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POISSONIAN STATISTICS IN THE EXTREMAL PROCESS OF BRANCHING BROWNIAN MOTION[1]

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As a first step toward a characterization of the limiting extremal process of branching Brownian motion, we proved in a recent work [*Comm. Pure Appl. Math.* **64** (2011) 1647–1676] that, in the limit of large time t, extremal particles descend with overwhelming probability from ancestors having split either within a distance of order 1 from time 0, or within a distance of order 1 from time t. The result suggests that the extremal process of branching Brownian motion is a randomly shifted cluster point process. Here we put part of this picture on rigorous ground: we prove that the point process obtained by retaining only those extremal particles which are also maximal inside the clusters converges in the limit of large t to a random shift of a Poisson point process with exponential density. The last section discusses the *Tidal Wave Conjecture* by Lalley and Sellke [*Ann. Probab.* **15** (1987) 1052–1061] on the full limiting extremal process and its relation to the work of Chauvin and Rouault [*Math. Nachr.* **149** (1990) 41–59] on branching Brownian motion with atypical displacement.

1. Introduction. Branching Brownian motion (BBM) is a continuous-time Markov branching process which plays an important role in the theory of partial differential equations [4, 5, 23], in particle physics [24] and in the theory of disordered systems [6, 14]. It is also widely used in biology to model the genealogies of evolving populations, the spread of advantageous genes, etc., [15, 19]. It is constructed as follows.

Start with a standard Brownian motion (BM) (we will often refer to Brownian motions as "particles"), $x(t)$, starting at 0. After an exponential random time, T, of mean 1, the BM splits into k independent BMs, independent of x and T, with probability p_k, where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$ and $K \equiv \sum_k k(k-1)p_k < \infty$.

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Each of these processes continues in the same way as first BM. Thus, after time $t > 0$, there will be $n(t)$ BMs located at $x_1(t), \dots, x_{n(t)}(t)$, with $n(t)$ being the random number of offspring generated up to that time [note that $\mathbb{E}n(t) = e^t$].

An interesting link between BBM and partial differential equations was observed by McKean [23]: denote by

$$u(t,x) \equiv \mathbb{P}\Big[\max_{1\le k\le n(t)} x_k(t) \le x\Big] \tag{1.1}$$

the law of the maximal displacement. Then, a renewal argument shows that $u(t,x)$ solves the Kolmogorov–Petrovsky–Piscounov or Fisher [F-KPP] equation,

$$\begin{gathered} u_t = \frac{1}{2}u_{xx} + \sum_{k=1}^{\infty} p_k u^k - u, \\ u(0,x) = \begin{cases} 1, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases} \end{gathered} \tag{1.2}$$

The F-KPP equation admits traveling waves: there exists a unique solution satisfying

$$u\big(t, m(t) + x\big) \to \omega(x) \qquad \text{uniformly in } x \text{ as } t \to \infty, \tag{1.3}$$

with the centering term, the *front* of the wave, given by

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t, \tag{1.4}$$

and $\omega(x)$ the unique (up to translation) distribution function which solves the ordinary differential equation

$$\tfrac{1}{2}\omega_{xx} + \sqrt{2}\omega_x + \omega^2 - \omega = 0. \tag{1.5}$$

The leading order of the front has been established by Kolmogorov, Petrovsky and Piscounov [20], whereas the logarithmic corrections have been obtained by Bramson [8], using the probabilistic representation given above.

The limiting law of the maximal displacement has been studied intensely. Let

$$Z(t) \equiv \sum_{k=1}^{n(t)} \big(\sqrt{2}t - x_k(t)\big) \exp -\sqrt{2}\big(\sqrt{2}t - x_k(t)\big) \tag{1.6}$$

denote the so-called *derivative martingale*. Lalley and Sellke [21] proved that $Z(t)$ converges almost surely to a strictly positive random variable, Z, and established the integral representation

$$\omega(x) = \mathbb{E}\big[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}x}}\big], \tag{1.7}$$

with $C > 0$ a constant. Thus the law of the maximum of BBM is a *random shift* of the Gumbel distribution. Moreover, it is known that

$$1 - \omega(x) \sim x\mathrm{e}^{-\sqrt{2}x}, \qquad x \to +\infty, \tag{1.8}$$

where $\sim$ means that the ratio of the terms converges to a positive constant; see, for example, Bramson [8] and Harris [17]. (There is emerging evidence that right-tails such as (1.8), manifestly different from those of the Gumbel, play an important role in a number of different fields, e.g., in models on spin glasses with logarithmic correlated potentials by Carpentier and Le Doussal [12], and Fyodorov and Bouchaud [16].)

Contrary to the maximal displacement, very little is known on the full statistics of the extremal configurations (first-, second-, third-, etc., largest) in BBM. Such statistics are completely encoded in the extremal process, which is the random point measure associated to the collection of points shifted by the expectation of their maximum, that is, the point process

$$\Xi(t) \equiv \sum_{i=1}^{n(t)} \delta_{\overline{x_i}(t)}, \qquad \overline{x_i}(t) \equiv x_i(t) - m(t). \tag{1.9}$$

The key issue of interest is to characterize the limit of this process, as $t \uparrow \infty$. It can be shown that the limit of the point process exists using Bramson's analysis [7] on the convergence of solutions of the KPP equations with appropriate initial conditions [9, 18].

