SCHÄL, MANFRED

Stochastic optimization for the ruin probability

The Cramér-Lundberg insurance model is studied where the risk process can be controlled by reinsurance and by investment in a financial market. The performance criterion is the ruin probability. The problem can be imbedded in the framework of discrete-time stochastic dynamic programming. Basic tools are the Howard improvement and the verification theorem. Explicit conditions are obtained for the optimality of employing no reinsurance and of not investing in the market.

1. Introduction

The control of the ruin probability is studied in a variant of the Cramér–Lundberg model with exponentially distributed claims. A period will be the time between two successive claims. At first view, the ruin probability is not a classical performance criterion for control problems. But it can be shown that one can write the ruin probability as some total cost without discounting where one has to pay one unit of cost when entering a ruin state. After this simple observation, the results from discrete–time dynamic programming apply. In spite of the lack of discounting, the model enjoys a kind of contraction property. This property is strong enough for the validity of the Howard improvement and a verification theorem. A more detailed description of the underlying stochastic optimization model and references to the literature are given in [2]. By use of the Howard improvement, one can look for a plan which is at least better than using no kind of control (where one is in the classical Cramér–Lundberg model). As an application of the verification theorem, one can answer the questions: When is it optimal (i) to have no reinsurance, (ii) not to invest in the financial market.

2. The optimization model

We will consider an insurance model where the occurrence of the claims is described by a Poisson process with rate λ. We write Z_n for the period length between the (n–1)–th and the n–th claim. The n–th *claim* itself will be described by an exponentially distributed random variable Y_n with mean μ. The process $\{X_n, n\geq 0\}$ is the risk process where $X_n \in \mathbb{R}$ describes the *surplus* (capital) of the insurance company immediately after the n–th claim. The process can be controlled by reinsurance, i.e. by choosing the *retention level* $b \in [\underline{b},\bar{b}]$ of a reinsurance for the period up to the next claim. The function h(b,y) specifies the part of the claim y paid by the insurer. Then h(b,y) depends on the retention level b (fixed in the risk sharing contract at the beginning of the respective period) where $0 \leq h(b,y) \leq y$. Hence $y - h(b,y)$ is the part paid by the reinsurer.

We will consider the following two cases: the case of an *excess of loss reinsurance* where

$$h(b,y) = \min(b,y) \quad \text{with retention level } 0 \leq \underline{b} \leq b \leq \bar{b} = \infty$$

and the case of a *proportional reinsurance* where

$h(b,y) = b \cdot y$ with retention level $0 \le \underline{b} \le b \le \bar{b} = 1$.

The *net income rate* $c(b)$ is calculated according to the *expected value principle* with *safety loadings* $\eta > 0$ and $\theta > \eta$ of insurer and reinsurer, respectively. For each retention level b, the insurer pays a premium rate to the reinsurer. This leads to

$$c(b) = (1+\eta) \cdot \lambda \cdot E[Y] - (1+\theta) \cdot \lambda \cdot E[Y - h(b,Y)] \quad \text{for } \underline{b} \le b \le \bar{b} .$$

There, the retention level $\bar{b}$ stands for the control action "no reinsurance", hence $h(\bar{b},y) = y$. The smallest retention level $\underline{b}$ will be chosen such that $c(b) \ge 0$ for $b \ge \underline{b}$.

In addition, the insurance company can invest the capital in some asset, say stock, described by the price process $\{S_n, n \ge 0\}$. There S_n is the price of one share of the stock at the occurrence of the n–th claim. We define the return process $\{R_n, n \ge 1\}$ by $S_n =: S_{n-1} \cdot (1 + R_n)$, where of course $1 + R_n > 0$. Investment in the stock is specified by a number $\vartheta \in [0,1]$ representing the proportion of the capital which is invested. Thus a control action $u = (b,\vartheta)$ will consist of two components where b and ϑ specify the retention level and the investment, respectively. Given the surplus X_n and the control action $u_n = (b_n,\vartheta_n)$, we can compute the surplus X_{n+1} according to

$$X_{n+1} = X_n \cdot [1 + \vartheta_n \cdot R_{n+1}] + c(b_n) \cdot Z_{n+1} - h(b_n,Y_{n+1}).$$

