq 0.$$

The proof follows from Proposition 3.8.

Lemma 3.12. *There is some* $\varepsilon > 0$ *such that* $\inf_{\theta\in\mathbb{R}^n} P\left[\, c\cdot Z + \langle\theta,R\rangle\cdot K - Y\cdot(1-K) < -\varepsilon\right] > 0$.

Proof. $P\left[c\cdot Z + \langle\theta,R\rangle\cdot K - Y\cdot(1-K) < -\varepsilon\right] \geq P\left[\, K=0,\ c\cdot Z - Y < -\varepsilon\right]$

$= P\left[\, K=0\right]\cdot P\left[c\cdot Z - Y < -\varepsilon\right] = P\left[\, K=0\right]\cdot \int_0^{\infty} \lambda\cdot e^{-\lambda z}\cdot P\left[Y > cz+\varepsilon\right] dz > 0$

for some $\varepsilon > 0$ since $P\left[Y > 0\right] > 0$. []

Proposition 3.13 . *Let* $M > 0$ *be arbitrary*.

(a) *There exists some* $n \in \mathbb{N}$ *such that* $\eta_{n,M} := \sup_{x\leq M,\pi} P\left[X_n^{x,\pi} \geq 0\right] < 1$.

(b) $P\left[0 \leq X_m^{x,\pi} \leq M \text{ for infinitely many } m\right] = 0$ *for all* x,π.

The proof is the same as the proof of Proposition 2 in Schäl 2004. From Proposition 3.13 we conclude that

(3.14) $\quad \lim_{n\to\infty} P\left[0 \leq X_n^{x,\pi} \leq M\right] = 0 \quad \forall\, x,\pi,\ M > 0.$

From (3.14) we immediately obtain a kind of contraction property (see Schäl 2004, Lemma 8).

Lemma 3.15. *If* $\xi : [-\infty,\infty) \mapsto [0,\infty)$ *is any bounded measurable function such that* $\xi(\infty) := \lim_{x\to\infty} \xi(x) = 0$ *and* $\xi(x) = 0$ *for* $x \le 0$, *then* $\lim_{n\to\infty} E[\xi(X_n^{x,\pi})] = 0 \quad \forall\, x,\pi$.

For an application of Theorems 3.9, 3.11, we choose $\xi = v - g$ in Lemma 3.15 and $v = \psi^o$ as the ruin probability under the stationary plan φ^∞ which describes the situation where the decision maker does not invest in stocks at all, i.e.,

(3.16) $\qquad \varphi(x) = 0$.

Thus we want to know how to improve φ^∞ and under what conditions φ^∞ is an optimal policy, i.e. can the insurance company do better than keeping all the funds as risk reserve.

The following first two properties are well–known. In fact, the condition $\xi(\infty) = \psi^o(\infty) = 0$ just means that the ruin probability tends to zero as the initial surplus tends to infinity.
Let us set for the classical ruin probability starting in 0:

(3.17) $\qquad q := \dfrac{\lambda \cdot E[Y]}{c} = \psi^o(0)$.

Lemma 3.18. *Assume that* $q < 1$. *In the situation of* (3.16) *we have for* $\psi^o := \psi^{\varphi^\infty}$:

(a) $\quad \psi^o(\infty) := \lim_{x\to\infty} \psi^o(x) = 0$.

(b) $\quad \psi^o(x) - g(x) = 0$ *for* $x \le 0$ *and* $\lim_{x\to\infty} \psi^o(x) - g(x) = 0$.

Proof. (a) follows from the law of large numbers (see Grandell 1991 p.5) and (b) is obvious. $\square$
Recall that $T_\varphi \psi^o = \psi^o$ by (3.7).

Corollary 3.19. *Assume that* $q < 1$. *Let* φ^∞ *be as in* (3.16) *and set* $\psi^o := \psi^{\varphi^\infty}$.

(a) (*Howard improvement*) *For each decision function* φ^* with $T\psi^o(x,\varphi^*(x)) = T^*\psi^o(x)$

one has $\psi^{\varphi^{*\infty}} \le T^*\psi(x) \le \psi(x)$.

(b) (*Verification theorem*). *If* $\psi^o = T^*\psi^o$, *i.e.*, $T_\varphi \psi^o = T^*\psi^o$, *then* φ^∞ *is an optimal plan.*

4 Lundberg inequalities

In this section we assume $d = 1$ as well as:

Classical Assumption for $m(t) = E[e^{tY}]$**:** *there is some* $r_\infty \in (0,\infty]$ *such that* $m(t) < \infty$ *for* $t < r_\infty$ *and* $m(t) \to \infty$ *for* $t \uparrow r_\infty$ (*the so-called small claims case*).

Notation: $\ell(t) := E[e^{-tR}]$; $\rho := \inf_{t\geq 0} \ell(t)$.

From $R \geq -1$ and $P[R < 0] > 0$ we get:

Lemma 4.1. (a) $\ell(t) < \infty$ for $t < \infty$, $\ell(t) \to \infty$ for $t \to \infty$.

(b) $\ell'(t) = -E[Re^{-tR}]$, $\ell'(0) = -E[R]$, $\ell''(t) = E[R^2e^{-tR}] > 0$.

(c) *There is a unique minimum point* $0 \leq t_o < \infty$ *such that* ℓ *is strictly decreasing in* $[0,t_o]$ *and strictly increasing in* $[t_o,\infty)$ *and* $\ell(t_o) = \rho \leq 1$.