For given realization of the branching, the positions $\{x_i(t)\}_{i\le n(t)}$ form a Gaussian process indexed by $i \in \{1, \dots, n(t)\} \equiv \Sigma_t$ with correlations given by the *genealogical distance*

$$Q_{ij}(t) \equiv \sup\{s \le t : x_i(s) = x_j(s)\} \tag{1.10}$$

(the time to first branching of the common ancestor). The information about the correlation structure of any subsets of particles in BBM is encoded in their genealogical distance. This applies, in particular, to the subset of extremal particles, for which the following result was proved in [3]: with probability tending to 1, branching can happen only at "very early times," smaller than r_d with $r_d = O(1)$ as $t \to \infty$, or at times "very close" to the age of the system, namely greater than $t - r_g$ for $r_g = O(1)$ as $t \to \infty$. (The reason for this notation, in particular the use of the subscripts, will be explained below.) More precisely, denoting by $\Sigma_t(D) \equiv \{i \in \Sigma_t : \overline{x_i}(t) \in D\}$ the set of particles in the subset $m(t) + D$, we have:

THEOREM 1 [3]. *For any compact $D \subset \mathbb{R}$,*

$$\lim_{r_d, r_g \to \infty} \sup_{t > 3\max\{r_d, r_g\}} \mathbb{P}[\exists i, j \in \Sigma_t(D) : Q_{ij}(t) \in (r_d, t - r_g)] = 0. \tag{1.11}$$

Figure 1 presents a graphical representation of the genealogies of extremal particles of BBM.

Theorem 1 gives insight into the limiting extremal process of BBM. In fact, it suggests the following picture, which holds with overwhelming probability in the limit when first $t \uparrow \infty$, and $r_d, r_g \to \infty$ after that.

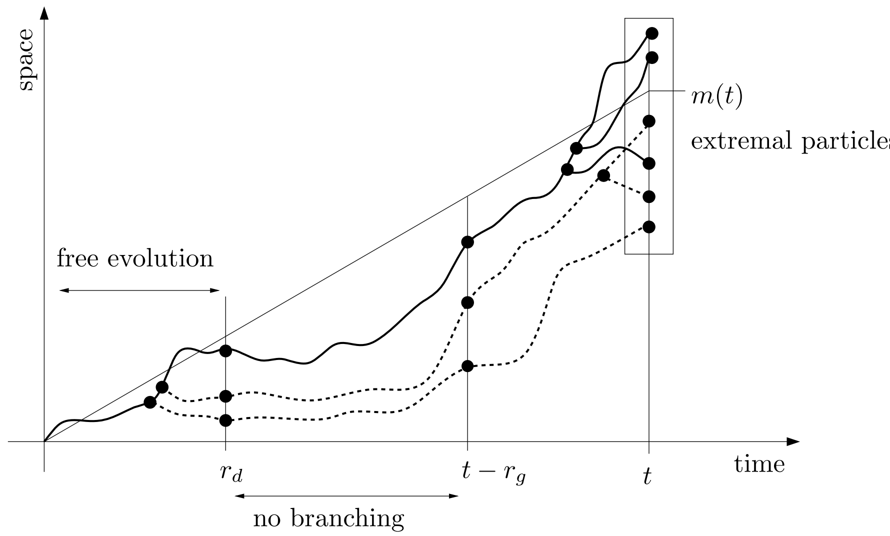

FIG. 1. *Genealogies of extremal particles.*

First, ancestries in the interval $[0, r_d]$ cannot be ruled out: this regime generates the derivative martingale appearing in the work of Lalley and Sellke [21]. Moreover, since the ancestors of the extremal particles evolved independently for most of the time (namely in the interval $[r_d, t - r_g]$), the extremal process must exhibit a structure similar to a Poisson process. Finally, since ancestors over the period $[t - r_g, t]$ also occur, it is natural to conjecture that *small grapes* of length at most $r_g = O(1)$, that is, clusters of particles with very recent common ancestor, appear at the end of the time-interval. (According to this picture, the subscript in r_d refers to *derivative martingale*, while that in r_g stands for *grape*.)

It is the purpose of this work to make part of this picture rigorous. In Section 2 we present our main result, which is proved in Section 3. In Section 4, we introduce a cluster point process, which we conjecture to correspond in the limit to the extremal process of BBM. We also discuss the cluster point process in relation to the work of Chauvin and Rouault [13] on BBM conditioned to perform unusually large displacements, and in relation to the *Tidal Wave Conjecture* of Lalley and Sellke [21]. Detailed properties of this cluster point process will be the subject of a subsequent paper [2].

2. Main results. Despite the rather clear image described above, a frontal attack on the extremal process appears to be difficult. This is in particular due to the fact that one has to take into account the self-similarity of BBM which is first and foremost detectable in the small clusters, an issue which remains rather elusive (see Section 4 for more on this). On the other hand, the picture naturally suggests the existence of an underlying point process obtained from the extremal particles by a thinning procedure, which we describe next.

Assume that the positions of particles at time t are ordered in decreasing order:

$$\overline{x}_1(t) \geq \overline{x}_2(t) \geq \cdots \geq \overline{x}_{n(t)}(t). \tag{2.1}$$

The inequalities will in fact be strict for almost all realizations of BBM for any deterministic time t. Define also

$$\overline{Q}(t) = \{\overline{Q}_{ij}(t)\}_{i,j \leq n(t)} \equiv \{t^{-1} Q_{ij}(t)\}_{i,j \leq n(t)}. \tag{2.2}$$

The pair $(\Xi(t), \overline{Q}(t))$ admits the following natural *thinning*. Since the matrix $\overline{Q}(t)$ is constructed from the branching of the BBM, the relation $\overline{Q}_{ij}(t) \geq q$ is transitive for any $q \geq 0$:

$$\overline{Q}_{ij}(t) \geq q \quad \text{and} \quad \overline{Q}_{jk}(t) \geq q \quad \Longrightarrow \quad \overline{Q}_{ik}(t) \geq q. \tag{2.3}$$