The sequence $\{(R_n,Y_n,Z_n), n \ge 1\}$ is assumed to be independent and identically distributed and, in addition, (R_n,Z_n) and Y_n are independent. It is reasonable to allow for a dependence of Z_n and R_n. At time 0 and after each claim, the decision about the control action will depend on the present size X_n of the capital (surplus). A (stationary) *plan* ψ is a (measurable) function $\psi : [0,\infty) \mapsto U := [\underline{b},\bar{b}] \times [0,1]$ such that $u_n = (b_n,\vartheta_n) = \psi(X_n)$, $n \ge 0$. Given the initial surplus x and a plan ψ, the risk process $(X_n) =: (X_n^{x,\psi})$ is well–defined on the underlying probability space and the ruin probability is

$$J^{\psi}(x) := P[X_n^{x,\psi} < 0 \text{ for some } n].$$

Then ψ is called *optimal* if $J^{\psi}(x) \le J^{\varphi}(x)$ for all plans φ and $x \ge 0$. We make the following well–known *no arbitrage assumption*: $P[\pm R_1 > 0 \mid Z_1 = z] > 0$ for all $z > 0$.

As usual in stochastic optimization it is convenient to introduce the following operators:

$$Tv(x,u) = Tv(x,b,\vartheta) = E[v(x \cdot [1 + \vartheta \cdot R_1] + c(b) \cdot Z_1 - h(b,Y_1))],$$

$$T^*v(x) = \min_{u \in U} Tv(x,u) \quad \text{for any } v : \mathbb{R} \mapsto \mathbb{R} \text{ (for which the expressions exist).}$$

An important role will be played by the plan φ recommending to do nothing, i.e. under φ the decision maker employs no reinsurance and does not invest in the stock. Then we have:

$\varphi(x) := (\bar{b},0)$ where $\bar{b}$ stands for "no reinsurance" as before.

In fact, φ will serve as a kind of benchmark for the other plans. Then under φ we have the usual uncontrolled Cramér–Lundberg model with exponentially distributed claims and thus we know:

$$J^{\varphi}(x) = (1-\kappa)\cdot e^{-\kappa x/\mu} \quad \text{with} \quad 1-\kappa := 1/(1+\eta),\ x \geq 0.$$

We remind that $TJ^{\varphi}(x,\varphi(x)) = J^{\varphi}(x)$, $x \geq 0$. In [2] the following result was obtained:

P r o p o s i t i o n. *Let* φ *be as above*, S *be any subset of* $[0,\infty)$ *and set*
$U(x,\varphi) := \{\, u \in U;\ TJ^{\varphi}(x,u) < J^{\varphi}(x) \,\}$, $x \geq 0$.

(a) ***Howard Improvement.*** *For each plan* ψ *with*

$$\psi(x) \in U(x,\varphi) \ \textit{for}\ x \in S \ \textit{and}\ \psi(x) = \varphi(x)\ \textit{for}\ x \notin S,$$

one has: $J^{\psi}(x) \leq J^{\varphi}(x)$, $x \geq 0$, *and* $J^{\psi}(x) \leq TJ^{\varphi}(x,\psi(x)) < J^{\varphi}(x)$, $x \in S$.

(b) ***Verification theorem.*** *If* $U(x,\varphi) = \emptyset$ *for* $x \geq 0$, *i.e.*, $J^{\varphi} = T^*J^{\varphi}$, *then* φ *is optimal*.

In the situation of (a) we call ψ a *Howard improvement*.

3. Optimality results

In this section we first assume that the decision maker can only employ reinsurance; then we only consider actions of the form $u = (b,0)$. From [2] we have:

T h e o r e m 1. In the model where the decision maker can only control by proportional reinsurance, it is optimal to have no reinsurance under the condition $\theta > 2\eta + \eta^2$.

