Local energy statistics in spin glasses *

Anton Bovier†
Irina Kurkova ‡

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Abstract

Recently, Bauke and Mertens conjectured that the local statistics of energies in random spin systems with discrete spin space should in most circumstances be the same as in the random energy model. We review some rigorous results confirming the validity of this conjecture. In the context of the SK models, we analyse the limits of the validity of the conjecture for energy levels growing with the volume of the system. In the case of the Generalised Random energy model, we give a complete analysis for the behaviour of the local energy statistics at all energy scales. In particular, we show that, in this case, the REM conjecture holds exactly up to energies $E_N < \beta_c N$, where β_c is the critical temperature. We also explain the more complex behaviour that sets in at higher energies.

1 Introduction

The canonical formalism has become the favorite tool to analyse models of statistical mechanics. The main reason for preferring it over the *micro-canonical formalism* is presumably its computational convenience; even in simple examples, the computation of the phase space volume of all states with a given energy appears as a rather complicated problem. In the case

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†Weierstraß–Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany, and Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, 12623 Berlin, Germany, bovier@wias-berlin.de

‡Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6, 4, place Jussieu, B.C. 188, 75252 Paris, Cedex 5, France, kourkova@ccr.jussieu.fr

of *disordered systems* the advantages of the canonical approach are even more apparent, and the recent advances in particular in the theory of mean-field spin-glasses highlight the computational power that this formalism can bring to the analysis of high-dimensional random processes (see these proceedings!).

Rather recently, however, Bauke and Mertens [1] have proposed a new and original look back at the micro-canonical scenario in precisely the case of disordered spin systems. This point of view consists studying very precisely the statistics of configurations whose energy is very close to a given value. In fact, in discrete spin systems, for a given system size, the Hamiltonian will take on a finite number of (random) values, and, at least if the distribution of the disorder is continuous, the probability that a given value E is attained will in fact be zero. One may, however, ask how close the "best" approximant to E will come when the system size grows. More generally, one may ask what the distribution of the energies that come closest to E is, and how the values of the corresponding configurations are distributed in configuration space.

The original motivation for this viewpoint came from a reformulation of a problem in combinatorial optimisation, the *number partitioning problem*, in terms of a spin system Hamiltonian [14, 15]. In the randomized version of this problem one is interested in finding an optimal partition of a set $\{1,\dots,N\}$ into (two) subsets, A, B, such that, for a given assignment of independent identically distributed random variables, X_i, $I = 1,\dots,N$, the sums $\sum_{i\in A} X_i$ and $\sum_{i\in B} X_i$, are as close to each other as possible. It is easy to see that this problem is equivalent to considering the random Hamiltonian

$$H_N(\sigma) = \sum_{i=1}^{N} X_i \sigma_i \tag{1}$$

with $\sigma_i \in \{-1, 1\}$, and searching for the configuration σ such that $H_N(\sigma)$ is as close as possible to the value 0. In this context, Mertens conjectured, that the distributions of the close to optimal values is the same as one would obtain if the random variables $H_N(\sigma)$ were replaced by the random variables $\sqrt{N}X_\sigma$ where X_σ are *independent* standard Gaussian random variables. This conjecture was proven to be correct in [2].

Some time later, [1], Bauke and Mertens generalised this conjecture in the following sense: Let $H_N(\sigma)$ be the Hamiltonian of any disordered spin system with discrete spins and with continuously distributed couplings, and let E be any given real number, then the distribution of the close to optimal approximants of the level $\sqrt{N}E$ is the same as if $H_N(\sigma)$ are replaced by *independent* Gaussian random variables with the same mean and variance as $H_N(\sigma)$. Moreover, they conjectured that the spin configurations realising these approximants are uniformly distributed on configuration space.

All these problems can naturally be considered as extreme value problems for random variables $|N^{-1/2}H_N(\sigma) - E|$, more precisely the minima of these random variables. The main statement of the Bauke-Mertens conjecture can then be formulated in the following form: For suitable normalisation constants $C(N, E)$, the sequence of point processes

$$\sum_{\sigma \in \mathcal{S}_N} \delta_{\{C(N,E)|N^{-1/2}H_N(\sigma)-E|\}}$$

converges, as $N \uparrow \infty$, to the Poisson point process $\mathcal{P}$ in $I\!R_+$ whose intensity measure is the Lebesgue measure.

Such types of results are quite well-known in the context of correlated random sequences, in particular in the case of stationary processes, where such results hold under certain mixing conditions [13]. In the present context, it appears rather surprising that such a result should hold in great generality. Indeed, it is quite well known that the correlations of the random variables are strong enough to modify, e.g. the behaviour of the maxima of the Hamiltonian. Thus there are two questions beyond the original conjecture that naturally pose themselves: (i) assume we consider instead of fixed E, N-dependent energy levels, say, $E_N = cN^\alpha$, $c \in I\!R$. How fast can we allow E_N to grow for the REM-like behaviour to hold? and (ii) what type of behaviour can we expect once E_N grows faster than this value? In this note we discuss some rigorous results around these questions.

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2 Criteria for REM behaviour.

