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Markov Processes

Exercise sheet 1 from 04/17/2009

Definition: Let $a, b \in \mathbb{R}$ and $T > 0$. A *one-dimensional Brownian Bridge* from $(0, a)$ to (T, b) is an $\mathbb{R}$-valued Gaussian Process $\{X_t\}_{t\in[0,T]}$ with expectation $\mathbb{E}X_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T}$ and covariance $\operatorname{Cov}(X_s, X_t) = (s \wedge t) - \frac{st}{T}$.

Exercise 1 : Brownian Bridge - SDE (10 points)

Let $\{B_t\}_{t\in[0,T]}$ denote a one-dimensional Brownian Motion, $a, b \in \mathbb{R}$ and $T > 0$. Consider the following stochastic differential equation on the Interval $[0, T)$:

$$\begin{cases} dX_t = \frac{b-X_t}{T-t}dt + dW_t, \; t \in [0, T) \\ X_0 = a \end{cases} \tag{1}$$

i) Show that (1) is solved by the process

$$X_t := a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T - t)\int_0^t \frac{dW_s}{T - s}.$$

ii) Set $X_T := b$, so that X_t is defined on the Interval $[0, T]$. Show that $\{X_t\}_{t\in[0,T]}$ is a Brownian Bridge from $(0, a)$ to (T, b).

Exercise 2 : Brownian Bridge - Finite dimensional distributions (10 points)

Let $0 = t_0 < \cdots < t_n < T$, $x_0 := a$ and $p_t(x) := (2\pi t)^{-\frac{1}{2}}\exp(-\frac{x^2}{2t})$. Show that the finite dimensional distributions of the Brownian Bridge $\{X_t\}_{t\in[0,T]}$ from $(0, a)$ to (T, b) are given by

$$\mathbb{P}\left[X_{t_1} \in dx_1 \ldots X_{t_n} \in dx_n\right] = \prod_{i=1}^{n} p_{t_i - t_{i-1}}(x_i - x_{i-1}) \frac{p_{T-t_n}(b - x_n)}{p_T(b - a)} dx_1 \ldots dx_n.$$

Exercise 3 : Brownian Bridge as conditioned Brownian Motion (10 points)

Let $\{X_t\}_{t\in[0,T]}$ be the canonical process on $\mathcal{C}([0, T])$, $\mathbb{P}^a$ the distribution of the Brownian Motion, starting in a, and $\mathbb{P}^{a,b}$ the distribution of the Brownian Bridge from $(0, a)$ to (T, b). Show that

$$\mathbb{P}^{a,b}(\bullet) = \mathbb{P}^a(\bullet | X_T = b).$$

(Interpretation: The Brownian Bridge from $(0, a)$ to (T, b) is a Brownian Motion, starting in a, which is forced to reach the point b at time T.)

Exercise 4 : Brownian Bridge as time-changed Brownian Motion (10 points)

i) Let $\{B_t\}_{t\in[0,T]}$ be a Brownian Motion, starting in 0. Show that

$$X_t := \begin{cases} a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + \frac{T-t}{\sqrt{T}} B_{\frac{t}{T-t}}, \; t \in [0, T) \\ b, \; t = T \end{cases}$$

defines a Brownian Bridge from $(0, a)$ to (T, b).

ii) Let $\{X_t\}_{t\in[0,T]}$ be a Brownian Bridge from $(0, 0)$ to $(1, 0)$. Show that

$$B_t := (1 + t)\, X_{\frac{t}{1+t}}, \; t \geq 0$$

defines a Brownian Motion, starting in 0.