In particular, for any $q > 0$, this relation defines an equivalence relation on the set $\{1, \ldots, n(t)\}$. The corresponding equivalence classes are just the particles at time t that had a common ancestor at a time later than tq. We want to select a representative of each class, namely the maximal particle within each class, and then consider the point process of these representatives. For any $q > 0$, the q*-thinning* of the process $(\Xi(t), \overline{Q}(t))$, denoted by $\Xi^{(q)}(t)$, is defined recursively as follows:

$$\begin{aligned} &i_1 = 1; \\ &i_k = \min\{j > i_{k-1} : \overline{Q}_{i_l j}(t) < q, \forall l \leq k-1\}; \end{aligned} \tag{2.4}$$

and

$$\Xi^{(q)}(t) \equiv \bigl(\Xi_k^{(q)}(t), k \in \mathbb{N}\bigr) \equiv \bigl(\overline{x}_{i_k}(t), k \in \mathbb{N}\bigr), \tag{2.5}$$

where it is understood that $\Xi_k^{(q)}(t) = 0$ when an index i_k in $\{1, \ldots, n(t)\}$ satisfying $\min\{j > i_{k-1} : \overline{Q}_{i_l j} < q \ \forall l \leq k-1\}$ can no longer be found. The procedure selects the maximal position in each equivalence class defined from the relation $\overline{Q}_{ij}(t) \geq q$. In addition, it is easily checked that the thinning map,

$$(\Xi(t), \overline{Q}(t)) \mapsto \Xi^{(q)}(t), \tag{2.6}$$

considered at the level of realizations, is a continuous function on the space of pairs (X, Q), where X is a sequence of ordered positions and Q is a symmetric matrix with entries in $[0, 1]$, satisfying (2.3) (when this space is equipped with the product topology in each coordinate of X and Q).

The thinning map can also be applied to t-dependent values of q. For example, take $q = q(t) = 1 - r_g/t$, where r_g is fixed t. In this case, the thinning effectively retains those particles which are extremal within the class defined by a "very recent" common ancestor, which we refer to as *cluster-extrema*. Figure 2 presents a graphical representation of the set of such particles.

Our main result states that all such thinned processes converge to the same randomly shifted Poisson Point Process (PPP for short) with exponential density.

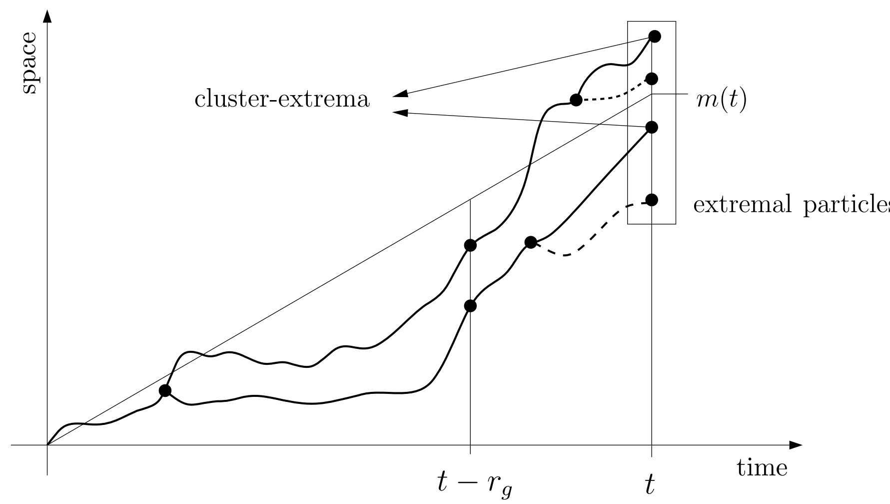

FIG. 2. *Cluster-extrema.*

THEOREM 2. *For any* $0 < q < 1$, *the processes* $\Xi^{(q)}(t)$ *converge in law to the same limit*, Ξ^0. *Also*,

$$\lim_{r_g \to \infty} \lim_{t \to \infty} \Xi^{(1-r_g/t)}(t) = \Xi^0. \tag{2.7}$$

Moreover, conditionally on Z, *the limit of the derivative martingale* (1.6),

$$\Xi^0 = \mathrm{PPP}\big(C \cdot Z \cdot \sqrt{2}\mathrm{e}^{-\sqrt{2}x}\, \mathrm{d}x\big), \tag{2.8}$$

where $C > 0$ *is the constant appearing in* (1.8).

The point process Ξ^0 has a fundamental connection with the limiting extremal process of BBM. To see this, suppose for simplicity that the processes $(\Xi(t), \overline{Q}(t))$ induced by the law of BBM converge, as $t \uparrow \infty$, to a process, $(\Xi, \overline{Q})$. (The laws of these processes are in fact tight because the law of $\Xi(t)$ is itself tight; see, e.g., Corollary 2.3 in [3], and that $\overline{Q}_{ij}(t) \in [0, 1]$ for any i, j. Convergence would evidently follow from a complete characterization of the extremal process.) It follows from Theorem 1 that $\overline{Q}_{ij}$ is either 0 or 1. This suggests:

(1) to define a cluster of particles as the maximal set of particles such that $\overline{Q}_{ij} = 1$ for all i, j in the set;
(2) to look at the process of the maxima of each cluster, denoted by, say, $\tilde{\Xi}^0$, defined as in (2.4), but where $i_k = \min\{j > i_{k-1} : \overline{Q}_{i_l j} = 0\ \forall l \le k-1\}$.