(d) *If* $E[R] > 0$, *then* $t_o > 0$ *and* $\rho < 1$.
If $E[R] \leq 0$, *then* $t_o = 0$ *and* $\rho = 1$.

Let $\hat{r}$ be the positive solution to the equation:

$$\frac{\nu}{\lambda+\nu}\rho + \frac{\lambda}{\lambda+\nu} m(r) = 1 + r\frac{c}{\lambda+\nu} \quad . \tag{4.2}$$

For $\rho = 1$ or $\nu = 0$, this the *classical Lundberg equation*: $m(r) = 1 + r\frac{c}{\lambda}$.

Lemma 4.3. *If* $E[R] > 0$, *then* (4.2) *has a positive solution* $\hat{r}$.

Proof. Set $\chi(r) := \frac{\nu}{\lambda+\nu}\rho + \frac{\lambda}{\lambda+\nu}\cdot m(r)$; $L(r) := 1 + r\frac{c}{\lambda+\nu}$.
Then $\chi(0) < 1 = L(0)$ since $\rho < 1$. Moreover we have as in the classical case:
$\chi(r_\infty) = \infty$ for $r_\infty < \infty$ and $\chi'(r_\infty) = \infty$ if $r_\infty = \infty$. Thus there exists always a solution. []

Proposition 4.4 . *Suppose that* $E[R] > 0$ *and* $q < 1$. *If* r_o *is the classical Lundberg adjustment coefficient, i.e.,* r_o *is the positive solution to* $m(r) - 1 = \frac{c}{\lambda} r$, *then one has* $\hat{r} > r_o$.

The proof is given in 4.9.

Definition 4.5. $A^* := t_o/\hat{r}$ *where* $A^* := 0$ *and* $\hat{r} := r_o$ *for* $E[R] \leq 0$;
$\varphi(x) := A^*$ for $x \geq 0$.

Since $E[e^{-\hat{r}cZ}] = \dfrac{\lambda+\nu}{\lambda+\nu+\hat{r}c}$ we obtain from (4.2):

$$(4.6)\qquad \left[\frac{\nu}{\lambda+\nu}\,E\left[e^{-\hat{r}\varphi(x)\cdot R}\right] + \frac{\lambda}{\lambda+\nu}\,E\left[e^{\hat{r}Y}\right]\right]\cdot E\left[e^{-\hat{r}cZ}\right] = 1.$$

For an upper bound for ψ we will prove the following result.

Proposition 4.7. $T_\varphi v \le v$ *for* $v(x) := \min(1, e^{-\hat{r}x}) = \mathbf{1}_{(-\infty,0)}(x) + \mathbf{1}_{[0,\infty)}(x)\cdot e^{-\hat{r}x}$.

Proof. Set $\hat{r} = r$ and let be $x \ge 0$. Then we obtain from 3.4

$$\begin{aligned}
Tv(x,\theta) &= \frac{\nu}{\lambda+\nu}\,P\left[x + c\cdot Z + \theta\cdot R < 0\right] + \frac{\lambda}{\lambda+\nu}\,P\left[x + c\cdot Z - Y < 0\right]\\
&\quad + \frac{\nu}{\lambda+\nu}\,E\left[\mathbf{1}_{[0,\infty)}(x + c\cdot Z + \theta\cdot R)\,)\cdot \exp\{-r\cdot(x + c\cdot Z + \theta\cdot R)\right]\\
&\quad + \frac{\lambda}{\lambda+\nu}\,E\left[\mathbf{1}_{[0,\infty)}(x + c\cdot Z - Y)\cdot \exp\{-r\cdot(x + c\cdot Z - Y)\right]\\
&\le \frac{\nu}{\lambda+\nu}\,E\left[\mathbf{1}_{(-\infty,0)}(x + c\cdot Z + \theta\cdot R\,)\cdot \exp\{-r\cdot(x + c\cdot Z + \theta\cdot R)\right]\\
&\quad + \frac{\lambda}{\lambda+\nu}\,E\left[\mathbf{1}_{(-\infty,0)}(x + c\cdot Z - Y)\cdot \exp\{-r\cdot(x + c\cdot Z - Y)\}\right]\\
&\quad + \frac{\nu}{\lambda+\nu}\,E\left[\mathbf{1}_{[0,\infty)}(x + c\cdot Z + \theta\cdot R)\cdot \exp\{-r\cdot(x + c\cdot Z + \theta\cdot R)\}\right]\\
&\quad + \frac{\lambda}{\lambda+\nu}\,E\left[\mathbf{1}_{[0,\infty)}(x + c\cdot Z - Y)\cdot \exp\{-r\cdot(x + c\cdot Z - Y)\}\right]\\
&= \frac{\nu}{\lambda+\nu}\,E\left[\exp\{-r\cdot(x + c\cdot Z + \theta\cdot R)\right] + \frac{\lambda}{\lambda+\nu}\,E\left[\exp\{-r\cdot(x + c\cdot Z - Y)\}\right]\\
&= e^{-rx}\cdot\left[\frac{\nu}{\lambda+\nu}\,E\left[e^{-r\,\theta\cdot R}\right] + \frac{\lambda}{\lambda+\nu}\cdot E\left[e^{rY}\right]\right]\cdot E\left[e^{-rcZ}\right],
\end{aligned}$$

hence by (4.6)

$$T_\varphi v(x) \le e^{-rx}\cdot\left[\frac{\nu}{\lambda+\nu}\,\rho + \frac{\lambda}{\lambda+\nu}\cdot m(r)\right]\cdot\frac{\lambda+\nu}{\lambda+\nu+rc} = e^{-rx} = v(x).$$

