Energy statistics in disordered systems: The local REM conjecture and beyond *

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Recently, Bauke and Mertens conjectured that the local statistics of energies in random spin systems with discrete spin space should in most circumstances be the same as in the random energy model. Here we give necessary conditions for this hypothesis to be true, which we show to hold in wide classes of examples: short range spin glasses and mean field spin glasses of the SK type. We also show that, under certain conditions, the conjecture holds even if energy levels that grow moderately with the volume of the system are considered. In the case of the Generalised Random energy model, we give a complete analysis for the behaviour of the local energy statistics at all energy scales. In particular, we show that, in this case, the REM conjecture holds exactly up to energies $E_N < \beta_c N$, where β_c is the

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critical temperature. We also explain the more complex behaviour that sets in at higher energies.

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1. Introduction

In a recent paper [1], Bauke and Mertens have formulated an interesting conjecture on the behaviour of local energy level statistics in disordered systems. Roughly speaking, their conjecture can be formulated as follows. Consider a random Hamiltonian, $H_N(\sigma)$, i.e. a random function from some discrete spin space, $\mathcal{S}^{\Lambda_N}$, of the form

$$H_N(\sigma) = \sum_{A \subset \Lambda_N} \Phi_A(\sigma) \tag{1.1}$$

where Λ_N are finite subsets of Z^d of cardinality, say, N, the sum runs over some set of finite subsets of Λ_N and Φ_A are random local interactions, typically of the form

$$\Phi_A(\sigma) = J_A \prod_{x \in A} \sigma_x \tag{1.2}$$

where J_A, $A \subset \mathbb{Z}^d$, is a family of (typically independent) random variables. In such a situation, for typical σ, $H_N(\sigma) \sim \sqrt{N}$, while $\sup_\sigma H_N(\sigma) \sim N$. Bauke and Mertens then ask the following question: Given a fixed number, E, what are the statistics of the values $N^{-1/2}H_N(\sigma)$ that are closest to this number E, and how are configurations, σ, for which these good approximants of E are realised, distributed on $\mathcal{S}^{\Lambda_N}$? Their conjectured answer is rather simple: find $\delta_{N,E}$ such that $\mathbb{P}(|N^{-1/2}H_N(\sigma) - E| \leq b\delta_{N,E}) \sim |\mathcal{S}|^{-N} b$; then the collection of points $\delta_{N,E}^{-1}|N^{-1/2}H_N(\sigma) - E|$ over all $\sigma \in \mathcal{S}^{\Lambda_N}$ converges to a Poisson point process on $\mathbb{R}_+$. Furthermore, for any finite k, the k-tuple of configurations $\sigma^1, \sigma^2, \ldots, \sigma^k$, where the k best approximations are realised, is such that all of its elements have maximal Hamming distance between each other. In other words, the asymptotic behavior of these best approximants of E is the same, as if the random variables $H_N(\sigma)$ were all independent Gaussian random variables with zero mean and variance N, i.e. as if we were dealing with the random energy model (REM) [10], that is why Bauke and Mertens call this "universal REM like behaviour".

A comparable result, which motivated this problem, had previously been found by Mertens [12] (for rigorous proofs see [2, 3]) in the particular case of the *number partitioning problem*, where

$$H_N(\sigma) = \sum_{i=1}^{N} X_i \sigma_i \tag{1.3}$$

with X_i i.i.d. random variables uniformly distributed on $[0,1]$, $\sigma_i \in \{-1,1\}$; one is interested in the distribution of energies near the value zero, which corresponds to an optimal partitioning of the N random variables X_i into two groups, such that their sum in each group is as similar as possible (a generalisation to other values of E was investigated in [4]).

In [7] we generalised this result to the case of the k-partitioning problem, where the random function to be considered is actually vector-valued (consisting of the vector of differences between the sums of the random variables in each of the k subsets of the partition). To be precise, we considered the special case where the subsets of the partition are required to have the same cardinality, N/k (restricted k-partitioning problem). The general approach to the proof we developed in that paper sets the path towards the proof of the conjecture by Bauke and Mertens that we will present here.