Our approach in [8, 9], and essentially also that of [2, 3, 4, 5] to the proof of the REM conjecture is based on the following general fact about random variables. Let $V_{i,M} \geq 0$, $i \in I\!N$, be a family of positive random variables with identical distributions, that are normalized, s.t.

$$I\!P(V_{i,M} < b) \sim \frac{b}{M}. \tag{2}$$

In the case of independent random variables, it would follow, that the number of $V_{i,M}$ that are smaller than b will have Binomial distribution with parameters M and (approximately) b/M. Assume that (2) holds for all $b \in \mathbb{R}_+$. If we plot the set of all points $V_{i,M}$, the number of points within any subset, $A \subset \mathbb{R}_+ \backslash \{0\}$ will converge to a random variable with Poisson distribution, with parameter the volume of the set A. Moreover, if A and B are two disjoint subsets, then the numbers of points within each respective set will be independent. This means that the random set of points $V_{i,M}$ converges to a *Poisson point process* on $\mathbb{R}_+ \backslash \{0\}$. The question is: under which conditions this is still true, if the random variables $V_{i,M}$ are correlated? It turns out that a very useful sufficient condition is the following:

Lemma 1 *If for any fixed number, ℓ, and any collection of $b_i \geq 0$,*

$$\lim_{M \uparrow \infty} \sum_{(i_1,\ldots,i_\ell) \subset \{1,\ldots,M\}} \mathbb{P}(\forall_{j=1}^{\ell} V_{i_j,M} < b_j) \to \prod_{j=1}^{\ell} b_j, \tag{3}$$

where the sum is taken over all sequences of different indices $(i_1, \ldots, i_\ell)$, then

$$\sum_{i=1}^{M} \delta_{V_{i,M}} \to \mathcal{P} \tag{4}$$

where $\mathcal{P}$ is the Poisson point process on $\mathbb{R}$ with intensity measure the Lebesgue measure.

A proof of this result can be found in Chapter 13 of [6]. Naturally, we would apply this theorem with $V_{i,M}$ given by $|N^{-1/2}H_N(\sigma) - E_N|$, properly normalised.

Remark 1 *As remarked in [5], these conditions are also almost necessary in the following sense. Since we are dealing with random variables, we may always find a proper normalisation $c(N, E_N)$, such that for the random variables $V_{i,M} = c(N, E_N)|N^{-1/2}H_N(\sigma) - E_N|$, (3) holds with $\ell = 1$. This term is also equal to the mean of the number of the $V_{i,M}$ that are smaller than b, i.e. the random variable $N_M(b) \equiv \sum_{i=1}^{M} \mathbb{1}_{V_{i,M}<b}$. Now note that, if we have convergence to a Poisson process, i.e. if $\sum_{i=1}^{M} \delta_{V_{i,M}} \to \mathcal{P}$, then $N_M(b)$ must converge to a Poisson random variable with parameter b. In particular, if a kth moment of $N_M(b)$ converges to a finite value as $M \to \infty$, then all lower-order moments must converge to those of the Poisson distribution with parameter b. It is trivial to check that this is equivalent to saying that,*

if for some $\ell_0 \in N$, the left-hand side of (3) converges to a finite value, than for all $\ell < \ell_0$, (3) must hold. In particular, to disprove the REM conjecture, it is enough to check that (3) does not hold for $\ell = 2$ while the left-hand side converges to some value for $\ell = 3$.

To understand how this lemma can be applied, it is useful to think of the random variables $N^{-1/2}H_N(\sigma)$ as Gaussian random variables with variance one. This holds, if the couplings are Gaussian. Otherwise, it is one of the main steps of the proof to show that they converge, in a strong sense, to Gaussians. If $E_N = E$ independently of N, this requires some continuity assumptions on the distributions of the couplings, otherwise it holds in great generality. If energies E_N that go to infinity are considered, this problem is much harder.

Our general setting is the following. We consider a product space $\mathcal{S}^N$ where $\mathcal{S}$ is a finite set. We define on $\mathcal{S}^N$ a real-valued random process, $N^{-1/2}H_N(\sigma)$, $\sigma \in \mathcal{S}^N$, where we assume that $\mathbb{E}H_N(\sigma) = 0$, $\mathbb{E}(H_N(\sigma))^2 = N$.

One problem we have to deal with from the outset are symmetries. We will not discuss this here in any detail and just replace in all considerations the state space $\mathcal{S}_N$ by the space Σ_N of residual classes modulo the group of automorphisms, G, of $\mathcal{S}_N$, that leave $H_N(\sigma)$ invariant. Let us consider energies

$$E_N = cN^\alpha, \quad c, \alpha \in \mathbb{R}, \quad 0 \leq \alpha < 1/2, \tag{5}$$

and define the sequence

$$\delta_N = \sqrt{\tfrac{\pi}{2}} e^{E_N^2/2} |\Sigma_N|^{-1}. \tag{6}$$

Note that δ_N is exponentially small in $N \uparrow \infty$, since $\alpha < 1/2$. This sequence is chosen such that for any $b \geq 0$,

$$\lim_{N \uparrow \infty} |\Sigma_N| \mathbb{P}(|Z - E_N| < b\delta_N) = b, \tag{7}$$

where Z is standard Gaussian random variable.

In [9] we have formulated a set of geometric conditions in this setting that imply that the hypothesis of Lemma 1 hold with $V_{i,M}$ given by $\delta_N^{-1}|N^{-1/2}H_N(\sigma) - E_N|$ (and $i \to \sigma$, $M \to |\Sigma_N|$), that is

$$\sum_{\substack{(\sigma^1,\dots,\sigma^\ell) \in \Sigma_N \\ \sigma^1,\dots,\sigma^\ell \ different}} \mathbb{P}\left(\forall_{i=1}^{\ell} : |N^{-1/2}H_N(\sigma^i) - E_N| < b_i\delta_N\right) \to b_1 \cdots b_\ell. \tag{8}$$

These assumptions of our theorem are verified in a wide class of physically relevant models. The examples we verified explicitly in [9] are: 1) the Gaussian p-spin SK models and 2) Gaussian short-range spin-glasses. One finds that there are two threshold values for the allowed growth of E_N.

(i) For short-range models and SK-models with $p = 1$ (which is essentially the number-partitioning problem), the REM-conjecture can be verified for $E_N \sim cN^\alpha$, with $\alpha < 1/4$. In the number-partitioning problem, [5] showed that this also holds for non-Gaussian couplings and suitable conditions on the finiteness of exponential moments.