For $x < 0$ we have $Tv(x,\theta) = g(x) = g(x) = v(x)$. []

The decision function φ above is **admissible**, i.e. $\varphi(x) \subset \Theta(x)\ \forall\ x$, if $A \ge A^*$.
Now we conclude from Propositions 3.8 and 4.7:

Theorem 4.8 . *If* $A \ge A^*$ *and* v *is as in* 4.7, *then* φ *is admissible and* $\psi^{\varphi^\infty}(x) \le Tv(x,A^*) \le e^{-\hat{r}x}$.

Here one has the important case that the decision maker holds a constant amount of the risky asset independent of the current level of capital (under φ). It was shown by Paulsen & Gjessing (1997),(1998) and Frovola, Kabanov & Pergamenshchikov (2002) that the asymptotic behaviour of the ruin probability is completely different if the decision maker holds a constant fraction of capital (under φ). In the latter case the ruin probability decreases only with some negative power of the initial reserve

Remark 4.9. Assume 0<A<A*, i.e. $\varphi \equiv A^*$ is not admissible, and $q < 1$. Then $E[R] > 0$ and $\rho < 1$.

(i) Define χ and L as in the proof of 4.3 and $f(r) := \frac{\nu}{\lambda+\nu}\,\ell(rA) + \frac{\lambda}{\lambda+\nu}\cdot m(r)$.

Then there is a positive solution $\tilde{r}$ to $f(r) = L(r)$ such that $r_o < \tilde{r} < \hat{r}$.

For a proof set $\tilde{\chi}(r) = \frac{\nu}{\lambda+\nu} + \frac{\lambda}{\lambda+\nu}\cdot m(r)$. Then for $0 < r \le \hat{r}$ we have:

$0 < rA < rA^* \le \hat{r}A^* = t_o$ and thus $1 > \ell(tA) > \ell(t_o) = \rho$ by Lemma 4.1.

Then we obtain $\tilde{\chi}(r) > f(r) > \chi(r)$ for $0 < r \le \hat{r}$ and $L(0) = \tilde{\chi}(0) = f(0) = 1 > \chi(0)$. From the classical case we know that $L(r) \ge \tilde{\chi}(r)$ for $0 \le r \le r_o$ and from Lemma 4.3 that $\chi(\hat{r}) = L(\hat{r})$.

(ii) Now set $\tilde{\varphi}(x) = A$ für $x \ge 0$. Then as in the proof of Proposition 4.7 we have

$$\left[\frac{\nu}{\lambda+\nu} E\left[e^{-\tilde{r}\tilde{\varphi}(x)\cdot R}\right] + \frac{\lambda}{\lambda+\nu} E\left[e^{\tilde{r}Y}\right]\right]\cdot E\left[e^{-\tilde{r}cZ}\right] = 1$$

Again as above, it can then be shown that

(4.10) $$\psi^{\tilde{\varphi}^\infty}(x) \le e^{-\tilde{r}x} \text{ where } r_o < \tilde{r} .$$

Thus it the present situation the plan $\tilde{\varphi}^\infty$ has a better Lundberg bound that the plan that does not invest at all. $\square$

For a lower bound for ψ we start with the following lemma.

Lemma 4.11. *Let* $x \ge 0$, $\theta \in \mathbb{R}$, $\varepsilon \ge 0$ *be given such that*

(4.12) $$\frac{\nu}{\lambda+\nu} P[x + c\cdot Z + \theta\cdot R < 0] + \frac{\lambda}{\lambda+\nu} P[x + c\cdot Z - Y < 0]$$
$$\ge \varepsilon\cdot\Big[\frac{\nu}{\lambda+\nu} E[\mathbf{1}_{(-\infty,0)}(x + c\cdot Z + \theta\cdot R)\cdot \exp\{-r\cdot(x + c\cdot Z + \theta\cdot R)]$$
$$+ \frac{\lambda}{\lambda+\nu} E[\mathbf{1}_{(-\infty,0)}(x + c\cdot Z - Y)\cdot \exp\{-r\cdot(x + c\cdot Z - Y)\}]\Big].$$

Then one has $Tv(x,\theta) \ge v(x)$ *for* $v(z) := \mathbf{1}_{(-\infty,0)}(z) + \varepsilon\cdot\mathbf{1}_{[0,\infty)}(z)\cdot e^{-\hat{r}z}$.

The proof is similar to the proof of Proposition 4.7. We set

(4.13) $$C^{-1} := \sup_{y\ge 0} E\left[e^{\hat{r}(Y-y)} \mid Y > y\right]$$

where $C \le 1$. As in Gaier, Grandits & Schachermayer 2003 one can say: Y has a uniform exponential moment in the tail distribution for $\hat{r}$ if $C > 0$.