The universality conjecture suggests that correlations are irrelevant for the properties of the local energy statistics of disordered systems for energies near "typical energies". On the other hand, we know that correlations must play a role for the extreme energies near the maximum of $H_N(\sigma)$. Thus there are two questions beyond the original conjecture that naturally pose themselves: (i) assume we consider instead of fixed E, N-dependent energy levels, say, $E_N = N^\alpha C$. How fast can we allow E_N to grow for the REM-like behaviour to hold? and (ii) what type of behaviour can we expect once E_N grows faster than this value? We will see that the answer to the first question depends on the properties of H_N, and we will give an answer in a class of examples related to the Sherrington-Kirkpatrick (SK) spin-glass models. The answer to question (ii) requires a detailed understanding of $H_N(\sigma)$ as a random process, and we will be able to give a complete answer on only in the case of the GREM, when H_N is a hierarchically correlated Gaussian process. In this note we explain why the local REM conjecture is true in a wide class of models, to what extend it can be generalised. In the case of the GREM, we give a complete analysis of the local energy statistics at all energy scales. We show that the REM conjecture holds for all energies $E_N < \beta_c N$, where β_c is the critical temperature. Beyond that, there appear more complicated limiting processes (compound Poisson processes) that reflect the onset of an intrinsic geometrical structure. We explain this phenomenon in detail. The emergence of β_c, the critical value for the REM conjecture in the GREM, suggests that possibly $\beta_c N$ is a general threshold for the REM-like behaviour in other models as well, in particular in the SK models. It will be a challenging problem for future work to investigate this possibility.

The technical details of our proofs will be presented elsewhere. Here we will concentrate on anon-technical exposition of the ideas behind these proofs.

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2. The local REM

Our approach to the proof of the REM conjecture is based on the following general fact about random variables. Let $V_{i,M} \geq 0$, $i \in I\!N$, be a family of positive random variables with identical distributions, that are normalized, s.t.

$$(V_{i_j,M} < b) \sim \frac{b}{M}. \tag{2.1}$$

In the case of independent random variables, it would follow, that the number of $V_{i,M}$ that are smaller than b will have Binomial distribution with parameters M, b/M. If (2.1) holds for all $b \in I\!R_+$, if we plot the set of all points $V_{i,M}$, the number of points within any subset, $A \subset I\!R_+ \backslash \{0\}$ will converge to a random variable with Poisson distribution, with parameter the volume of the set A. Moreover, if A and B are two disjoint subsets, then the numbers of points within each respective set will be independent. This means that the random set of points $V_{i,M}$ converges to a *Poisson point process* on $I\!R_+ \backslash \{0\}$. The question is under which conditions this is still true, if the random variables $V_{i,M}$ are correlated. It turns out that a very useful sufficient condition is that, for any fixed number, ℓ, average over all choices of collections of ℓ different the variables $V_{i_1,M}, \dots, V_{i_\ell,M}$, the probabilities that all ℓ are below thresholds, b_i, behaves as in the independent case, when M goes to infinity, i.e.

$$\lim_{M \uparrow \infty} \sum_{(i_1,\dots,i_\ell) \subset \{1,\dots,M\}} I\!P(\forall_{j=1}^{\ell} V_{i_j,M} < b_j) \to \prod_{j=1}^{\ell} b_j. \tag{2.2}$$

where the sum is taken over all sequences of *different* indices $(i_1, \dots, i_\ell)$.

Remark: A proof of this result can be found in Chapter 13 of [5]. Naturally, we would apply this theorem with $V_{i,M}$ given by $|N^{-1/2}H_N(\sigma) - E_N|$, properly normalised.

It is useful to think of the random variables $Y_N(\sigma) \equiv N^{-1/2}H_N(\sigma)$ as Gaussian random variables with variance one. This holds, if the couplings, J_A, are Gaussian. Otherwise, it is one of the main steps of the proof to show that they converge, in a strong sense, to Gaussians. This requires some continuity assumptions on the distributions of the couplings, but then holds in great generality, if $E_N = E$ is independent of N. If energies E_N that go to infinity are considered, this problem is much harder.

Let us see what needs to be checked in order for the above results to apply. Consider a product space $\mathcal{S}^N$ where $\mathcal{S}$ is a finite set. We define on $\mathcal{S}^N$ a real-valued random process, $Y_N(\sigma)$, $\sigma \in \mathcal{S}^N$. Assume for simplicity that $\mathbb{E}Y_N(\sigma) = 0$, $\mathbb{E}(Y_N(\sigma))^2 = 1$. Define on $\mathcal{S}^N$

$$b_N(\sigma, \sigma') \equiv \text{cov}(Y_N(\sigma), Y_N(\sigma')). \tag{2.3}$$

One problem we have to deal with from the outset are symmetries. Let G be the group of automorphisms on $\mathcal{S}_N$, such that, for $g \in G$, $Y_N(g\sigma) = Y_N(\sigma)$. Then it is clear that we should only consider the residual classes of $\mathcal{S}^N$ modulo this group[1], which we denote by Σ_N. Let us consider energies

$$E_N = cN^\alpha, \quad c, \alpha \in \mathbb{R}, \quad 0 \leq \alpha < 1/2, \tag{2.4}$$

and define the sequence

$$\delta_N = \sqrt{\tfrac{\pi}{2}} e^{E_N^2/2} |\Sigma_N|^{-1}. \tag{2.5}$$

Note that δ_N is exponentially small in $N \uparrow \infty$, since $\alpha < 1/2$. This sequence is chosen such that for any $b \geq 0$,

$$\lim_{N \uparrow \infty} |\Sigma_N| \mathbb{P}(|Z_N(\sigma) - E_N| < b\delta_N) = b. \tag{2.6}$$