(ii) In the SK-models with $p \geq 2$, the conjecture is valid for $E_N \sim cN^\alpha$, with $\alpha < 1/2$.

As pointed out in [5], these thresholds are sharp in the SK-models with $p = 1$ and $p = 2$, in the sense that the REM conjecture fails if $E_N = cN^{1/4}$, respectively $E_N = cN^{1/2}$, with c small. This can be verified by checking that the conditions (3) fail for $\ell = 2$ in these cases, while the left-hand side converges to some finite number for $\ell = 3$. As pointed out above, this implies that Poisson convergence cannot hold. For $p \geq 3$ we strongly believe that the REM conjecture is still true at the level $E_N = cN^{1/2}$ with c small enough. In this paper we add the discussion of this case.

We have also proved the REM conjecture for short-range spin glass models or p-spin SK models with non-Gaussian couplings, $p \geq 1$, and/or non-zero mean, under assumption $\alpha = 0$, see [9].

3 The REM conjecture in the SK models.

Let us now turn in more details to the REM conjecture in the context of the Sherrington-Kirkpatrick models. Here $\mathcal{S}_N = \{-1, 1\}^N$, and the Hamiltonian is given by

$$H_N(\sigma) = \frac{\sqrt{N}}{N^{p/2}} \sum_{1 \leq i_1, i_2, \ldots, i_p \leq N} J_{i_1,\ldots,i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p} \tag{9}$$

where $J_{i_1,\ldots,i_p}$ are independent standard Gaussian random variables.

3.1 p-spin Sherrington-Kirkpatrick models, $0 \leq \alpha < 1/2$

In [9] we have derived the following theorem.

Theorem 1 *If $p=1$ and $\alpha \in [0,1/4[$, or, if $p \geq 2$, and $\alpha \in [0,1/2[$, then, for any constant $c \in I\!R \setminus \{0\}$ the sequence of point processes*

$$\mathcal{P}_N \equiv \sum_{\sigma \in \mathcal{S}_N} \delta_{\{\delta_N^{-1} |N^{-1/2} H_N(\sigma) - cN^{\alpha}|\}} \tag{10}$$

where $\delta_N = 2^{-N} e^{+c^2 N^{2\alpha}/2} \sqrt{\frac{\pi}{2}}$ converges weakly to the standard Poisson point process, $\mathcal{P}$, on $I\!R_+$.

3.2 p-spin Sherrington-Kirkpatrick models, $\alpha = 1/2$

The obvious question is to know whether the bound on the growth of E_N in the preceding theorem is sharp. In particular, one would want to know whether in the case $p \geq 2$, the REM-conjecture can hold for *extensive* energies, i.e. for $E_N = cN^{1/2}$.

As we will see, there will be a difference between the cases $p = 2$ and $p \geq 3$. As noted in [5], the answer is negative in the former case (see below), while we will show here that it is probably yes in the case $p \geq 3$. Below we give a proof of this fact modulo a technical conjecture.

Let $\sigma^1, \dots, \sigma^\ell$, be ℓ spin configurations. We denote by $m_{ij} \equiv \frac{\sigma^i \cdot \sigma^j}{N} = N^{-1} \sum_{k=1}^N \sigma_k^i \sigma_k^j$. Consider the product space $\mathcal{S}_N^{\otimes \ell}$ endowed with the uniform measure. Let $I(m) \equiv I(m_{1,2}, \dots, m_{\ell-1,\ell})$ be the entropy of the overlaps $\sigma^1 \cdot \sigma^2 / N, \dots, \sigma^{\ell-1} \cdot \sigma^\ell / N$, i.e.

$$I(m_{1,2}, \dots, m_{\ell-1,\ell}) = \ln 2 - \lim_{N \uparrow \infty} \frac{1}{N\ell} \ln \# \left\{ \sigma^1, \dots, \sigma^\ell : \frac{\sigma^i \cdot \sigma^j}{N} = m_{ij}, \forall_{i<j} \right\}. \tag{11}$$

It is useful to note that

$$I(m_{1,2}, \dots, m_{\ell-1,\ell}) = -\ell^{-1} \inf_{\substack{h_{i,j} \geq 0 \\ 1 \leq i < j \leq \ell}} \ln I\!E_\eta \exp\Big(\sum_{1 \leq i < j \leq \ell} (\eta_i \eta_j - m_{i,j}) h_{i,j} \Big) \tag{12}$$

where the η_i are i.i.d. random variables taking the values plus and minus one with equal probability. Let $B_{\ell,p} \equiv B_{\ell,p}(m)$ be the $\ell \times \ell$ the symmetric matrix with the elements $m_{i,j}^p$, for $i < j$, and $m_{i,i} = 1$ on the diagonal. We define the set

$$\mathcal{M}_N^\ell \equiv \left\{ m \in [-1,1]^{\ell(\ell-1)/2} : \exists (\sigma^1, \dots, \sigma^\ell) \in \mathcal{S}_N^\ell : m_{i,j} = \frac{\sigma^i \cdot \sigma^j}{N}, \forall_{i<j} \right\}$$

and

$$\mathcal{M}^\ell \equiv \overline{\lim_{N \uparrow \infty} \mathcal{M}_N^\ell}.$$

Note that the matrix $B_{\ell,p}(m) \geq 0$, for all $m \in \mathcal{M}^\ell$. We set

$$\widetilde{\mathcal{M}}^\ell \equiv \{m \in \mathcal{M}^\ell : B_{\ell,p}(m) > 0\}.$$

For $m \in \widetilde{\mathcal{M}}^\ell$, we denote by $B_{\ell,p}^{-1} \equiv B_{\ell,p}^{-1}(m)$ the inverse of this matrix and we write $b_{i,j}^{-1} = b_{i,j}^{-1}(m)$ for its elements.

