Institute for Applied Mathematics
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“Markov Processes”, Problem Sheet 5.

Hand in solutions on Thursday 26.11. during the lecture.

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1. (Brownian semigroup) (4+4* points)
Consider $S = \mathbb{R}$ and $X(t)$ a Brownian motion.

(i) Show that $T(t)$ defined by $T(t)f(x) := \mathbb{E}^x[f(X(t))]$ is a probability semigroup.
Hint: Recall that functions in $C(\mathbb{R})$ are uniformly continuous.

(ii)* Explain why this would not be the case if $C(\mathbb{R})$ would be replaced by the space of bounded continuous functions (without the requirement that they vanish at infinity).

2. (Markov Chains vs Feller Processes) (3+5 points)
Let $p_t(x, y)$ be a transition function of a Markov chain on a finite or countable state space S. Let $T(t)$ be defined by

$$T(t)f(x) = \sum_{y \in S} p_t(x, y)f(y).$$

(i) For finite S, show that $T(t)$ is a probability semigroup.

(ii) For infinite S, show that $T(t)$ is a probability semigroup if and only if

$$\lim_{x\to\infty} p_t(x, y) = 0, \quad \text{for all } y \in S, t > 0. \tag{1}$$

Remark: On a general countable space, $\lim_{x\to\infty} p_t(x, y) = 0$ is meant as a short notation for the function $p_t(x, y)$ to vanish at infinity (in x), i.e. it is arbitrarily small outside of compact subsets.

3. (Generators of Markov chains) (3+5 points)
Suppose that $q(x, y)$ is a Q-matrix on a finite or countable state space S. Let $\mathcal{L}$ be defined by

$$\mathcal{L}f(x) = \sum_{y\in S} q(x, y)(f(y) - f(x)) \tag{2}$$

for those $f \in C(S)$ for which the series converges for each x, and the resulting $\mathcal{L}f$ is in $C(S)$.

(i) Show that if S is finite, then $\mathcal{L}$ is a probability generator.

(ii) Take $S = \{0, 1, 2, ...\}$ and the Q-matrix for a pure death process with resurrection, given by

$$q(0,1) = 1, \quad q(0,0) = -1, \quad q(i,i-1) = \delta_i, \quad q(i,i) = -\delta_i, \;\; i \geq 1$$

and $q(i,j) = 0$ otherwise. Assume that for all $i \geq 1$, $0 < \delta_i \leq M < \infty$.
Show that properties (a), (b), and (d) of the definition of a probability generator are satisfied and that $\mathcal{R}(I - \lambda\mathcal{L})$ contains all functions with finite support (for all $\lambda > 0$).