For $\ell \in \mathbb{N}$ and any collection, $\sigma^1, \ldots, \sigma^\ell \in \Sigma_N^{\otimes \ell}$, we denote by $B_N(\sigma^1, \ldots, \sigma^\ell)$ the covariance matrix with elements

$$b_{i,j}(\sigma^1, \ldots, \sigma^\ell) \equiv b_N(\sigma^i, \sigma^j). \tag{2.7}$$

(i) Let $\mathcal{R}_{N,\ell}^\eta$ denote the set where all covariances are small, i.e. with $\eta > 0$

$$\mathcal{R}_{N,\ell}^\eta \equiv \{(\sigma^1, \ldots, \sigma^\ell) \in \Sigma_N^{\otimes \ell} : \forall_{1 \leq i < j \leq \ell} \ |b_N(\sigma^i, \sigma^j)| \leq N^{-\eta}\}. \tag{2.8}$$

This set will in any reasonable model exhaust almost the entire configuration space. We will assume that there exists a continuous decreasing function, $\rho(\eta) > 0$, on $]\eta_0, \tilde{\eta}_0[$ (for some $\tilde{\eta}_0 \geq \eta_0 > 0$), and a function, $\mu(\eta) > 0$, on $]\eta_0, \tilde{\eta}_0[$, such that

$$|\mathcal{R}_{N,\ell}^\eta| \geq \left(1 - \exp\left(-\mu(\eta) N^{\rho(\eta)}\right)\right) |\Sigma_N|^\ell. \tag{2.9}$$

[1] In the special case $E = 0$, we need even consider a larger group that leaves the modulus of Y_N invariant.

(ii) Our main worry would then be degenerate situations where the covariance matrix is singular. We need that the number of such configurations is very small, in the following sense. For $\ell \geq 2$, $r = 1, \dots, \ell-1$, set

$$\mathcal{L}^{\ell}_{N,r} = \Big\{(\sigma^1,\dots,\sigma^\ell) \in \Sigma_N^{\otimes \ell} \;:\; \forall_{1\leq i<j\leq \ell} |Y_N(\sigma^i)| \neq |Y_N(\sigma^j)|, \\ \operatorname{rank}(B_N(\sigma^1,\dots,\sigma^\ell)) = r\Big\}$$

Then there exists $d_{r,\ell} > 0$, such that, for all N large enough,

$$|\mathcal{L}^{\ell}_{N,r}| \leq |\Sigma_N|^r e^{-d_{r,\ell} N}. \tag{2.10}$$

This is the most critical assumption that is not always easy to verify.

(iii) Finally, we must ensure that even in the degenerate cases, the probabilities considered are under control. We ask that for $\ell \geq 1$, any $r = 1, 2, \dots, \ell$, and any $b_1, \dots, b_\ell \geq 0$, there exist constants, $p_{r,\ell} \geq 0$ and $Q < 0$, such that, for any $\sigma^1, \dots, \sigma^\ell \in \Sigma_N^{\otimes \ell}$, such that $\operatorname{rank}(B_N(\sigma^1,\dots,\sigma^\ell)) = r$,

$$I\!P(\forall_{i=1}^{\ell} : |Y_N(\sigma^i) - E_N| < \delta_N b_i) \leq Q \delta_N^r N^{p_{r,\ell}}. \tag{2.11}$$

Theorem *Under the assumptions above, if $\alpha \in [0,1/2[$ is such that, for some $\eta_1 \leq \eta_2 \in]\eta_0, \tilde{\eta}_0[$,*

$$\alpha < \eta_2/2, \tag{2.12}$$

$$\alpha < \eta/2 + \rho(\eta)/2,\ \forall \eta \in]\eta_1, \eta_2[, \tag{2.13}$$

and

$$\alpha < \rho(\eta_1)/2. \tag{2.14}$$

Furthermore, assume that, for any $\ell \geq 1$, any $b_1, \dots, b_\ell > 0$, and $(\sigma^1, \dots, \sigma^\ell) \in \mathcal{R}^{\eta_1}_{N,\ell}$,

$$\begin{aligned} & I\!P\left(\forall_{i=1}^{\ell} : \ |Y_N(\sigma^i) - E_N| < \delta_N b_i\right) \\ = \ & I\!P\left(\forall_{i=1}^{\ell} : \ |Z_N(\sigma^i) - E_N| < \delta_N b_i\right) + o(|\Sigma_N|^{-\ell}). \end{aligned} \tag{2.15}$$

Then, the point process,

$$P_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1} |Y_N(\sigma) - E_N|\}} \to \mathcal{P} \tag{2.16}$$

converges weakly to the standard Poisson point process $\mathcal{P}$ on $I\!R_+$.