Define

$$\rho_p \equiv \sup_{\ell \geq 2} \sup_{m \in \widetilde{\mathcal{M}}^\ell} \frac{1 - \ell^{-1} \sum_{i,j=1}^{\ell} b_{i,j}^{-1}}{I(m)}. \tag{13}$$

Conjecture 1 *For any $p \geq 3$, $\rho_p < \infty$. Moreover, $\lim_{p \uparrow \infty} \rho_p = 1/\ln 2$.*

Remark 2 *It is not difficult to derive from (12) that*

$$I(m) = \frac{1}{2\ell} \sum_{1 \leq i < j \leq \ell} m_{i,j}^2 + o(\|m\|^2), \tag{14}$$

as

$$\|m\| \equiv \max_{1 \leq i < j \leq \ell} |m_{i,j}| \to 0.$$

It is also easy to see from the construction of the inverse matrix that

$$b_{i,j}^{-1} = -m_{i,j}^p(1 + o(1)), \;\; i \neq j, \quad b_{i,i}^{-1} = 1 + O(\|m\|^{2p}), \quad \|m\| \to 0. \tag{15}$$

Consequently,

$$1 - \ell^{-1} \sum_{i,j=1}^{\ell} b_{i,j}^{-1} = \frac{2}{\ell} \sum_{1 \leq i < j \leq \ell} m_{i,j}^p (1 + o(1)) + O(\|m\|^{2p}). \tag{16}$$

Moreover, $\det B_{\ell,p} = 1 + O(\|m\|^{2p})$. It follows form (14) and (16) that, for $p \geq 3$,

$$\frac{1 - \ell^{-1} \sum_{i,j=1}^{\ell} b_{i,j}^{-1}}{I(m)} \leq 5 \max_{i<j} |m_{ij}|^{p-2} \to 0, \quad \|m\| \to 0$$

(we put 5 instead of 4 to absorb the extra error terms). Let us also note that, since $B_{\ell,p}^{-1}$ is a positively defined matrix, $1 - \ell^{-1} \sum_{i,j=1}^{\ell} b_{i,j}^{-1} \leq 1$, for $m \in \mathcal{M}^\ell$. These arguments imply that for any value of $\ell \geq 2$ and any $p \geq 3$

$$\rho_{\ell,p} \equiv \sup_{m \in \widetilde{\mathcal{M}}^\ell} \frac{\ell - \sum_{i,j=1}^{\ell} b_{i,j}^{-1}}{I(m_{i,j})} < \infty.$$

One also can see that

$$\rho_{2,p} = \sup_{-1<m<1} \frac{m^p/(1+m^p)}{(1/4)[(1+m)\ln(1+m)+(1-m)\ln(1-m)]} \geq \frac{1}{\ln 2},$$

thus $\rho_p \geq 1/\ln 2$. *Moreover, we expect* ρ_p *to tend to* $1/\ln 2$, *as* $p \uparrow \infty$. *In fact, in the formal limit, when the matrix* $B_{\ell,\infty}$ *has only elements zero and* ± 1, *depending on whether* $|m_{i,j}| < 1$, *or* $m_{i,j} = \pm 1$, *one may show easily that* $\rho_\infty = 1/\ln 2$.

We will now show the validity of the REM conjecture for $p \geq 3$ for energies $E_N < \sqrt{2\rho_p^{-1} N}$ if $\rho_p < \infty$.

Theorem 2 *Let* $H_N(\sigma)$ *be given by (9),* $p \geq 3$. *The assertion (10) holds true for* $E_N = c\sqrt{N}$ *with* $c^2/2 < \rho_p^{-1}$.

Remark 3 *The theorem is of course void for any p for which Conjecture 1 is false.*

Proof. We have to prove that the limit of the sum

$$\sum_{\substack{\sigma^1,\ldots,\sigma^\ell \in \mathcal{S}_N \\ \sigma^1,\ldots,\sigma^\ell \ different}} I\!P(\forall_{i=1}^{\ell} \ : \ |N^{-1/2} H_N(\sigma^i) - cN| \leq \delta_N b_i) \tag{17}$$

as $N \uparrow \infty$ converges to $b_1 \cdots b_\ell$.

The elements of the matrix $B_{\ell,p} = B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$ are:

$$\mathrm{cov}(N^{-1/2} H_N(\sigma^i), N^{-1/2} H_N(\sigma^j)) = \Big(\frac{\sigma^i \cdot \sigma^j}{N}\Big)^p = m_{i,j}^p.$$

First, let us consider a part of the sum (17) over $\sigma^1, \ldots, \sigma^\ell$ such that the matrix $B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$ is non-degenerate. From the representation of the matrix elements, $\det B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$ is a finite polynomial in the variables $\sigma^i \cdot \sigma^j / N$, thus its inverse can grow at most polynomially. Hence, for any $\sigma^1, \ldots, \sigma^\ell$ with $B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$ non-degenerate, we have the estimate $\det B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)^{-1} \leq N^d$ with some constant $d > 0$ depending on ℓ and p only. Since, in addition to this, under the assumption $c^2/2 < \rho_p^{-1} \leq \ln 2$, δ_N is exponentially small in N and $H_N(\sigma)$ are Gaussian random variables with zero mean and covariance matrix $B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$, it follows that this part of the sum (17) equals

$$\sum_{\substack{\sigma^1,\ldots,\sigma^\ell \ different \\ det\, B_{l,p,N}(\sigma^1,\ldots,\sigma^\ell)>0}} (2\delta_N)^\ell b_1\cdots b_\ell (2\pi)^{-\ell/2} (\det B_{\ell,p,N}(\sigma^1,\ldots,\sigma^\ell))^{-1/2}$$

$$\times \exp\Big(-(\vec{c}\vec{N}) B^{-1}_{\ell,p,N}(\sigma^1,\ldots,\sigma^\ell)(\vec{c}\vec{N})/2\Big)(1+o(1))$$

$$= \sum_{\substack{\sigma^1,\ldots,\sigma^\ell \ different \\ det\, B_{l,p,N}(\sigma^1,\ldots,\sigma^\ell)>0}} |\mathcal{S}_N|^\ell (b_1\cdots b_\ell)(\det B_{\ell,p,N}(\sigma^1,\ldots,\sigma^\ell))^{-1/2}$$

$$\times \exp\Big(c^2 N\Big(\ell - \sum_{i,j=1}^{l} b^{-1}_{i,j}\Big)/2\Big)(1+o(1)) \tag{18}$$

where $(\vec{c}\vec{N})$ is the vector of ℓ coordinates equal to cN, $(o(1))$ is uniform over $\sigma^1,\ldots,\sigma^\ell$ with $\det B_{\ell,p,N}(\sigma^1,\ldots,\sigma^\ell)>0$ as $N\uparrow\infty$.