We claim that $\tilde{\Xi}^0$ is in fact the limit Ξ^0 of $\Xi^{(q)}(t)$ in Theorem 2. Indeed, in view of the continuity of the thinning map (2.6), $\Xi^{(q)}(t)$ converges to the q-thinned process, $\Xi^{(q)}$, constructed from $(\Xi, \overline{Q})$ for all q. But, for any $0 < q < 1$, the q-thinned processes, $\Xi^{(q)}$, constructed from $(\Xi, \overline{Q})$ using (2.4) are equal trivially to $\tilde{\Xi}^0$, since $\overline{Q}_{ij}$ is either 0 or 1. The claim then follows from Theorem 2. The point process describing the particles at the frontier of BBM in the limit of large times is thus formed by two "types" of particles: those coming from the randomly shifted PPP with exponential density, the cluster-extrema; and the second type of particles, those forming the clusters. Clearly, particles in the same cluster always lie on the left of the corresponding Poissonian particles, by the very definition of the cluster-extrema. It remains an open question to characterize the law of the clusters (see Section 4 for some conjectures).

We remark that, since $r_g = O(1)$ as $t \to \infty$, the thinned process $\Xi_t^{(1-r_g/t)}$ is obtained from the extremal one by removing only a small number of particles, those which have genealogical distance smaller than $t - r_g$ from the maximum in their class. It is rather surprising at first sight (but not quite when seen under the light of Theorem 1) that such a point process converges, despite the high correlations among the branching Brownian particles, to a PPP with exponential density.

Theorem 2 also provides insights into a result by Bovier and Kurkova [6], who addressed the weak limit of the Gibbs measure of BBM, the random probability measure on Σ_t attaching weights

$$\mathcal{G}_{\beta,t}(k) \equiv \frac{\exp(\beta x_k(t))}{\mathcal{Z}_t(\beta)}, \qquad \mathcal{Z}_t(\beta) \equiv \sum_{j \in \Sigma_t} \exp(\beta x_j(t)), \tag{2.9}$$

where $\beta > 0$ is the inverse of temperature. To see this, let us first recall the following.

Consider the random set $(\xi_i, i \in \mathbb{N})$ where the ξ's are generated according to a PPP with density $CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x$ on the real axis, C and Z as in Theorem 2. Construct then a new random set $(\rho_i, i \in \mathbb{N})$ where $\rho_i \equiv \exp(\beta\xi_i)$. For $\beta > \sqrt{2}$, it is easily seen that $\mathcal{N}(\rho) \equiv \sum_j \rho_j < \infty$ almost surely, in which case the normalization $\hat{\rho}_i \equiv \rho_i/\mathcal{N}(\rho)$ is well defined, and the law of the normalized collection $(\hat{\rho}_i, i \in \mathbb{N})$ is the Poisson–Dirichlet distribution with parameter $m(\beta) = \sqrt{2}/\beta$, which we shall denote by $\mathrm{PD}(m(\beta))$.

In a somewhat indirect way (by means of the so-called Ghirlanda–Guerra identities, which avoid the need to first identify the limiting extremal process) Bovier and Kurkova proved that, in the low temperature regime $\beta > \sqrt{2}$, the Gibbs measure $\mathcal{G}_{\beta,t}$ converges, in the limit of large times, to the $\mathrm{PD}(m(\beta))$; together with our Theorem 2, this naturally suggests that the Gibbs measure of BBM is concentrated, in fact, on the *cluster-extrema*.

Finally, Theorem 2 sheds light on a property of the extremal process of BBM that was conjectured by Brunet and Derrida [10]. They suggested that the statistics of the leading particles are invariant under superposition in the sense that the extremal process of two independent branching Brownian motions has the same law, up to a random shift, as the extremal process of a single one. This property at the level of the entire process is likely to involve specific features of the laws of the individual clusters. On the other hand, at the level of the thinned process, it is a straightforward consequence of Theorem 2, since the law is Poisson with exponential density.

COROLLARY 3. *Let $\Xi(t)$ and $\Xi'(t)$ be the extremal processes (1.9) of two independent branching Brownian motions. Denote by Z and Z' the pointwise limit of their respective derivative martingale. Then, for any $0 < q < 1$, the law of the q-thinning of $\Xi(t) + \Xi'(t)$ conditionally on Z and Z' converges to*

$$\mathrm{PPP}\big(C \cdot (Z + Z') \cdot \sqrt{2}\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x\big). \tag{2.10}$$

In particular, the thinned process of $\Xi(t) + \Xi'(t)$ has the same law in the limit as the thinned process Ξ^0 of a single branching Brownian motion, up to a random shift.

As mentioned before, Theorem 2 is a natural consequence of Theorem 1. The main ingredient is the following lemma, which allows to compare thinning processes on a set of large probability. We use the notation

$$\Xi^{(q)}(t)|_{(y,\infty)} \equiv \big\{\Xi_i^{(q)}(t) : \Xi_i^{(q)}(t) > y\big\}. \tag{2.11}$$

LEMMA 4. *For any $y \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $r_0 = r_0(y, \varepsilon)$ such that for $r_d, r_g > r_0$ and $t > 3\max\{r_g, r_d\}$, on a set of probability $1 - \varepsilon$,*

$$\Xi^{(q)}(t)|_{(y,\infty)} = \Xi_t^{(r_d/t)}|_{(y,\infty)}, \tag{2.12}$$

for any $\frac{r_d}{t} < q < 1 - \frac{r_g}{t}$.

Theorem 2 is then proved by a standard Poisson convergence argument which exploits the weak correlations between the cluster-extrema in classes of the $\frac{r_d}{t}$-thinning.