Moreover, for any $\epsilon > 0$ and any $b \in \mathbb{R}_+$, the probability that there exists two configuarations, σ, σ', such that $|b_N(\sigma,\sigma')| > \epsilon$ and $|Y_N(\sigma) - E_N| \leq |Y_N(\sigma') - E_N| \leq \delta_N b$, tends to zero, as $N \uparrow \infty$.

Theorem 2 is proven by verifying that the conditions (2.2) are verified for $V_{i,M}$ given by $\delta_N^{-1}|Y_N(\sigma) - E_N|$, i.e. that

$$\sum_{(\sigma^1,\dots,\sigma^\ell)\in\Sigma_N^{\otimes l}} \mathbb{P}\left(\forall_{i=1}^{\ell} : |Y_N(\sigma^i) - E_N| < b_i\delta_N\right) \to b_1 \cdots b_\ell. \tag{2.17}$$

The formal proof is given in [8].

The second assertion of the theorem is elementary: the sum of terms $\mathbb{P}(\forall_{i=1}^2 : |Y_N(\sigma^i) - E_N| < \delta_N b)$ over all pairs, $(\sigma^1,\sigma^2) \in \Sigma_N^{\otimes 2} \setminus \mathcal{R}_{N,2}^{\eta_1}$, such that $\sigma^1 \neq \sigma^2$, converges to zero exponentially fast.

Finally, we remark that the results can be extended to the case when $\mathbb{E}Y_N(\sigma) \neq 0$, if $\alpha = 0$, i.e. $E_N = c$. Note that, e.g., the unrestricted number partitioning problem falls into this class. To see this, let $Z_N(\sigma)$ be the Gaussian process with the same mean and covariances as $Y_N(\sigma)$. Let us consider both the covariance matrix, B_N, and the mean of Y_N, $\mathbb{E}Y_N(\sigma)$, as random variables on the probability space $(\Sigma_N, \mathcal{B}_N, \mathbb{E}_\sigma)$, where $\mathbb{E}_\sigma$ is the uniform law on Σ_N. Assume that, for any $\ell \geq 1$,

$$B_N(\sigma^1,\dots,\sigma^\ell) \xrightarrow{\mathcal{D}} I_d, \quad N \uparrow \infty, \tag{2.18}$$

where I_d denotes the identity matrix, and

$$\mathbb{E}Y_N(\sigma) \xrightarrow{\mathcal{D}} D, \quad N \uparrow \infty, \tag{2.19}$$

where D is some random variable D. Let

$$\widetilde{\delta}_N = \sqrt{\tfrac{\pi}{2}} K^{-1} |\Sigma_N|^{-1}. \tag{2.20}$$

where

$$K \equiv \mathbb{E}e^{-(c-D)^2/2}. \tag{2.21}$$

Theorem *Assume that, for some $R > 0$, $|\mathbb{E}Y_N(\sigma)| \leq N^R$, for all $\sigma \in \Sigma_N$. Assume that (2.9) holds for some $\eta > 0$ and that (ii) and (iii) of Assumption A are valid. Assume that there exists a set, $\mathcal{Q}_N \subset \mathcal{R}_{N,\ell}^\eta$, such that (2.16) is valid for any $(\sigma^1,\dots,\sigma^\ell) \in \mathcal{Q}_N$, and that $|\mathcal{R}_{N,\ell}^\eta \setminus \mathcal{Q}_N| \leq |\Sigma_N|^\ell e^{-N^\gamma}$, with some $\gamma > 0$. Then, the point process*

$$P_N \equiv \sum_{\sigma\in\Sigma_N} \delta_{\widetilde{\delta}_N^{-1}|Y_N(\sigma)-E_N|} \to \mathcal{P} \tag{2.22}$$

converges weakly to the standard Poisson point process $\mathcal{P}$ on $\mathbb{R}_+$.

3. Examples

The assumptions of our theorem are verified in a wide class of physically relevant models. The examples we verified explicitly in [8] are: 1) the Gaussian p-spin SK models, 2) SK-models with non-Gaussian couplings, and 3) short-range spin-glasses. In the last two examples we consider only the case $\alpha = 0$.

3.1. p-spin Sherrington-Kirkpatrick models, $0 \leq \alpha < 1/2$

In this subsection we illustrate our general theorem in the class of Sherrington-Kirkpatrick models. Consider $\mathcal{S} = \{-1, 1\}$.

$$H_N(\sigma) = \frac{\sqrt{N}}{\sqrt{\binom{N}{p}}} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq N} J_{i_1,\ldots,i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p} \tag{3.1}$$

is the Hamiltonian of the p-spin Sherrington-Kirkpatrick model, where $J_{i_1,\ldots,i_p}$ are independent standard Gaussian random variables.

The following elementary proposition concerns the symmetries to the Hamiltonian.