We split the sum (18) into two sums I_N^1 and I_N^2 : the first one is over $(\sigma^1,\ldots,\sigma^\ell)$ with $\|m\|<N^{-\zeta}$ and the second one with $\|m\|>N^{-\zeta}$ for some fixed $1/3<\zeta<1/2$. Then by (16)

$$I_N^1 = \sum_{\sigma^1,\ldots,\sigma^\ell:\ \|m\|<N^{-\zeta}} |\mathcal{S}_N|^{-\ell} b_1\cdots b_\ell (1+o(1)) \exp\Big(c^2 N \sum_{1\le i<j\le \ell} m^p_{i,j}(1+o(1))\Big). \tag{19}$$

Since $\zeta>1/3$ and $p\ge 3$, we have that $Nm^p_{i,j}=o(1)$ as $N\uparrow\infty$. So, each term in the sum (19) is of the order $|\mathcal{S}_N|^{-\ell} b_1\cdots b_\ell(1+o(1))$, where $|\mathcal{S}_N|^{-\ell}=2^{-N\ell}$. Since $\zeta<1/2$, the number of terms in I_N^1 is bounded from below by $2^{N\ell}(1-\exp(-hN^{1-2\zeta}))$ with some $h>0$ and from above by $2^{N\ell}$. Hence, I_N^1 converges to $b_1\cdots b_\ell$.

To treat the sum I_N^2 over $\|m\|>N^{-\zeta}$, let us estimate it as

$$I_N^2 \le \mathrm{const} N^d \sum_{\substack{m\in\widetilde{\mathcal{M}}^\ell_N: \\ \|m\|>N^{-\zeta}}} \exp\Big(c^2 N(\ell - \sum_{i,j=1}^{l} b^{-1}_{i,j})/2\Big)\exp(-N\ell I(m)). \tag{20}$$

Here $\widetilde{\mathcal{M}}^\ell_N$ is a part of $\mathcal{M}^\ell_N$ where $B_{\ell,p,N}(m)$ is non-degenerate.

Again, by (16), for any $\theta>0$ one can choose $\delta>0$ so small that for all $(\sigma^1,\ldots,\sigma^\ell)$ with $\|m\|<\delta$ we have $\ell-\sum_{i,j} b^{-1}_{i,j}\le(2+\theta)\sum_{1\le i<j\le\ell} m^p_{i,j}$. It follows then from (14), that each term in the sum (20) with $N^{-\zeta}<\|m\|<\delta$ is of the order at most $\exp(-hN^{1-2\zeta})$ with some constant $h>0$. Now, each term in I_N^2 with $\|m\|>\delta$ is bounded by $\exp(-hN)$ with some $h>0$ due to the Conjecture 1, $\rho_p < \infty$, and the fact that $c^2/2 < \rho_p^{-1}$. The number of terms in (20) being polynomial, this completes the analysis of the part of the sum (17) over $\sigma^1, \ldots, \sigma^\ell$ with $\det B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell) > 0$.

It remains to consider parts of the sum (17) where the rank of the matrix $B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell)$ equals $r < \ell$ for $r = 1, 2, \ldots, \ell - 1$. This means that there exists an r-tuple of spin configurations, such that the covariance matrix of their Hamiltonians $B_{r,p,N}$ is non-degenerate and, moreover, the Hamiltonians of the remaining $\ell - r$ spin configurations can be represented as linear combinations of the Hamiltonians of these r configurations. It is well-known [12, 2] that the $N \times r$ matrix of these r spin configurations of the basis can not contain all 2^r different rows, but at most $2^r - 1$: otherwise one of the remaining $\ell - r$ configurations would be equal to one of those r, which is impossible since the sum in (17) is taken over different ℓ configurations. Moreover, it is also known, [2], that there exists an N-independent number of possibilities to complete it by $\ell - r$ appropriate configurations up to $N \times \ell$ matrix. Thus, the part of the sum (17) where $\mathrm{rank} B_{\ell,p,N}(\sigma^1, \ldots, \sigma^\ell) = r < \ell$ is bounded by

$$\binom{r}{\ell} \sum_{\substack{\sigma^1,\ldots,\sigma^\ell \ different \\ det\, B_{r,p,N}(\sigma^1,\ldots,\sigma^r)>0 \\ det\, B_{k,p,N}(\sigma^1,\ldots,\sigma^k)=0,\ k=r+1,\ldots,\ell}} I\!\!P(\forall_{i=1}^r \ : \ |N^{-1/2} H_N(\sigma^i) - cN| \leq \delta_N b_i) \tag{21}$$

where the number of terms in the sum is $O((2^r - 1)^N)$. The part of this sum over $\max_{i,j \in \{1,\ldots,r\}} |m_{i,j}| > N^{-\zeta}$ converges to zero by the same arguments as in the case of $B_{\ell,p,N}$ non-degenerate. The part over $\max_{i,j \in \{1,\ldots,r\}} |m_{i,j}| < N^{-\zeta}$ have terms of order $2^{-Nr}(1 + o(1))$ by the same arguments as in the case of $B_{\ell,p,N}$ non-degenerate as well. But since the number of terms in this the sum is $O((2^r - 1)^N)$, it converges to zero exponentially fast. This completes the proof.