Theorem 4.14. For C as in (4.13) one has

$$\psi^\pi(x) \ge \min(C, e^{-\hat{r}A})\cdot e^{-\hat{r}x} \text{ for all plans } \pi.$$

If $A = 0$ we have $\min(C, e^{-\hat{r}A}) = C$ since $C \le 1$. The theorem is also interesting for $E[R] \le 0$ where $\hat{r} = r_o$. For $A > 0$, the constant $\min(C, e^{-rA})$ is worse than in the case of the Black–Scholes model in Gaier, Grandits & Schachermayer (2003). The reason is that for $A > 0$ we here have some overshooting of the boundary 0 caused by investment in the stock.

Proof. Set $r = \hat{r}$ and let $x \ge 0$ and θ be given such that $0 \le \theta \le \alpha \cdot x + A$. Then

$$0 \le \cdot \theta \cdot (1+R) = \theta + \theta \cdot R \le \alpha \cdot x + A + \theta \cdot R \le x + \theta \cdot R + A.$$

Set $\eta := e^{-rA}$. Now we obtain $1 \le e^{r \cdot (x + \theta R + A)}$, i.e.,

$\eta \cdot e^{-r \cdot (x + \theta R)} \le 1$ for $x \ge 0$, which implies $\eta \cdot e^{-r \cdot (x + cZ + \theta R)} \le 1$.

From this relation we get

$$\eta \cdot E\left[1_{\{x + cZ + \theta R < 0\}} \cdot e^{-r \cdot (x + cZ + \theta R)}\right] \le P\left[x + cZ + \theta \cdot R < 0\right].$$

Now we want to prove (4.12) for $\varepsilon := \eta \wedge C$:

$$\begin{aligned}
&\frac{\nu}{\lambda+\nu} P\left[x + c \cdot Z + \theta \cdot R < 0\right] + \frac{\lambda}{\lambda+\nu} P\left[x + c \cdot Z - Y < 0\right] \\
&\ge \frac{\nu}{\lambda+\nu} \cdot \eta \cdot E\left[1_{\{x + cZ + \theta R < 0\}} \cdot e^{-r \cdot (x + cZ + \theta R)}\right] \\
&+ \frac{\lambda}{\lambda+\nu} \cdot C \cdot E\left[\mathbf{1}_{(-\infty,0)}(x + c \cdot Z - Y) \cdot \exp\{-r \cdot (x + c \cdot Z - Y)\}\right] \\
&\ge \varepsilon \cdot \Big[\frac{\nu}{\lambda+\nu} E\left[\mathbf{1}_{(-\infty,0)}(x + c \cdot Z + \theta \cdot R) \cdot \exp\{-r \cdot (x + c \cdot Z + \theta \cdot R)\right] \\
&\qquad + \frac{\lambda}{\lambda+\nu} E\left[\mathbf{1}_{(-\infty,0)}(x + c \cdot Z - Y) \cdot \exp\{-r \cdot (x + c \cdot Z - Y)\}\right]\Big] .
\end{aligned}$$

We set $v(z) := \mathbf{1}_{(-\infty,0)}(z) + \varepsilon \cdot \mathbf{1}_{[0,\infty)}(z) \cdot e^{-rz}$, then we know from Lemma 4.11 that

$$Tv(x,\theta) \ge v(x) \quad \text{for all } \theta \in \Theta(x), \text{ i.e., } T^*v \ge v.$$

For $x < 0$ we have $Tv(x,\theta) = \mathbf{1}_{(-\infty,0)}(x) \ge v(x)$. Now Propositon 3.8 applies since $E\left[v(X_n) - g(X_n)\right] = \varepsilon \cdot E\left[\mathbf{1}_{[0,\infty)}(X_n) \cdot e^{-rX_n}\right] \to 0$ by Lemma 3.15. []

Corollary 4.15. *For* $A \ge A^*$ *one has*:

$$\min(C, e^{-\hat{r}A}) \cdot e^{-\hat{r}x} \le \inf_\pi \psi^\pi(x) \le e^{-\hat{r}x}. \quad , x \ge 0.$$

5 Exponentially distributed claims

Assumption: $0 < \alpha \le 1$, $A = 0$; $Y \sim E_1$, *i.e.*, Y *is exponentially distributed, where for convenience and w.l.o.g. we assume* $E[Y] = 1$.

For the classical ruin probability we now write $\psi = \psi^o$ and for the classical Lundberg coefficient we now write $\delta = r_o$. Then the following relations are well known:

(5.1) $$\psi(x) = q \cdot e^{-\delta \cdot x} \quad \text{for } x \ge 0 \text{ with } q = 1 - \delta = \frac{\lambda}{c} > 0,$$
$$\psi(x) = 1 \quad \text{for } -\infty < x < 0.$$

This is the ruin probability when the insurer does not invest at all. We want to compute $Tv(x,\theta)$ for some arbitrary v. For $x \ge 0$ we have $x + c \cdot Z + <\theta,R> \ge 0$ in view of (2.5).
Now from 3.4 we get

(5.2) $$Tv(x,\theta) = \nu \int_0^\infty E[v(x + c \cdot z + <\theta,R>))]\, e^{-(\lambda+\nu)z}\, dz + \lambda \int_0^\infty E[v(x + c \cdot z - Y)]\, e^{-(\lambda+\nu)z}\, dz.$$

We want to study $T\psi(x,\theta)$ for $x \ge 0$. Then the second term in (5.2) does not depend on θ and it is easy to show that