Proposition *Assume that, for any* $0 < i_1 < \ldots < i_p \leq N$, $\sigma_{i_1} \cdots \sigma_{i_p} = \sigma'_{i_1} \cdots \sigma'_{i_p}$. *Then, if* p *is pair, either* $\sigma_i = \sigma'_i$, *for all* $i = 1, \ldots, N$, *or* $\sigma_i = -\sigma'_i$, *for all* $i = 1, \ldots, N$, *and, if* p *is odd, then* $\sigma_i = \sigma'_i$, *for all* $i = 1, \ldots, N$. *Assume that, for any* $0 < i_1 < \ldots < i_p \leq N$, $\sigma_{i_1} \cdots \sigma_{i_p} = -\sigma'_{i_1} \cdots \sigma'_{i_p}$. *This is impossible, if* p *is pair and* $\sigma_i = -\sigma'_i$, *for all* $i = 1, \ldots, N$, *if* p *is odd.*

This proposition allows us to construct the space Σ_N: If p is odd and $c \neq 0$, $\Sigma_N = \mathcal{S}^N$, thus $|\Sigma_N| = 2^N$. If p is even, or $c = 0$, Σ_N consists of equivalence classes where configurations σ and $-\sigma$ are identified, thus $|\Sigma_N| = 2^{N-1}$.

Theorem *Let* $p \geq 1$ *be odd. Let* $\Sigma_N = \mathcal{S}^N$. *If* $p = 1$ *and* $\alpha \in [0, 1/4[$, *and, if* $p = 3, 5, \ldots,$, *and* $\alpha \in [0, 1/2[$, *for any constant* $c \in I\!R \setminus \{0\}$ *the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1} |N^{-1/2} H_N(\sigma) - cN^\alpha|\}} \tag{3.2}$$

where $\delta_N = 2^{-N} e^{+c^2 N^{2\alpha}/2} \sqrt{\frac{\pi}{2}}$, *converges weakly to the standard Poisson point process,* $\mathcal{P}$, *on* $I\!R_+$.

Let p *be even. Let* Σ_N *be the space of equivalence classes of* $\mathcal{S}^N$ *where* σ *and* $-\sigma$ *are identified. For any* $\alpha \in [0, 1/2[$ *and any constant,* $c \in I\!R$, *the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{(2\delta_N)^{-1} |N^{-1/2} H_N(\sigma) - cN^\alpha|\}} \tag{3.3}$$

converges weakly to the standard Poisson point process, $\mathcal{P}$, on $I\!R_+$. The result (3.3) holds true as well in the case of $c = 0$, for p odd.

3.2. Short range spin glasses

As a final example, we consider short-range spin glass models. To avoid unnecessary complications, we will look at models on the d-dimensional torus, Λ_N, of length N. We consider Hamiltonians of the form

$$H_N(\sigma) \equiv -N^{-d/2} \sum_{A\subset\Lambda_N} r_A J_A \sigma_A \tag{3.4}$$

where e $\sigma_A \equiv \prod_{x\in A} \sigma_x$, r_A are given constants, and J_A are random variables. We will introduce some notation:

(a) Let $\mathcal{A}_N$ denote the set of all $A \subset \Lambda_N$, such that $r_A \neq 0$.

(b) For any two subsets, $A, B \subset \Lambda_N$, we say that $A \sim B$, iff there exists $x \in \Lambda_N$ such that $B = A + x$. We denote by $\mathcal{A}$ the set of equivalence classes of $\mathcal{A}_N$ under this relation.

We will assume that the constants, r_A, and the random variables, J_A, satisfy the following conditions:

(i) $r_A = r_{A+x}$, for any $x \in \Lambda_N$;

(ii) there exists $k \in I\!N$, such that any equivalence class in $\mathcal{A}$ has a representative $A \subset \Lambda_k$; we will identify the set $\mathcal{A}$ with a uniquely chosen set of representatives contained in Λ_k.

(iii) $\sum_{A\subset\Lambda_N:} r_A^2 = N^d$.

(iv) J_A, $A \in \mathbb{Z}^d$, are a family of independent random variables, such that

(v) J_A and J_{A+x} are identically distributed for any $x \in \mathbb{Z}^d$;

(vi) $I\!E J_A = 0$ and $I\!E J_A^2 = 1$, and $I\!E J_A^3 < \infty$;

(vii) For any $A \in \mathcal{A}$, the Fourier transform $\phi_A(s) \equiv I\!E \exp(isJ_A)$, of J_A satisfies $|\phi_A(s)| = O(|s|^{-1})$ as $|s| \to \infty$.

Observe that $I\!E H_N(\sigma) = 0$,

$$b(\sigma, \sigma') \equiv N^{-d} I\!E H_N(\sigma) H_N(\sigma') = N^{-d} \sum_{A\in\Lambda_N} r_A^2 \sigma_A \sigma'_A \leq 1 \tag{3.5}$$

where equality holds, if $\sigma = \sigma'$.