PROPOSITION 5. *With $C > 0$ and Z the limiting derivative martingale, conditionally on Z,*

$$\lim_{r_d\to\infty}\lim_{t\to\infty} \Xi^{(r_d/t)}(t) = \mathrm{PPP}\big(CZ\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x\big). \tag{2.13}$$

3. Proofs.

PROOF OF LEMMA 4. Theorem 1 describes the genealogies of particles which fall into compact sets around the level of the maximum but for the proof of Lemma 4 we need a slight extension in order to cover the case of sets which are only bounded from below; more precisely, we claim that for $y \in \mathbb{R}$,

$$\lim_{r_d,r_g\to\infty}\ \sup_{t>3\max\{r_d,r_g\}} \mathbb{P}[\exists i, j \in \Sigma_t(y,\infty) : Q_{ij}(t) \in (r_d, t - r_g)] = 0. \tag{3.1}$$

To see this, we recall the following estimate proved by Bramson [8], Proposition 3:

$$\mathbb{P}\Big[\max_{k\le n(t)} \overline{x_k}(t) \ge Y\Big] \le \kappa(Y+1)^2 \mathrm{e}^{-\sqrt{2}Y}, \tag{3.2}$$

which is valid for $t \ge 2, 0 < Y < \sqrt{t}$ and $\kappa > 0$ a numerical constant. The bound (3.2) implies in particular that

$$\lim_{Y\to\infty} \sup_{t\ge 2} \mathbb{P}[\sharp\Sigma_t(Y,\infty) > 0] = 0. \tag{3.3}$$

For $Y > y$, using the splitting $\Sigma_t(y,\infty) = \Sigma_t(y,Y) \cup \Sigma_t(Y,\infty)$, we have the bound

$$\begin{aligned} &\mathbb{P}[\exists i,j \in \Sigma_t(y,\infty): Q_{ij}(t) \in (r_d, t-r_g)] \\ &\quad \le \mathbb{P}[\exists i,j \in \Sigma_t(y,Y): Q_{ij}(t) \in (r_d, t-r_g)] + \mathbb{P}[\sharp\Sigma_t(Y,\infty) > 0]. \end{aligned} \tag{3.4}$$

The first term on the right-hand side vanishes, by Theorem 1, in the limit $t \to \infty$ first and $r_d, r_g \to \infty$ next, whereas the second term vanishes, by (3.3), in the limit $t \to \infty$ first and $Y \to \infty$ next: this proves (3.1).

Let us denote by $A_{t,r_d,r_g}(y,\varepsilon)$ the event

$$\{\exists i,j \in \Sigma_t(y,\infty): Q_{ij}(t) \in [r_d, t-r_g]\}. \tag{3.5}$$

By (3.1), there exists $r_0 = r_0(y,\varepsilon)$ such that, for $r_d, r_g > r_0$, $\mathbb{P}[A^c_{t,r_d,r_g}] > 1-\varepsilon$. By definition, on the event A^c_{t,r_d,r_g}, the following equivalence holds for any $\frac{r_d}{t} \le q \le 1 - \frac{r_g}{t}$:

$$\overline{Q}_{ij}(t) < q \quad \Longleftrightarrow \quad \overline{Q}_{ij}(t) < \frac{r_d}{t}. \tag{3.6}$$

The assertion of the lemma is now a direct consequence of the definition of the thinning $\Xi^{(q)}(t)$ in (2.4). $\square$

To prove Proposition 5, we need some control on the derivative martingale.

LEMMA 6. *Let*

$$Z^{(2)}(t) \equiv \sum_{k\le n(t)} \{\sqrt{2}t - x_k(t)\}^2 \exp[-2\sqrt{2}\{\sqrt{2}t - x_k(t)\}]. \tag{3.7}$$

For any given $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\big[Z^{(2)}(t) \ge \varepsilon\big] = 0. \tag{3.8}$$

PROOF. First, by Bramson's estimate [8], we may find $Y = Y(\varepsilon)$ large enough, s.t.

$$\mathbb{P}\Big[\max_k x_k(t) - m(t) > Y\Big] \le (1+Y)^2 \mathrm{e}^{-\sqrt{2}Y} \le \varepsilon/2. \tag{3.9}$$

Using this bound, and the Markov inequality, we get

$$\begin{aligned}&\mathbb{P}\big[Z^{(2)}(t)\geq\varepsilon\big]\\ &\quad\leq\frac{\mathrm{e}^t}{\varepsilon}\mathbb{E}\big[\{\sqrt{2}t-x(t)\}^2\mathrm{e}^{-2\sqrt{2}\{\sqrt{2}t-x(t)\}};x(t)\leq m(t)+Y\big]+\varepsilon/2.\end{aligned}\tag{3.10}$$

The first term on the right-hand side is bounded from above by

$$\begin{aligned}&\frac{\mathrm{e}^t}{\varepsilon}\int_{(3/(2\sqrt{2}))\log t-Y}^{\infty}x^2\mathrm{e}^{-2\sqrt{2}x}\exp\biggl\{-\frac{(\sqrt{2}t-x)^2}{2t}\biggr\}\frac{\mathrm{d}x}{\sqrt{2\pi t}}\\ &\quad\leq\frac{1}{\varepsilon}\int_{(3/(2\sqrt{2}))\log t-Y}^{\infty}x^2\mathrm{e}^{-\sqrt{2}x}\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi t}}\\ &\quad\leq\frac{\exp-\sqrt{2}(3/(2\sqrt{2})\log t-Y)}{\varepsilon}\int_{(3/(2\sqrt{2}))\log t-Y}^{\infty}x^2\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi t}}\\ &\quad\leq\frac{\rho\cdot t^{-3/2}}{\varepsilon}\int_0^{\infty}x^2\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi t}}\\ &\quad\leq\frac{\rho\cdot t^{-3/2}}{\varepsilon}t\to 0\qquad\text{as } t\to\infty.\end{aligned}\tag{3.11}$$

This proves the lemma. □

PROOF OF PROPOSITION 5. We will show the convergence of the Laplace functionals. For $\phi:\mathbb{R}\to\mathbb{R}_+$ measurable with compact support, we claim that

$$\begin{aligned}&\lim_{r_d\to\infty}\lim_{t\to\infty}\mathbb{E}\biggl[\exp-\int\phi(x)\,\Xi^{(r_d/t)}(t)(\mathrm{d}x)\biggr]\\ &\qquad=\mathbb{E}\biggl[\exp-CZ\int\bigl(1-\mathrm{e}^{-\phi(x)}\bigr)\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x\biggr],\end{aligned}\tag{3.12}$$

from which the proposition would evidently follow.