(5.3) $$E[\psi(x - Y)] = e^{-\delta \cdot x}.$$

Furthermore we have

$$I(x,\theta) := \int_0^\infty E[\psi(x + c \cdot z + <\theta,R>)]\, e^{-(\lambda+\nu)z}\, dz$$
$$= (1-\delta) \cdot E[\exp\{-\delta \cdot (x + <\theta,R>\})] \int_0^\infty e^{-(\delta \cdot c + \lambda + \nu) \cdot z}\, dz$$
$$= \frac{1 - \delta}{\delta c + \lambda + \nu} \cdot E[\exp\{-\delta \cdot (x + <\theta,R>)\}].$$

Thus we obtain from (5.3) and from $\delta c + \lambda = c$:

$$T\psi(x,\theta) = \nu \cdot I(x,\theta) + \lambda \cdot \int_0^\infty E[\psi(x + c \cdot z - Y)]\, e^{-(\lambda+\nu)z}\, dz$$
$$= \nu \cdot \frac{1 - \delta}{\delta c + \lambda + \nu} \cdot E[\exp\{-\delta \cdot (x + <\theta,R>)\}] + \lambda \cdot \int_0^\infty e^{-\delta(x+cz)}\, e^{-(\lambda+\nu)z}\, dz$$
$$= \frac{1}{c + \nu} \cdot (1-\delta) \cdot e^{-\delta x} \cdot \left[\nu \cdot E[e^{-\delta \cdot <\theta,R>}] + \frac{\lambda}{1-\delta}\right] = \frac{1}{\nu + c} \cdot \psi(x) \cdot \left[\nu \cdot E[e^{-\delta \cdot <\theta,R>}] + c\right],$$

hence

(5.4) $$T\psi(x,\theta) = \frac{1}{\nu + c} \cdot \left[\nu \cdot E[e^{-\delta \cdot <\theta,R>}] + c\right] \cdot \psi(x).$$

Then $T\psi(x,0) = \psi(x)$ as expected. Now we consider a special case:

Assumption: $d = 1$, hence $<\theta,R> = \theta \cdot R$.

Then

(5.5) $$T\psi(x,\theta) = \frac{1}{\nu + c} \cdot \left[\nu \cdot \ell(\delta \cdot \theta) + c\right] \cdot \psi(x) .$$

From Lemma 4.1 we know that $\ell\,'$ is strictly increasing, ℓ is strictly convex with $\ell(\infty) = \infty$.

On the one hand for $E[R] \le 0$ we have: $\ell\,'(t) > \ell\,'(0) \ge 0,\ t > 0$. Thus:

$\theta \mapsto \ell(\kappa \cdot \theta)$ is increasing and attains the minimum for $\theta=0$. This minimum is unique.

On the other side for $E[R] > 0$ we have:

$\ell\,'(0) < 0$, ℓ has a unique minimum point $0 < t_o < \infty$ and
a unique point $0 < t_1 < \infty$ with $\ell(t_1) = \ell(0) = 1$.

Thus:

$\theta = 0$ is **never** a minimum point for $x > 0$; more exactly

$$\inf_{0\le\theta\le\alpha x} \ell(\delta \cdot \theta) = \ell(\delta \cdot (\alpha x \wedge \tfrac{1}{\delta} t_o)), \quad \ell(\delta \cdot \theta) \le \ell(0) \quad \text{for } 0 \le \theta \le \alpha x \wedge \tfrac{1}{\delta} t_1.$$

Theorem 5.6. (a) *If* $E[R] \le 0$ *then it is optimal not to invest.*
(b) *If* $E[R] > 0$ *there exists an optimal Howard improvement* φ^* *of* φ *such that*

$$\varphi^*(x) = \alpha \cdot x \wedge \frac{1}{\delta} \cdot t_o \quad \textit{where } \ell(t_o) = \rho \text{ with } 0 < t_o < \infty.$$

(c) *If* $E[R] > 0$ *then* $\tilde\varphi$ *is a Howard improvement of* φ *if and only if*

$$\tilde\varphi \ne 0 \;\; \textit{and} \;\; 0 \le \tilde\varphi(x) \le \alpha \cdot x \wedge \frac{1}{\delta} \cdot t_1 \;\textit{where } \ell(t_1) = 1 \textit{ with } 0 < t_1 < \infty.$$

As a corollary one obtains that for any Howard improvement $\tilde\varphi$ of φ one has

$$\tilde\psi(x) \le \psi(x) = q \cdot e^{-\delta \cdot x} \quad \text{where} \quad \tilde\psi = \psi^{\tilde\varphi^\infty} .$$

Thus investment according to $\tilde\varphi$ is not dangerous in the sense of Frovola, Kabanov & Pergamenshchikov (2002).

6 Erlang distributed claims

Assumption: $0 < \alpha \le 1$, $A = 0$; $Y \sim E_2$ *i.e.* Y *is Erlang distributed,*
where for convenience and w.l.o.g. $E[Y] = 2$.