Note that $Y_N(\sigma) = Y_N(\sigma')$ (resp. $Y_N(\sigma) = -Y_N(\sigma')$), if and only if, for all $A \in \mathcal{A}_N$, $\sigma_A = \sigma'_A$ (resp. $\sigma_A = -\sigma'_A$). E.g., in the standard Edwards-Anderson model, with nearest neighbor pair interaction, if σ_x differs from σ'_x on every second site, x, then $Y_N(\sigma) = -Y_N(\sigma')$, and if $\sigma' = -\sigma$, $Y_N(\sigma) = Y_N(\sigma')$. In general, we will consider two configurations, $\sigma, \sigma' \in S^{\Lambda_N}$, as equivalent, iff for all $A \in \mathcal{A}_N$, $\sigma_A = \sigma'_A$. We denote the set of these equivalence classes by Σ_N. We will assume in the sequel that there is a finite constant, $\Gamma \geq 1$, such that $|\Sigma_N| \geq 2^{N^d}\Gamma^{-1}$. In the special case of $c = 0$, the equivalence relation will be extended to include the case $\sigma_A = -\sigma'_A$, for all $A \in \mathcal{A}_N$. In most reasonable examples (e.g. whenever nearest neighbor pair interactions are included in the set $\mathcal{A}$), the constant $\Gamma \leq 2$ (resp. $\Gamma \leq 4$, if $c = 0$).

Theorem *Let $c \in I\!R$, and Σ_N be the space of equivalence classes defined before. Let $\delta_N \equiv |\Sigma_N|^{-1} e^{c^2/2}\sqrt{\frac{\pi}{2}}$. Then the point process*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1}|H_N(\sigma) - c|\}} \tag{3.6}$$

converges weakly to the standard Poisson point process on $I\!R_+$.
If, moreover, the random variables J_A are Gaussian, then, for any $c \in I\!R$, and $0 \leq \alpha < 1/4$, with $\delta_N \equiv |\Sigma_N|^{-1} e^{N^{2\alpha}c^2/2}\sqrt{\frac{\pi}{2}}$ the point process

$$\mathcal{P}_N \equiv \sum_{\sigma \in \Sigma_N} \delta_{\{\delta_N^{-1}|H_N(\sigma) - cN^\alpha|\}} \tag{3.7}$$

converges weakly to the standard Poisson point process on $I\!R_+$.

4. Beyond REM behaviour

Let us briefly recall the definition of the GREM. We consider parameters $\alpha_0 = 1 < \alpha_1, \ldots, \alpha_n < 2$ with $\prod_{i=1}^n \alpha_i = 2$, $a_0 = 0 < a_1, \ldots, a_n < 1$, $\sum_{i=1}^n a_i = 1$. Let $X_{\sigma_1 \cdots \sigma_l}$, $l = 1, \ldots, n$, be independent standard Gaussian random variables indexed by $\sigma_1 \ldots \sigma_l \in \{-1, 1\}^{N \ln(\alpha_1 \cdots \alpha_l)/\ln 2}$. The Hamiltonian of the GREM is $H_N(\sigma) \equiv \sqrt{N} X_\sigma$, with

$$X_\sigma \equiv \sqrt{a_1} X_{\sigma_1} + \cdots + \sqrt{a_n} X_{\sigma_1 \cdots \sigma_n}. \tag{4.1}$$

Then $\mathrm{cov}\,(X_\sigma, X_{\sigma'}) = A(d_N(\sigma, \sigma'))$, where $d_N(\sigma, \sigma') = N^{-1}[\min\{i : \sigma_i \neq \sigma'_i\} - 1]$, and $A(x)$ is a right-continuous step function on $[0, 1]$ with $A(x) = a_0 + \cdots + a_i$, if $x \in [\ln(\alpha_0\alpha_1, \cdots \alpha_i)/\ln 2\,,\, \ln(\alpha_0\alpha_1, \cdots \alpha_{i+1})/\ln 2)$. We will assume here for simplicity that the linear envelope of the function A is convex.

To formulate our results, we also need to recall from [6] (Lemma 1.2) the point process of Poisson cascades $\mathcal{P}^l$ on $I\!R^l$. It is best understood in terms of the following iterative construction. If $l = 1$, $\mathcal{P}^1$ is the Poisson point process on $I\!R^1$ with the intensity measure $e^{-x}dx$. To construct $\mathcal{P}^l$, we place the process $\mathcal{P}^{l-1}$ on the plane of the first $l-1$ coordinates and through each of its points draw a straight line orthogonal to this plane. Then we put on each of these lines independently a Poisson point process with intensity measure $e^{-x}dx$. These points on $I\!R^l$ form the process $\mathcal{P}^l$.