Let us finally turn to the negative results in the cases $p = 2$ and $p = 1$.

Case $p = 2$**.** We proceed along the lines of the proof of the previous theorem. Using (14) and (16), the sum I_N^1 is

$$I_N^1 = (2\pi N)^{-\ell(\ell-1)/4} \sum_{\substack{m_{i,j} \in \mathcal{M}_N^\ell: \\ |m| < N^{-\zeta}}} (b_1 \cdots b_\ell) \exp\Big(c^2 N \sum_{1 \leq i < j \leq \ell} m_{i,j}^2 (1 + o(1))\Big)$$
$$\times \exp\Big(- N \sum_{1 \leq i < j \leq \ell} m_{i,j}^2 (1 + o(1))/2\Big). \tag{22}$$

The change of variables $s_{i,j} = m_{i,j}\sqrt{N}$ shows that when $2c^2 < 1$, I_N^1 converges to the integral

$$\int_{\mathbb{R}^{\ell(\ell-1)/2}} (2\pi)^{-\ell(\ell-1)/4}(b_1 \cdots b_\ell) \prod_{1\le i<j\le \ell} \exp\Big(s_{i,j}^2(2c^2-1)/2\Big) ds_{i,j}$$

which is $(b_1\cdots b_\ell)(1-2c^2)^{-\ell(\ell-1)/4}$. In addition, for given ℓ, if $c^2/2 < \rho_{2,\ell}^{-1}$ where $\rho_{2,\ell} = \sup_{m\in\widetilde{\mathcal{M}}^\ell} \frac{1-\ell^{-1}\sum_{i,j=1}^{\ell} b_{i,j}^{-1}}{I(m)}$, then the same arguments as in the case $p \ge 3$ prove that the remaining part of the sum (17) converges to 0. Hence, in the case $p = 2$, for $c^2 < 2\rho_{2,\ell}^{-1}$, the sum (17) converges to $(b_1\cdots b_\ell)(1-2c^2)^{-\ell(\ell-1)/4}$. In view of the remark after Lemma 1, this implies that for small enough c, we cannot have convergence to a Poisson point process. This was first observed in [5].

Case $p = 1$. The case $p = 1$ is the simplest one. Here we can exclude Poisson convergence for $E_N = cN^{1/4}$, for any $c < \infty$, not just for small enough c. Using (14) and (16), the sum I_N^1 is

$$I_N^1 = (2\pi N)^{-\ell(\ell-1)/4} \sum_{\substack{m\in\mathcal{M}_N^\ell \\ |m|>N^{-\zeta}}} (b_1\cdots b_\ell)\exp\Big(c^2\sqrt{N}\sum_{1\le i<j\le\ell} m_{i,j}(1+o(1))\Big)$$
$$\times \exp\Big(-N\sum_{1\le i<j\le\ell} m_{i,j}^2(1+o(1))/2\Big). \quad (23)$$

The same change of variables $s_{i,j} = m_{i,j}\sqrt{N}$ shows that I_N^1 converges to the integral

$$\int_{\mathbb{R}^{\ell(\ell-1)/2}} (2\pi)^{-\ell(\ell-1)/4}(b_1\cdots b_\ell) \prod_{1\le i<j\le\ell}\prod_{1\le i<j\le\ell} \exp(s_{i,j}c^2 - s_{i,j}^2/2) ds_{i,j}$$

which is $(b_1\cdots b_\ell)\exp(c^4/2)^{\ell(\ell-1)/2}$. Using (14) for $\|m\|$ small enough and the fact that $\alpha = 1/4$ for $\|m\|$ bounded from below, it is easy to see that the remaining part of the sum (17) converges to 0.

In both cases we see that the probability that ℓ variables are small is larger by a quickly increasing factor const$^{\ell(\ell-1)}$ over the Poisson case, which indicates that “good approximants of E_N” will tend to “lump” together, which is the effect we should expect from the increasing importance of correlations. It would be very nice to have a more complete and explicit description of the limiting process. Unfortunately, it seems that even in the case

$p = 1$, where we have all moments, these grow too fast with ℓ to determine e.g. the distribution of the number of points in an interval.

In the next section we turn to another class of models, where we are able to get a complete picture of what happens after the REM conjecture fails.

4 Beyond REM behaviour

In the particular case of the Generalised Random Energy models (GREMs), it is possible to analyse completely not only what the precise threshold for the validity of the REM-conjecture is, but also what happens for higher energy levels. We briefly summarize these results now. As we will see, in these models there will be a rather clear link between properties of the local energy statistics and the properties of the Gibbs measures, quite in contrast to the findings on the SK models or in short-range models, where non-REM behaviour sets in already at energy levels that are not linked to equilibrium phase transitions.

Let us briefly recall the definition of the GREM. We consider parameters $\alpha_0 = 1 < \alpha_1, \dots, \alpha_n < 2$ with $\prod_{i=1}^n \alpha_i = 2$, $a_0 = 0 < a_1, \dots, a_n < 1$, $\sum_{i=1}^n a_i = 1$. Let $X_{\sigma_1 \cdots \sigma_\ell}$, $\ell = 1, \dots, n$, be independent standard Gaussian random variables indexed by $\sigma_1 \dots \sigma_\ell \in \{-1, 1\}^{N \ln(\alpha_1 \cdots \alpha_\ell)/\ln 2}$. The Hamiltonian of the GREM is $H_N(\sigma) \equiv \sqrt{N} X_\sigma$, with

$$X_\sigma \equiv \sqrt{a_1} X_{\sigma_1} + \cdots + \sqrt{a_n} X_{\sigma_1 \cdots \sigma_n}. \tag{24}$$

Then $\operatorname{cov}(X_\sigma, X_{\sigma'}) = A(d_N(\sigma, \sigma'))$, where $d_N(\sigma, \sigma') = N^{-1}[\min\{i : \sigma_i \neq \sigma'_i\} - 1]$, and $A(x)$ is a right-continuous step function on $[0, 1]$ with $A(x) = a_0 + \cdots + a_i$, if $x \in [\ln(\alpha_0 \alpha_1, \cdots \alpha_i)/\ln 2\,, \ln(\alpha_0 \alpha_1, \cdots \alpha_{i+1})/\ln 2)$. We will assume here for simplicity that the linear envelope of the function A is convex.