For the classical ruin probability $\psi = \psi^o$ we have (see Appendix):

(6.1) $$\psi(x) = \Delta \cdot e^{-\delta \cdot x} - \Gamma \cdot e^{-\gamma \cdot x} \text{ for } x \ge 0 \text{ with } \Gamma > 0,\ \Delta > 0,\ 0 < \delta < 1 < \gamma,$$
$$\psi(x) = 1 \quad \text{for } -\infty < x < 0.$$

This is the ruin probability when the insurer does not invest at all. There δ is again the classical Lundberg coefficient r_o and γ is the virtual Lundberg coefficient, i.e. δ and γ are the non–zero solution to the Lundberg equation:

$m(r) = E[e^{rY}] = 1 + \frac{c}{\lambda}\cdot r \;\Leftrightarrow\; (\frac{1}{1-r})^2 = 1 + \frac{c}{\lambda}\cdot r \;\Leftrightarrow\; r = 0 \text{ or } r^2 + (\frac{1}{2}q - 2)\cdot r + 1 - q = 0.$

For special values, formula (6.1) is given by Dickson & Hipp (1998) and proved by the theory of phase–type distributions. The following properties will be proved in the Appendix.

Lemma 6.2 .

(a) $\Delta\delta > \Gamma\gamma$ *where* $\Delta\delta - \Gamma\gamma = \frac{1}{2}q\cdot(1-q)$

(b) $\Delta\cdot\delta^2 < \Gamma\cdot\gamma^2$ *where* $\Delta\cdot\delta^2 - \Gamma\cdot\gamma^2 = -\frac{1}{4}q^2\cdot(1-q)$.

(c) $\gamma\delta = 1-q$.

We again set

(6.3) $$I(x,\theta) := \int_0^\infty E[\psi(x + c\cdot z + <\theta,R>))]\, e^{-(\lambda+\nu)z}\, dz.$$

Then we have by (5.2)

(6.4) $$T\psi(x,\theta) = \nu\cdot I(x,\theta) + \lambda \int_0^\infty E[\psi(x + c\cdot z - Y)]\, e^{-(\lambda+\nu)z}\, dz$$

where the last term is independent of θ.

Lemma 6.5. $I(x,\theta) = \frac{\Delta}{\delta c+\lambda+\nu}\, e^{-\delta x}\cdot E[\exp\{-\delta\cdot<\theta,R>\}] - \frac{\Gamma}{\gamma c+\lambda+\nu}\, e^{-\gamma x}\cdot E[\exp\{-\gamma\cdot<\theta,R>\}]$

$= \frac{\Delta}{\delta c+\lambda+\nu}\cdot E[\exp\{-\delta\cdot(x + <\theta,R>)\}] - \frac{\Gamma}{\gamma c+\lambda+\nu}\, e^{-\gamma x}\cdot E[\exp\{-\gamma\cdot(x + <\theta,R>)\}]$

where $x + <\theta,R> \geq 0$ *for* $\theta \leq \alpha\cdot x$.

Proof. For k > 0 we have: $\int_0^\infty E[\exp\{-k\cdot(x + c\cdot z + <\theta,R>))\}]\, e^{-(\lambda+\nu)z}\, dz$

$= E[\exp\{-k\cdot(x + <\theta,R>)\}]\cdot\int_0^\infty e^{-(k\cdot c\cdot+\lambda+\nu)z}\, dz = \frac{1}{kc+\lambda+\nu}\, e^{-kx}\cdot E[\exp\{-k\cdot<\theta,R>\}]$. []

Again we now consider a special case:

Assumption. d = 1.

Now we can compute

$\frac{\partial}{\partial\theta} I(x,\theta) = \frac{\Delta\delta}{\delta c+\lambda+\nu}\, e^{-\delta x}\cdot E[(-R)\cdot e^{-\delta\theta R}] - \frac{\Gamma\gamma}{\gamma c+\lambda+\nu}\, e^{-\gamma x}\cdot E[(-R)\cdot e^{-\gamma\theta R}]$,

$\frac{\partial}{\partial\theta} I(x,0) = E[(-R)]\cdot\left[\frac{\Delta\delta}{\delta c+\lambda+\nu}\, e^{-\delta x} - \frac{\Gamma\gamma}{\gamma c+\lambda+\nu}\, e^{-\gamma x}\right]$.

Since by Lemma 6.2 $\Delta\delta > \Gamma\gamma$ and $\delta < \gamma$ we know that : $\frac{\Delta\delta}{\delta c+\lambda+\nu}\, e^{-\delta x} > \frac{\Gamma\gamma}{\gamma c+\lambda+\nu}\, e^{-\gamma x}$. Thus

(6.6) $$\text{sign}\, \frac{\partial}{\partial\theta} I(x,0) = -\,\text{sign}\, E[R].$$

We will here concentrate on the case $E[R] \geq 0$. Then we obtain:

Theorem 6.7. *If* $E[R] > 0$*, then there is an optimal Howard improvement* φ^* *of* φ *with* $\varphi^*(x) > 0$ *for all* $x \geq 0$.