Let us define the constants d_l, $l = 0, 1, \dots, n$, where $d_0 = 0$ and

$$d_l \equiv \sum_{i=1}^{l} \sqrt{a_i 2 \ln \alpha_i}. \tag{4.2}$$

Finally, set, for $l = 0, \dots, k-1$, as

$$D_l \equiv d_l + \sqrt{\frac{2\ln \alpha_{l+1}}{a_{l+1}}} \sum_{j=l+1}^{k} a_j. \tag{4.3}$$

It is not difficult to verify that $D_0 < D_1 < \cdots < D_{n-1}$. Interestingly, the border of D_0 is the point β_c, that is the critical temperature of the respective model. We are now ready to formulate the main result.

Theorem *If $|c| < D_0 = \beta_c$, then, the point process*

$$\mathcal{M}_N^0 = \sum_{\sigma \in \Sigma_N} \delta_{\left\{2^{N+1}(2\pi)^{-1/2} e^{-c_N^2 N/2} \left|X_\sigma - c_N\sqrt{N}\right|\right\}} \tag{4.4}$$

converges to the Poisson point process with intensity measure the Lebesgue measure.

Theorem *If for $l = 1, \dots, n-1$, $D_{l-1} \le c < D_l$, set*

$$\tilde{c}_l = |c| - d_l, \tag{4.5}$$

$$\beta_l = \frac{\tilde{c}_l}{a_{l+1} + \cdots + a_n}, \qquad \gamma_i = \sqrt{a_i/(2\ln\alpha_i)}, \quad i = 1, \dots, l, \tag{4.6}$$

and

$$R_l(N) = \frac{2(\alpha_{l+1}\cdots\alpha_k)^N \exp(-N\tilde{c}_l\beta_l/2)}{\sqrt{2\pi(a_{l+1} + \cdots + a_k)}} \prod_{j=1}^{l} (4N\pi \ln \alpha_j)^{-\beta_l\gamma_j/2}. \tag{4.7}$$

Then, the point process

$$\mathcal{M}_N^l = \sum_{\sigma \in \Sigma_N} \delta_{\left\{R_l(N)\left|\sqrt{a_1}X_{\sigma_1} + \cdots + \sqrt{a_n}X_{\sigma_1\dots\sigma_n} - c\sqrt{N}\right|\right\}} \tag{4.8}$$

converges to mixed Poisson point process on $[0, \infty[$*: given a realization of the random variable* Λ_l*, its intensity measure is* $\Lambda_l dx$*. The random variables* Λ_l *is defined in terms of the Poisson cascades* $\mathcal{P}_l$ *via*

$$\Lambda_l = \int_{\mathbb{R}^l} e^{\beta_l(\gamma_1 x_1 + \cdots \gamma_l x_l)} \mathcal{P}^l(dx_1, \ldots, dx_l). \tag{4.9}$$

The proof of this theorem is given in [9]. Here we give a heuristic interpretation of the main result.

Let us first look at (4.4). This statement corresponds to the REM-conjecture of Bauke and Mertens [1]. It is quite remarkable that this conjecture holds in the case of the GREM for energies of the form cN (namely for $c \in \mathcal{D}_0$).

In the REM [10], X_σ are 2^N *independent* standard Gaussian random variables and a statement (4.4) would hold for all c with $|c| < \sqrt{2 \ln 2}$: it is a well known result from the theory of independent random variables [11]. The value $c = \sqrt{2 \ln 2}$ corresponds to the maximum of 2^N independent standard Gaussian random variables, i.e., $\max_{\sigma \in \Sigma_N} N^{-1/2} X_\sigma \to \sqrt{2 \ln 2}$ a.s. Therefore, at the level $c = \sqrt{2 \ln 2}$, one has the emergence of the extremal process. More precisely, the point process

$$\sum_{\sigma \in \Sigma_N} \delta_{\left\{\sqrt{2N \ln 2}\left(X_\sigma - \sqrt{2N \ln 2} + \ln(4\pi N \ln 2)/\sqrt{8N \ln 2}\right)\right\}}, \tag{4.10}$$

that is commonly written as $\sum_{\sigma \in \Sigma_N} \delta_{u_N^{-1}(X_\sigma)}$ with

$$u_N(x) = \sqrt{2N \ln 2} - \frac{\ln(4\pi N \ln 2)}{2\sqrt{2N \ln 2}} + \frac{x}{\sqrt{2N \ln 2}}, \tag{4.11}$$

converges to the Poisson point process $\mathcal{P}^1$ defined above (see e.g. [11]). For $c > \sqrt{2 \ln 2}$, the probability that one of the X_σ will be outside of the domain $\{|x| < c\sqrt{N}\}$, goes to zero, and thus it makes no sense to consider such levels.