To formulate our results, we also need to recall from [7] (Lemma 1.2) the point process of Poisson cascades $\mathcal{P}^\ell$ on $\mathbb{R}^\ell$. It is best understood in terms of the following iterative construction. If $\ell = 1$, $\mathcal{P}^1$ is the Poisson point process on $\mathbb{R}^1$ with the intensity measure $e^{-x}dx$. To construct $\mathcal{P}^\ell$, we place the process $\mathcal{P}^{\ell-1}$ on the plane of the first $\ell - 1$ coordinates and through each of its points draw a straight line orthogonal to this plane. Then we put on each of these lines independently a Poisson point process with intensity measure $e^{-x}dx$. These points on $\mathbb{R}^\ell$ form the process $\mathcal{P}^\ell$.

Let us define the constants d_ℓ, $\ell = 0, 1, \dots, n$, where $d_0 = 0$ and

$$d_\ell \equiv \sum_{i=1}^{\ell} \sqrt{a_i 2 \ln \alpha_i}. \tag{25}$$

Finally, set, for $\ell = 0, \ldots, k-1$, as

$$D_\ell \equiv d_\ell + \sqrt{\frac{2\ln\alpha_{\ell+1}}{a_{\ell+1}}} \sum_{j=\ell+1}^{k} a_j. \tag{26}$$

It is not difficult to verify that $D_0 < D_1 < \cdots < D_{n-1}$. Interestingly, the border of D_0 is the point β_c, that is the critical temperature of the respective model. We are now ready to formulate the main result.

Theorem 3 *If $|c| < D_0 = \beta_c$, then, the point process*

$$\mathcal{M}_N^0 = \sum_{\sigma \in \mathcal{S}_N} \delta_{\left\{2^{N+1}(2\pi)^{-1/2} e^{-c_N^2 N/2}\left|X_\sigma - c_N\sqrt{N}\right|\right\}} \tag{27}$$

converges to the Poisson point process with intensity measure the Lebesgue measure.

Theorem 4 *If for $\ell = 1, \ldots, n-1$, $D_{\ell-1} \le c < D_\ell$, set*

$$\tilde{c}_\ell = |c| - d_\ell, \tag{28}$$

$$\beta_\ell = \frac{\tilde{c}_\ell}{a_{\ell+1} + \cdots + a_n}, \qquad \gamma_i = \sqrt{a_i/(2\ln\alpha_i)}, \quad i = 1, \ldots, \ell, \tag{29}$$

and

$$R_\ell(N) = \frac{2(\alpha_{\ell+1}\cdots\alpha_k)^N \exp(-N\tilde{c}_\ell\beta_\ell/2)}{\sqrt{2\pi(a_{\ell+1} + \cdots + a_k)}} \prod_{j=1}^{\ell} (4N\pi\ln\alpha_j)^{-\beta_\ell\gamma_j/2}. \tag{30}$$

Then, the point process

$$\mathcal{M}_N^\ell = \sum_{\sigma \in \mathcal{S}_N} \delta_{\left\{R_\ell(N)\left|\sqrt{a_1}X_{\sigma_1} + \cdots + \sqrt{a_n}X_{\sigma_1\cdots\sigma_n} - c\sqrt{N}\right|\right\}} \tag{31}$$

converges to mixed Poisson point process on $[0,\infty[$: given a realization of the random variable Λ_ℓ, its intensity measure is $\Lambda_\ell dx$. The random variables Λ_ℓ is defined in terms of the Poisson cascades $\mathcal{P}_\ell$ via

$$\Lambda_\ell = \int_{\mathbb{R}^\ell} e^{\beta_\ell(\gamma_1 x_1 + \cdots \gamma_\ell x_\ell)} \mathcal{P}^\ell(dx_1, \ldots, dx_\ell). \tag{32}$$

The proof of these theorems is given in [10]. Here we give a heuristic interpretation of the main result.

Let us first look at (27). This statement corresponds to the REM-conjecture of Bauke and Mertens [1]. It is quite remarkable that this conjecture holds in the case of the GREM for energies of the form cN (namely for $c \in \mathcal{D}_0$).

In the REM [11], X_σ are 2^N *independent* standard Gaussian random variables and a statement (27) would hold for all c with $|c| < \sqrt{2\ln 2}$: it is a well known result from the theory of independent random variables [13]. The value $c = \sqrt{2\ln 2}$ corresponds to the maximum of 2^N independent standard Gaussian random variables, i.e., $\max_{\sigma\in\mathcal{S}_N} N^{-1/2} X_\sigma \to \sqrt{2\ln 2}$ a.s. Therefore, at the level $c = \sqrt{2\ln 2}$, one has the emergence of the extremal process. More precisely, the point process

$$\sum_{\sigma\in\Sigma_N} \delta_{\left\{\sqrt{2N\ln 2}\left(X_\sigma - \sqrt{2N\ln 2} + \ln(4\pi N\ln 2)/\sqrt{8N\ln 2}\right)\right\}}, \tag{33}$$

that is commonly written as $\sum_{\sigma\in\mathcal{S}_N} \delta_{u_N^{-1}(X_\sigma)}$ with

$$u_N(x) = \sqrt{2N\ln 2} - \frac{\ln(4\pi N\ln 2)}{2\sqrt{2N\ln 2}} + \frac{x}{\sqrt{2N\ln 2}}, \tag{34}$$

converges to the Poisson point process $\mathcal{P}^1$ defined above (see e.g. [13]). For $c > \sqrt{2\ln 2}$, the probability that one of the X_σ will be outside of the domain $\{|x| < c\sqrt{N}\}$, goes to zero, and thus it makes no sense to consider such levels.