Proof. If $E[R] > 0$, then $\frac{\partial}{\partial\theta} I(x,0) < 0$, and $\theta = 0$ is never a minimum point of $T\psi(x,\theta)$. $\square$

For $E[R] = 0$ we have $\frac{\partial}{\partial\theta} I(x,0) = 0$ and we are looking for conditions for $G(x,0) := \frac{\partial^2}{\partial\theta^2} I(x,\theta)$ to be negative. We can compute

$$G(x,\theta) := \frac{\partial^2}{\partial\theta^2} I(x,\theta) = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} I(x,\theta)$$

$$= \frac{\Delta\delta^2}{\delta c+\lambda+\nu}\ e^{-\delta x}\cdot E[R^2\cdot e^{-\delta\theta R}] - \frac{\Gamma\gamma^2}{\gamma c+\lambda+\nu}\ e^{-\gamma x}\cdot E[R^2\cdot e^{-\gamma\theta R}]\ .$$

Then we obtain

$$G(x,\theta) = \frac{\Delta\delta^2}{\delta c+\lambda+\nu}\cdot E[R^2\cdot e^{-\delta(x+\theta R)}] - \frac{\Gamma\gamma^2}{\gamma c+\lambda+\nu}\cdot E[R^2\cdot e^{-\gamma(x+\theta R)}] \quad \text{where}$$

(6.8) $$G(0,0) = \left[\frac{\Delta\delta^2}{\delta c+\lambda+\nu} - \frac{\Gamma\gamma^2}{\gamma c+\lambda+\nu}\right]\cdot E[R^2] < 0 \quad \text{for large } \nu$$

since $\frac{k^2}{kc+\lambda+\nu} = \frac{1}{\nu}\cdot\frac{k^2}{(kc+\lambda)/\nu\ +\ 1} \approx \frac{k^2}{\nu}$ for large ν and since where $\Delta\cdot\delta^2 < \Gamma\cdot\gamma^2$ by Lemma 6.2.

Now assume that ν is large, then it is not difficult to show that

$$\sup_{0\leq\theta\leq\alpha x} |G(x,\theta) - G(0,0)| \to 0 \quad \text{for } x \to 0$$

and thus

$$\sup_{0\leq\theta\leq\alpha x} \frac{\partial^2}{\partial\theta^2} I(x,\theta) \leq \tfrac{1}{2}G(0,0) < 0\ ,\ x \leq x_o, \text{ for some } x_o > 0.$$

For $E[R] \geq 0$ we have $\frac{\partial}{\partial\theta} I(x,0) \leq 0$ and hence we obtain

$$\frac{\partial}{\partial\theta} I(x,\theta) < 0 \quad \text{for } 0 < \theta \leq \alpha\cdot x \text{ and } x \leq x_o.$$

Thus $\theta = \alpha\cdot x$ is a minimum point of $I(x,\theta)$ for $x \leq x_o$. Then there exists an optimal Howard improvement φ^* with

(6.9) $$\varphi^*(x) = \alpha\cdot x \quad \text{for } x \leq x_o\ , \text{ for } E[R] \geq 0 \text{ and for large } \nu.$$

There ν is large and we will write $\nu \geq \nu_o$, if $\frac{\Delta\delta^2}{\delta c+\lambda+\nu} < \frac{\Gamma\gamma^2}{\gamma c+\lambda+\nu}$.

We have in view of Lemma 6.2

$$\frac{\Delta\delta^2}{\delta c+\lambda+\nu} < \frac{\Gamma\gamma^2}{\gamma c+\lambda+\nu} \Leftrightarrow \frac{q}{4}\cdot q\cdot(1-q)\cdot(\lambda+\nu) > \frac{q}{2}\cdot(1-q)\cdot\gamma\delta c \quad \Leftrightarrow \tfrac{1}{2}q\cdot(\lambda+\nu) > \gamma\delta c = c\cdot(1-q)$$

$$\Leftrightarrow \nu > \frac{1}{\lambda}\,[c^2 - 2c\lambda - \lambda^2] = \frac{1}{\lambda}\,[(c-\lambda)^2 - 2\lambda^2]\ , \quad \text{hence}$$

(6.10) $$\nu_o = \frac{1}{\lambda}\,[(c-\lambda)^2 - 2\lambda^2].$$

Then $\nu_o = 0$ if $(c-\lambda)^2 \leq 2\lambda^2 \Leftrightarrow \frac{c}{\lambda} - 1 \leq \sqrt{2}$ since $\frac{c}{\lambda} - 1 > 0$.
The condition $\nu \geq \nu_o$ is always fulfilled if $\nu_o = 0$, i.e., if

(6.11) $\qquad 2 < \frac{c}{\lambda} \le 1 + \sqrt{2} \iff 0{,}8284 = 2 \cdot (\sqrt{2} - 1) \le q < 1.$

Theorem 6.12. *Assume* $E[R] \ge 0$ *and* $\nu \ge \nu_o$. *Then there exists an optimal Howard improvement* φ^* *with* $\varphi^*(x) = \alpha \cdot x$, $x \le x_o$ (*for some* $x_o > 0$). *The assumption* $\nu \ge \nu_o$ *always holds under condition* (6.11).

As a consequence, even in the case where $E[R] = 0$, i.e. where the price process is a martingale, investment (with no short selling and no borrowing of money) reduces the ruin probability in the situation where the insurer is poor.

Appendix

Assumption: $Y \sim E_2$ where w.l.o.g. $E[Y] = 2$, hence $P[Y > z] = e^{-z} \cdot (1 + z)$.