In the GREM, $N^{-1/2} \max_{\sigma \in \Sigma_N} X_\sigma$ converges to the value $d_k \in \partial D_{k-1}$ (4.2) (see Theorem 1.5 of [6]) that is generally smaller than $\sqrt{2 \ln 2}$. Thus it makes no sense to consider levels with $c \notin \overline{D}_{k-1}$. However, the REM-conjecture is not true for all levels in $\mathcal{D}_{k-1}$, but only in the smaller domain $\mathcal{D}_0$.

To understand the statement of the theorem outside $\mathcal{D}_0$, we need to recall how the extremal process in the GREM is related to the Poisson cascades introduced above. Let us set $\Sigma_{Nw_l} \equiv \{-1, 1\}^{Nw_l}$ where

$$w_l = \ln(\alpha_1 \cdots \alpha_l)/\ln 2 \tag{4.12}$$

and define the functions

$$U_{l,N}(x) \equiv N^{1/2}d_l - N^{-1/2}\sum_{i=1}^{l}\gamma_i \ln(4\pi N \ln\alpha_i)/2 + N^{-1/2}x \tag{4.13}$$

Set

$$\widehat{X}_\sigma^j \equiv \sum_{i=1}^{j}\sqrt{a}_i X_{\sigma_1\ldots\sigma_i}, \qquad \check{X}_\sigma^j \equiv \sum_{i=j+1}^{n}\sqrt{a}_i X_{\sigma_1\ldots\sigma_i}. \tag{4.14}$$

From what was shown in [6], for any $l = 1,\ldots,n$, the point process,

$$\mathcal{E}_{l,N} \equiv \sum_{\hat\sigma\in\Sigma_{Nw_l}} \delta_{U_{l,N}^{-1}(\widehat{X}_{\hat\sigma}^{J_l})} \tag{4.15}$$

converges in law to the Poisson cluster process, $\mathcal{E}_l$, given in terms of the Poisson cascade, $\mathcal{P}^l$, as

$$\mathcal{E}_l \equiv \int_{\mathbb{R}^l} \mathcal{P}^{(l)}(dx_1,\ldots,dx_l)\delta_{\sum_{i=1}^{l}\gamma_i x_i}. \tag{4.16}$$

In view of this observation, we can re-write the definition of the process $\mathcal{M}_N^l$ as follows:

$$\mathcal{M}_N^l = \sum_{\hat\sigma\in\Sigma_{w_lN}}\sum_{\check\sigma\in\Sigma_{(1-w_l)N}} \delta_{\left\{R_l(N)\left|\check{X}_{\hat\sigma\check\sigma}^{J_l} - \sqrt{N}\left[|c|-d_l-N^{-1}(\Gamma_{l,N}-U_{l,N}^{-1}(\widehat{X}_{\hat\sigma}^{J_l}))\right]\right|\right\}}, \tag{4.17}$$

with the abbreviation

$$\Gamma_{l,N} \equiv \sum_{i=1}^{l}\gamma_i \ln(4\pi N \ln\alpha_i)/2 \tag{4.18}$$

(c is replaced by $|c|$ due to the symmetry of the standard Gaussian distribution). The normalizing constant, $R_l(N)$, is chosen such that, for any finite value, U, the point process

$$\sum_{\check\sigma\in\Sigma_{(1-w_l)N}} \delta_{\left\{R_l(N)\left|\check{X}_{\hat\sigma\check\sigma}^{J_l} - \sqrt{N}\left[|c|-d_l-N^{-1}(\Gamma_{l,N}-U)\right]\right|\right\}}, \tag{4.19}$$

converges to the Poisson point processes on $\mathbb{R}_+$, with intensity measure given by e^U times Lebesgue measure, which is possible because $c \in \mathcal{D}_l \backslash \overline{\mathcal{D}_{l-1}}$, that is $|c| - d_l$ is smaller that the limit of $N^{-1/2}\max_{\check\sigma\in\Sigma_{(1-w_l)N}} \check{X}_{\hat\sigma\check\sigma}^{J_l}$. This is completely analogous to the analysis in the domain $\mathcal{D}_0$. Thus each term in the sum over $\hat{\sigma}$ in (4.17) that gives rise to a "finite" $U_{l,N}^{-1}(\widehat{X}_{\hat{\sigma}}^{l})$, i.e., to an element of the extremal process of $\widehat{X}_{\hat{\sigma}}^{l}$, gives rise to one Poisson process with a random intensity measure in the limit of $\mathcal{M}_N^l$. This explains how the statement of the theorem can be understood, and also shows what the geometry of the configurations realizing these mixed Poisson point processes will be.

Let us add that, if $c \in \partial\mathcal{D}_{k-1}$, i.e. $|c| = d_k$, then one has the emergence of the extremal point process (4.15) with $l = k$, i.e.

$$\sum_{\sigma\in\Sigma_N} \delta_{\{\sqrt{N}(X_\sigma - d_k\sqrt{N} + N^{-1/2}\Gamma_{k,N})\}} \to \mathcal{E}_k, \tag{4.20}$$

see [6].

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