We will prove (3.12) for simple step functions, that is, of the form $\phi(x)=\sum_{i=1}^N a_i 1_{A_i}$ for $a_i>0, i=1,\dots,N$ and $A_i=[\underline{A_i},\overline{A_i}], i=1\dots N$ disjoint compact subsets. The extension to the general case of measurable ϕ follows by a standard monotone class argument.

We will make use of the splitting

$$m(t)=\sqrt{2}r_d+m(t-r_d)+R_t\tag{3.13}$$

for some $R_t=o(1)$ as $t\uparrow\infty$.

We also introduce, for $j\leq n(r_d)$, independent BBMs $\{x_k^{(j)}(t-r_d)\}_{k\leq n_j(t-r_d)}$, and use the abbreviation

$$M_j(t-r_d)\equiv\max_{k\leq n_j(t-r_d)}x_k^{(j)}(t-r_d)-m(t-r_d).\tag{3.14}$$

Conditionally on everything that happened up to time r_d, the following equality holds in law due to the Markov property and the definition of the extrema in the (r_d/t)-thinning class:

$$\Xi^{(r_d/t)}(t) \stackrel{(d)}{=} \{x_j(r_d) - \sqrt{2}r_d + M_j(t-r_d) + R_t\}_{j=1\dots n(r_d)}. \tag{3.15}$$

Since the M_j's are i.i.d., with $\mathbb{E}_{M(t-r_d)}$ standing for expectation with respect to $M(t-r_d)$,

$$\begin{aligned} &\mathbb{E}\left[\exp - \int \phi(x)\,\Xi^{(r_d/t)}(t)(\mathrm{d}x)\right] \\ &\quad = \mathbb{E}\left[\prod_{j=1}^{n(r_d)} \mathbb{E}_{M(t-r_d)}\big[\mathrm{e}^{-\phi(x_j(r_d)-\sqrt{2}r_d+M(t-r_d)+R_t)}\big]\right]. \end{aligned} \tag{3.16}$$

As $t \to \infty$, the variable $M(t-r_d)$ converges weakly to M with law ω by (1.3). Hence

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}\left[\exp - \int \phi(x)\,\Xi^{(r_d/t)}(t)(\mathrm{d}x)\right] \\ &\quad = \mathbb{E}\left[\prod_{j=1}^{n(r_d)} \mathbb{E}_{M}\big[\mathrm{e}^{-\phi(x_j(r_d)-\sqrt{2}r_d+M)}\big]\right]. \end{aligned} \tag{3.17}$$

Define $y_j(r_d) \equiv \sqrt{2}r_d - x_j(r_d)$. We write

$$\begin{aligned} \mathbb{E}_M\big[\mathrm{e}^{-\phi(-y_j(r_d)+M)}\big] &= 1 - \mathbb{E}_M\big[1 - \mathrm{e}^{-\phi(-y_j(r_d)+M)}\big] \\ &=: 1 - F(-y_j(r_d)), \end{aligned} \tag{3.18}$$

and

$$(3.17) = \mathbb{E}\left[\exp\left\{\sum_{j\le n(r_d)} \log[1 - F(-y_j(r_d))]\right\}\right]. \tag{3.19}$$

Observe that

$$\min_{j\le n(r_d)} y_j(r_d) \uparrow \infty \qquad \text{a.s.} \tag{3.20}$$

as $r_d \uparrow \infty$. This implies that

$$\max_{j\le n(r_d)} F(-y_j(r_d)) \downarrow 0. \tag{3.21}$$

Using that $-x - x^2 < \log(1-x) < -x$ for $0 < x < 1/2$, for r_d large enough, we obtain (up to a vanishing error) upper and lower bounds of the form

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{-\sum_{j\le n(r_d)} F(-y_j(r_d))\Big\}\Big] \\
&\quad \ge (3.19) \\
&\quad \ge \mathbb{E}\Big[\exp\Big\{-\sum_{j\le n(r_d)} F(-y_j(r_d)) - F(-y_j(r_d))^2\Big\}\Big].
\end{aligned} \tag{3.22}
$$

Since ϕ is chosen to be a simple step function,

$$
F(-y_j(r_d)) = \sum_{i=1}^{N} (1 - \mathrm{e}^{-a_i}) \int_{A_i + y_j(r_d)} \mathrm{d}\omega. \tag{3.23}
$$