For the survival probability $\Phi(u) = 1 - \psi(u)$ the following identity is well known (see Grandell 1991, (4) p. 5):

(A1) $\qquad \Phi(u) = \Phi(0) + \frac{\lambda}{c} \int_0^u \Phi(u-z)\,(1 - P[Y \le z])\,dz.$

Thus we have:

$$\Phi(u) = \Phi(0) + \frac{\lambda}{c} \int_0^u \Phi(u-z) \cdot e^{-z}\,(1 + z)\,dz$$
$$= \Phi(0) + \frac{\lambda}{c} \cdot e^{-u} \cdot \left[(1+u) \cdot \int_0^u \Phi(t)\,e^t\,dt - \int_0^u \Phi(t)\,e^t \cdot t\,dt\right] \quad \text{where } t = u-z\,.$$

Therefore we know that Φ is continuous and thus differentiable. Now we obtain:

(A2) $\qquad \Phi'(u) = \Phi(0) - \left[1 - \frac{\lambda}{c}\right] \cdot \Phi(u) + \frac{\lambda}{c} \cdot e^{-u} \cdot \int_0^u \Phi(t)\,e^t\,dt.$

Proceeding in the same way we obtain:

$$\Phi''(u) = -\left[2 - \frac{\lambda}{c}\right] \cdot \Phi'(u) - \left[1 - 2\frac{\lambda}{c}\right] \cdot \Phi(u) + \Phi(0), \quad \text{hence}$$

(A3) $\qquad \Phi''(u) + b \cdot \Phi'(u) + a \cdot \Phi(u) = \Phi(0)$

where $q := 2\frac{\lambda}{c}$, $b = 2 - \frac{1}{2}q$, $a = 1 - q$ and $b^2 = 4 - 2q + \frac{1}{4}q^2 > 4a = 4 - 4q$.

Now we get for $\psi(x) := 1 - \Phi(x)$:

Theorem A4. $\psi''(u) + b \cdot \psi'(u) + a \cdot \psi(u) = 0.$

The characteristic polynom is

$$\ell^2 + b \cdot \ell + a = 0 \iff (\ell + \tfrac{1}{2}b)^2 = (\tfrac{1}{2}b)^2 - a > 0 \text{ with roots}$$

(A5) $\quad -\gamma = -\tfrac{1}{2}b - \{(\tfrac{1}{2}b)^2 - a\}^{\frac{1}{2}} < -\beta = -\tfrac{1}{2}b + \{(\tfrac{1}{2}b)^2 - a\}^{\frac{1}{2}} < 0.$

The general solution of the homogeneous equation is $\psi(u) = -\Gamma\, e^{-\gamma u} + \Delta\, e^{-\delta u}$. There we have

(A6) $\quad -\Gamma + \Delta = q,$

$$\psi'(u) = \gamma\Gamma\, e^{-\gamma u} - \delta\Delta\, e^{-\delta u};$$

$$\psi''(u) = -\gamma^2\Gamma\, e^{-\gamma u} + \delta^2\Delta\, e^{-\delta u}.$$

From (A2) we get: $\psi'(0) = \gamma\Gamma - \delta\Delta = -\Phi'(0) = -\left[\Phi(0) - \lceil 1 - \frac{\lambda}{c}\rceil \cdot \Phi(0)\right] = -\frac{\lambda}{c}\cdot\Phi(0)$, hence

(A7) $\quad \gamma\Gamma - \delta\Delta = -\tfrac{1}{2}q\cdot(1-q).$

From $\psi''(u) = \delta^2\Delta\, e^{-\delta u} - \gamma^2\Gamma\, e^{-\gamma u}$ and Theorem A4 we obtain

$\psi''(0) = \delta^2\Delta - \gamma^2\Gamma = -(2 - \tfrac{1}{2}q)\cdot\psi'(0) - (1-q)\cdot\psi(0) - \frac{q}{4}\cdot q\cdot(1-q) < 0$, hence

(A8) $\quad \delta^2\Delta - \gamma^2\Gamma = -\frac{q}{4}\cdot q\cdot(1-q) < 0.$

Now we want to compute γ, δ, Γ, and Δ. From (A6) and (A7) we get

(A9) $\quad \Gamma = -q\cdot\frac{1\text{-}q-2\delta}{2(\gamma-\delta)} = q\cdot\frac{2\delta+q-1}{2(\gamma-\delta)}, \quad \Delta = -q\cdot\frac{1\text{-}q-2\gamma}{2(\gamma-\delta)} = q\cdot\frac{2\gamma+q-1}{2(\gamma-\delta)}.$

Furthermore we can compute from (A3) and (A5) that

(A10a) $\quad \gamma = 1 - \frac{q}{4} + \tfrac{1}{2}\sqrt{2q + \frac{q^2}{4}}$

(A10b) $\quad \delta = 1 - \frac{q}{4} - \tfrac{1}{2}\sqrt{2q + \frac{q^2}{4}}.$

It is clear that $\delta < 1$ and $\gamma > 1$. From (A10) and $q < 1$ we easily get:

(A11) $\quad \Gamma > 0 \;, \Delta > 0.$

Since $-\gamma$, $-\delta$ are solutions to $\ell^2 + b\cdot\ell + a = 0$, we know that

$$\frac{1}{(1-\ell)^2} = 1 + \frac{2}{q}\cdot\ell \quad \text{for } \ell = \gamma,\delta$$

which is Lundberg–equation. We have the following situation:

$$\psi''(x) < 0 \Leftrightarrow x < \frac{1}{\gamma - \delta}\ln\left[\frac{\gamma^2\cdot\Gamma}{\delta^2\cdot\Delta}\right] \text{ i.e. } \psi \text{ is concav–convex.}$$

Finally, from (A10) we obtain

(A12) $\quad \gamma\delta = 1 - q.$

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