Now we want to study the case of excess of loss reinsurance; then $\underline{b} \leq b \leq \bar{b} = \infty$. By a straightforward but lengthy computation one obtains the following formula:

L e m m a 1. *In the model where the decision maker can only control by excess of loss reinsurance, one has for* $TJ^{\varphi}(x,(b,0)) =: TJ^{\varphi}(x,b)$:

$$T^{\varphi}J(x,b) = \frac{\lambda\mu}{\lambda\mu + \kappa c(b)}\cdot \exp\{-\kappa\cdot\frac{x}{\mu}\}\cdot\left[1 - \kappa\cdot\exp\{-(1-\kappa)\frac{b}{\mu}\}\right] \quad \textit{for}\ b \leq x,$$

$$T^{\varphi}J(x,b) = \frac{\lambda\mu}{\lambda\mu + \kappa c(b)}\cdot\left[\exp\{-\kappa\cdot\frac{x}{\mu}\} - \kappa\cdot\exp\{-[1 + \frac{\lambda\mu}{c(b)}]\cdot\frac{b}{\mu}\}\cdot\exp\{\frac{\lambda\mu}{c(b)}\cdot\frac{x}{\mu})\}\right] \quad \textit{for}\ b \geq x$$

where $c(b) = \lambda\mu\cdot[1+\eta-(1+\theta)\cdot e^{-b/\mu}]$.

The present situation is completely different from that of Theorem 1. Let us set:

$$x_o := \frac{\mu}{\kappa}\cdot\ln\{\frac{1+\theta}{1+\eta}\}, \quad x^o > x_o \text{ is the minimizer of the function}$$

$$B(x) := \frac{\lambda\mu}{\lambda\mu + \kappa c(x)}\cdot\left[1 - \kappa\cdot\exp\{-(1-\kappa)\cdot\frac{x}{\mu}\}\right] \quad \text{on } [x_o,\infty).$$

T h e o r e m 2. *In the situation of Lemma 1, it is never optimal to have no reinsurance for all* $x \geq 0$. *More exactly* ψ *is a Howard improvement of* φ *if*
(i) $\psi(x) = \infty$ for $x \leq x_o$, (ii) $\psi(x) = \min(x, x^o)$ for $x > x_o$.

For the proof of the first statement it is sufficient to show that $TJ^{\varphi}(x,\infty) \leq TJ^{\varphi}(x,x) \Leftrightarrow x \leq x_o$ which is easy. The more exact statements are obtained by minimizing the function $b \mapsto TJ^{\varphi}(x,b)$.

The statements of the theorems are similar to numerical results in [1] where only constant plans $\psi(x) = b$ (for all x) are studied.

Finally, we consider the situation where the decision maker wants to minimize the ruin probability by investing in the stock; then we only consider actions of the form $u = (\bar{b},\vartheta)$.

L e m m a 2. *In the model where the decision maker can only control by investing in the stock one has*: $TJ^{\varphi}(x,(\bar{b},\vartheta)) =: TJ^{\varphi}(x,\vartheta) = \exp\{-\frac{1}{\mu}\kappa \cdot x\} \cdot E\left[\exp\{-\frac{1}{\mu}\kappa \cdot [c \cdot Z_1 + x \cdot \vartheta \cdot R_1]\}\right]$.

The following quantity will play an important role: $q := E\left[R_1 \cdot e^{-\eta\lambda Z_1}\right]$.

T h e o r e m 3. *In the situation of Lemma 2, one has*:
(a) *If* $q \leq 0$, *then it is optimal not to invest at all, i.e.* φ *is optimal*.
(b) *If* $q > 0$, *then there exists a Howard improvement* ψ *of* φ *investing for all* $x > 0$ *and furthermore investing all the capital*, i.e. $\psi(x) = (\bar{b},1)$, *for all* $x \leq x_1$ *and some* $x_1 > 0$.

For the proof it is important to notice that $\frac{\partial}{\partial\vartheta} TJ(x,\vartheta)$ is strictly increasing in ϑ.

4. References

1 Dickson, D.C.M.; Waters, H.R. : Reinsurance and ruin. Insurance: Mathematics and Economics **19**, 1996, 61–80.

2 Schäl, M.: On discrete–time dynamic programming in insurance: exponential utility and minimizing the ruin probability. Scand. Actuar. J. 2003.

MANFRED SCHÄL, Inst. Angew. Math., Univ. Bonn, Wegelerstr. 6, D–53115 Bonn, Germany.