**Institute for Applied Mathematics**
**Winter term 2015/16**
Prof. Dr. Joe Neeman,
Kaveh Bashiri, Anna Kraut

# “Markov Processes”, Problem Sheet 4.

**Hand in solutions on Thursday 19.11. during the lecture.**

---

**1. (Stochastic versus sub-stochastic)** (6 points)
Take $S = \{0, 1, 2, \dots\}$ and for $\beta \geq 0, \delta \geq 0$ and $\alpha > 0$ consider the following $Q$-matrix:

$$q(i,j) = \begin{cases} \delta i^\alpha, & \text{if } j = i-1, \\ \beta i^\alpha, & \text{if } j = i+1, \\ -(\beta+\delta) i^\alpha, & \text{if } j = i. \end{cases}$$

For which values of $\beta, \delta$ and $\alpha$ is the minimal solution to the KBE stochastic?

**2. (Homogeneous vs non-homogeneous jump rates)** (6 points)
Let $X(t)$ be a Markov chain with exponential holding time parameters 1 for all sites, i.e. $c(x) = 1, \forall x \in S$. Let $b(\cdot)$ be a strictly positive and bounded function on $S$. Define

$$\lambda(t) = \int_0^t \frac{1}{b(X(s))} ds,$$

and let $\gamma(\cdot)$ be its inverse function. Show that $X(\gamma(t))$ is a Markov chain with the same embedded discrete time chain as $X(t)$, but holding time parameter $b(x)$ at $x \in S$.

*Hint: You may proceed as follows. Let $q(x,y)$ be the $Q$-matrix of $X$ and define $\tilde{q}(x,y) := b(x)q(x,y)$. From this new $Q$-matrix construct a Markov processes as in the lecture. As waiting times choose $\tilde{\tau}_i = \tau_i / b(Z_i)$, where $\tau_i$ are the waiting times of $X$. Compare the resulting process to $X$. Justify each step!*

**3. (Stationary measures)** (4 points)
Suppose that $\sum_x c(x)\pi(x) < \infty$ for some Markov chain on $S$ with transition function $p_t(x,y)$. Prove that $\pi$ is stationary if and only if

$$\sum_x \pi(x) q(x,y) = 0 \text{ for all } y \in S.$$

**4. (Stationary Markov chains and reversibility)** (2+2 points)
Let $\pi$ be a probability measure on $S$ and $\{X(t)\}_{t\geq 0}$ a Markov chain with transition probabilities $p_t(x,y)$ and initial distribution $\pi$.

(i) Show that the Process $X(t)$ is stationary if and only if $\pi$ is stationary.

(ii) If $\pi$ is stationary, the stationary process $X(t)$ can be extended to negative $t$ by using the Kolmogorov extension theorem. Show that $\pi$ is reversible if and only if the processes $X(t)$ and $X(-t)$ have the same distribution.