In the GREM, $N^{-1/2}\max_{\sigma\in\mathcal{S}_N} X_\sigma$ converges to the value $d_k \in \partial D_{k-1}$ (25) (see Theorem 1.5 of [7]) that is generally smaller than $\sqrt{2\ln 2}$. Thus it makes no sense to consider levels with $c \notin \overline{D}_{k-1}$. However, the REM-conjecture is not true for all levels in $\mathcal{D}_{k-1}$, but only in the smaller domain $\mathcal{D}_0$.

To understand the statement of the theorem outside $\mathcal{D}_0$, we need to recall how the extremal process in the GREM is related to the Poisson cascades introduced above. Let us set $\mathcal{S}_{Nw_\ell} \equiv \{-1,1\}^{Nw_\ell}$ where

$$w_\ell = \ln(\alpha_1\cdots\alpha_\ell)/\ln 2 \tag{35}$$

and define the functions

$$U_{\ell,N}(x) \equiv N^{1/2}d_\ell - N^{-1/2}\sum_{i=1}^{\ell}\gamma_i\ln(4\pi N\ln\alpha_i)/2 + N^{-1/2}x \tag{36}$$

Set

$$\widehat{X}_\sigma^j \equiv \sum_{i=1}^{j} \sqrt{a}_i X_{\sigma_1\dots\sigma_i}, \quad \check{X}_\sigma^j \equiv \sum_{i=j+1}^{n} \sqrt{a}_i X_{\sigma_1\dots\sigma_i}. \tag{37}$$

From what was shown in [7], for any $\ell = 1, \dots, n$, the point process,

$$\mathcal{E}_{\ell,N} \equiv \sum_{\hat{\sigma}\in\mathcal{S}_{Nw_\ell}} \delta_{U_{\ell,N}^{-1}(\widehat{X}_{\hat{\sigma}}^{J_\ell})} \tag{38}$$

converges in law to the Poisson cluster process, $\mathcal{E}_\ell$, given in terms of the Poisson cascade, $\mathcal{P}^\ell$, as

$$\mathcal{E}_\ell \equiv \int_{\mathbb{R}^\ell} \mathcal{P}^{(\ell)}(dx_1, \dots, dx_\ell)\delta_{\sum_{i=1}^\ell \gamma_i x_i}. \tag{39}$$

In view of this observation, we can re-write the definition of the process $\mathcal{M}_N^\ell$ as follows:

$$\mathcal{M}_N^\ell = \sum_{\hat{\sigma}\in\mathcal{S}_{w_\ell N}} \sum_{\check{\sigma}\in\mathcal{S}_{(1-w_\ell)N}} \delta_{\left\{R_\ell(N)\left|\check{X}_{\hat{\sigma}\check{\sigma}}^{J_\ell} - \sqrt{N}\left[|c|-d_\ell - N^{-1}(\Gamma_{\ell,N} - U_{\ell,N}^{-1}(\widehat{X}_{\hat{\sigma}}^{J_\ell}))\right]\right|\right\}}, \tag{40}$$

with the abbreviation

$$\Gamma_{\ell,N} \equiv \sum_{i=1}^{\ell} \gamma_i \ln(4\pi N \ln \alpha_i)/2 \tag{41}$$

(c is replaced by $|c|$ due to the symmetry of the standard Gaussian distribution). The normalizing constant, $R_\ell(N)$, is chosen such that, for any finite value, U, the point process

$$\sum_{\check{\sigma}\in\mathcal{S}_{(1-w_\ell)N}} \delta_{\left\{R_\ell(N)\left|\check{X}_{\hat{\sigma}\check{\sigma}}^{J_\ell} - \sqrt{N}\left[|c|-d_\ell - N^{-1}(\Gamma_{\ell,N} - U)\right]\right|\right\}}, \tag{42}$$

converges to the Poisson point processes on $\mathbb{R}_+$, with intensity measure given by e^U times Lebesgue measure, which is possible because $c \in \mathcal{D}_\ell \backslash \overline{\mathcal{D}_{\ell-1}}$, that is $|c| - d_\ell$ is smaller that the limit of $N^{-1/2} \max_{\check{\sigma}\in\mathcal{S}_{(1-w_\ell)N}} \check{X}_{\hat{\sigma}\check{\sigma}}^{J_\ell}$. This is completely analogous to the analysis in the domain $\mathcal{D}_0$. Thus each term in the sum over $\hat{\sigma}$ in (40) that gives rise to a "finite" $U_{\ell,N}^{-1}(\widehat{X}_{\hat{\sigma}}^\ell)$, i.e., to an element of the extremal process of $\widehat{X}_{\hat{\sigma}}^\ell$, gives rise to one Poisson process with a random intensity measure in the limit of $\mathcal{M}_N^\ell$. This explains how the statement of the theorem can be understood, and also shows what the geometry of the configurations realizing these mixed Poisson point processes will be.

Let us add that, if $c \in \partial\mathcal{D}_{k-1}$, i.e. $|c| = d_k$, then one has the emergence of the extremal point process (38) with $\ell = k$, i.e.

$$\sum_{\sigma \in \mathcal{S}_N} \delta_{\{\sqrt{N}(X_\sigma - d_k\sqrt{N} + N^{-1/2}\Gamma_{k,N})\}} \to \mathcal{E}_k, \tag{43}$$

see [7].

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