Hence we can make use of the asymptotics (1.8) to obtain

$$
\begin{aligned}
F(-y_j(r_d)) \sim \sum_{i=1}^{N} (1 - \mathrm{e}^{-a_i}) C \big\{ &\big(\underline{A_i} + y_j(r_d)\big)\mathrm{e}^{-\sqrt{2}(\underline{A_i} + y_j(r_d))} \\
&- \big(\overline{A_i} + y_j(r_d)\big)\mathrm{e}^{-\sqrt{2}(\overline{A_i} + y_j(r_d))}\big\},
\end{aligned} \tag{3.24}
$$

with $\sim$ meaning that the ratio of the left- and right-hand sides converges to 1, in the limit $r_d \to \infty$, $\mathbb{P}$-a.s. We regroup the terms on the right-hand side to get

$$
\begin{aligned}
F(-y_j(r_d)) &\sim C y_j(r_d) \mathrm{e}^{-\sqrt{2} y_j(r_d)} \\
&\quad \times \sum_{i=1}^{N} (1 - \mathrm{e}^{-a_i}) \{\mathrm{e}^{-\sqrt{2}\underline{A_i}} - \mathrm{e}^{-\sqrt{2}\overline{A_i}}\} + \mathcal{R}(y_j(r_d)),
\end{aligned} \tag{3.25}
$$

with $\mathcal{R}$ containing all the remaining terms; clearly,

$$
|\mathcal{R}(y_j(r_d))| \le \rho \cdot \mathrm{e}^{-\sqrt{2} y_j(r_d)}, \tag{3.26}
$$

where ρ depends on the a_i and A_i, but not on $y_j(r_d)$. By the convergence of the derivative martingale as $r_d \uparrow \infty$ [cf. (1.6)], and the fact that, in the same limit,

$$
\sum_{j\le n(r_d)} \mathrm{e}^{-\sqrt{2} y_j(r_d)} \to 0, \tag{3.27}
$$

$\mathbb{P}$-almost surely, we get that

$$
\begin{aligned}
\lim_{r_d \uparrow \infty} \sum_{j \le n(r_d)} F\big(x_j(r_d) - \sqrt{2} r_d\big) &= C Z \sum_{i=1}^{N} (1 - \mathrm{e}^{-a_i}) \{\mathrm{e}^{-\sqrt{2}\underline{A_i}} - \mathrm{e}^{-\sqrt{2}\overline{A_i}}\} \\
&= C Z \int (1 - \mathrm{e}^{-\phi(x)}) \sqrt{2} \mathrm{e}^{-\sqrt{2} x} \, \mathrm{d}x,
\end{aligned} \tag{3.28}
$$

$\mathbb{P}$-almost surely. This yields the correct asymptotics for the upper bound in 3.22.

The lower bound in (3.22) involves exactly the same term as the left-hand side of (3.28), and the additional term

$$\sum_{j\le n(r_d)} F\big(x_j(r_d)-\sqrt{2}r_d\big)^2. \tag{3.29}$$

It is straightforward to see that (3.29) converges to zero, as $r_d\uparrow\infty$. In fact, by the same argument as in (3.23)–(3.26), one sees that

$$|(3.29)| = O\biggl(\sum_{j\le n(r_d)} y_j(r_d)^2 \mathrm{e}^{-2\sqrt{2}y_j(r_d)}\biggr), \qquad r_d\uparrow\infty. \tag{3.30}$$

With the notation of Lemma 6,

$$\sum_{j\le n(r_d)} y_j(r_d)^2\mathrm{e}^{-2\sqrt{2}y_j(r_d)} = Z^{(2)}(r_d), \tag{3.31}$$

and this converges to zero in probability, by Lemma 6. Hence, in the limit of large r_d, the lower and upper bounds in (3.22) coincide, which concludes the proof of the proposition. □

PROOF OF THEOREM 2. Let $\phi:\mathbb{R}\to\mathbb{R}_+$ be measurable, with compact support. We need to show that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\biggl[\exp-\int\phi(x)\,\Xi^{(q)}(t)(\mathrm{d}x)\biggr]\\ =\mathbb{E}\biggl[\exp -CZ\int\big(1-\mathrm{e}^{-\phi(x)}\big)\sqrt{2}\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x\biggr],\end{aligned} \tag{3.32}$$

for any $\frac{r_d}{t}\le q\le 1-\frac{r_g}{t}$. This is straightforward in view of Lemma 4 and Proposition 5 by taking y smaller than the minimum of the support of ϕ and ε arbitrarily small. □

4. Open problems.

4.1. *On the extremal process of BBM.* We consider the following cluster point process. Let $(\Omega',\mathcal{F}',P)$, $C>0$ be a probability space, and $Z:\Omega'\to\mathbb{R}_+$ with distribution as in Theorem 2. (Expectation w.r.t. P will be denoted by E.) Conditionally on a realization of Z, let $(\eta_i; i\in\mathbb{N})$ be the position of particles generated according to a Poisson point process with density

$$CZ\big(-x\mathrm{e}^{-\sqrt{2}x}\big)\,\mathrm{d}x \tag{4.1}$$

on the negative axis. For each $i\in\mathbb{N}$, consider independent branching Brownian motions with drift $-\sqrt{2}$, that is, $\{x_k^{(i)}(r)-\sqrt{2}r; k\le n_i(r)\}$, issued on $(\Omega',\mathcal{F}',P)$. ("Time" is denoted here by r.)

Remark that for given $i \in \mathbb{N}$,

$$\max_{k \le n_i(r)} x_k^{(i)}(r) - \sqrt{2}r \to -\infty, \tag{4.2}$$

P-almost surely. The branching Brownian motions with drift are then superimposed on the Poissonian points, that is, the cluster point process is given by

$$\Pi_r \equiv \{\pi_{i,k}(r); i \in \mathbb{N}, k = 1 \dots n_i(r)\}, \qquad \pi_{i,k}(r) \equiv \eta_i + x_k^{(i)}(r) - \sqrt{2}r. \tag{4.3}$$

The existence of the large time limit of Π_r is not straightforward. Due to (4.2), only those Poissonian points whose attached branching Brownian motion performs an unusually large displacement can contribute to the limiting object. It is thus not clear that one finds any Poissonian points at all which, together with their cluster of particles, achieve this feat. The fundamental observation here is that, in virtue of (4.1), the density of the Poissonian points on the negative axis grows (slightly faster than) exponentially when $x \to -\infty$. Together with the work of Chauvin and Rouault [13] on branching Brownian motions conditioned to perform unusually large displacements, this observation can be exploited to rigorously establish the existence of the point process Π_r in the limit of large times, as well as some of its statistical properties. We will report on this in a subsequent paper [2].

Here, we only put forward the following conjecture, which appears rather natural in the light of Theorem 1 and the results on the paths of extremal particles in BBM established in [3]:

CONJECTURE 7. In the limit of large times, the distribution of the extremal process of BBM, $\Xi(t)$ and that of Π_r coincide, that is,

$$\lim_{t\to\infty} \Xi(t) \overset{(d)}{=} \lim_{r\to\infty} \Pi_r. \tag{4.4}$$

In particular, with $\phi : \mathbb{R} \to \mathbb{R}_+$ a measurable function with compact support,

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{E}\left[\exp\left(-\sum_{k\le n(t)} \phi\big(x_k(t) - m(t)\big)\right)\right]\\ &\quad = \lim_{r\to\infty} E\left[\exp -CZ\int_{-\infty}^{0} (1 - \mathrm{e}^{-\psi_r(x)})\{-x\mathrm{e}^{-\sqrt{2}x}\}\,\mathrm{d}x\right],\end{aligned} \tag{4.5}$$

where

$$\mathrm{e}^{-\psi_r(x)} \equiv E\left[\exp\left(-\sum_{k\le n(r)} \phi\big(x + x_k(r) - \sqrt{2}r\big)\right)\right]. \tag{4.6}$$

We remark that densities of the form $-x\exp(-\sqrt{2}x)\,\mathrm{d}x$ on the negative axis have been conjectured to play an important role in the recent work by Brunet and

Derrida [10], where the average size of the gaps between the nth- and $(n+1)$th-leading particle at the edge of BBM is numerically shown to behave as

$$\frac{1}{n} - \frac{1}{n \log n} + \cdots, \tag{4.7}$$

(which is indeed "close" to the average size of the gaps in a PPP with density $-x\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x$ on the negative axis).

4.2. *On a conjecture by Lalley and Sellke.* Conjecture 7 is similar but fundamentally different from the *Tidal Wave Conjecture* formulated by Lalley and Sellke [21]. Lalley and Sellke suggested that the Poisson point process entering into the construction of Π_r should have density $CZ\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x$ conditionally on a realization of Z where C is some constant. However, this cannot be correct. We will show that such a point process does not exist in the limit $r \to \infty$: the density of the Poissonian component cannot compensate (4.2) and all the particles are bound to drift off to $-\infty$. To formulate this precisely, consider the point process

$$\widetilde{\Pi}_r \equiv \left(\tilde{\eta}_i + x_k^{(i)}(r) - \sqrt{2}r; i \in \mathbb{N}, k = 1, \ldots, n_i(r)\right), \tag{4.8}$$

where the $\tilde{\eta}$'s are points of a PPP with density $CZ\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x$, and the $x^{(i)}$'s independent BBMs.

PROPOSITION 8. *For given $y \in \mathbb{R}$,*

$$\lim_{r\to\infty} P\left[\widetilde{\Pi}_r[y,\infty) \geq 1 | Z\right] = 0. \tag{4.9}$$

In order to prove Proposition 8, we make use of the following bound established by Bramson:

PROPOSITION 9 ([7], Proposition 8.2). *Let $y_0 < 0$ (strictly). There exists $r_0 = r_0(y_0)$ such that for $r \geq r_0$, $x \geq m(r) + 1$ and $z \equiv x - m(r)$,*

$$P\Big[\max_{k\leq n(r)} x_k(r) \geq x\Big] \leq \rho \cdot \mathrm{e}^r \int_{y_0}^{0} \frac{\mathrm{e}^{-(x-y)^2/2r}}{\sqrt{2\pi r}} \left(1 - \mathrm{e}^{-2(y-y_0)z/r}\right) \mathrm{d}y, \tag{4.10}$$

where $\rho > 0$ is a numerical constant.

Using this with $y_0 := -1$, we obtain the following corollary. (Here and below, $\rho > 0$ denotes a numerical constant, not necessarily the same at different occurrences.)

COROLLARY 10. *For $X > 1$, and $r \geq r_o = r_o(-1)$,*

$$\begin{aligned} &P\Big[\max_{k\leq n(t)} x_k(r) - m(r) \geq X\Big] \\ &\quad \leq \rho \cdot X \cdot \exp\left(-\sqrt{2}X - \frac{X^2}{2r} + \frac{3}{2\sqrt{2}} X \frac{\log r}{r}\right). \end{aligned} \tag{4.11}$$

PROOF. According to Proposition 9, for $X > 1$,

$$\begin{aligned} &P\Big[\max_{k\le n(t)} x_k(r) - m(r) \ge X\Big] \\ &\quad \le \rho \cdot \mathrm{e}^r \int_{-1}^0 \frac{\mathrm{e}^{-(X+m(r)-y)^2/2r}}{\sqrt{2\pi r}} \big(1 - \mathrm{e}^{-2(y+1)X/r}\big)\,\mathrm{d}y. \end{aligned} \tag{4.12}$$

Since $y + 1 > 0$ we have that $1 - \mathrm{e}^{-2(y+1)X/r} \le 2(y + 1)X/r$. Using this, the right-hand side of (4.12) is at most

$$\rho \cdot \frac{X\mathrm{e}^r}{r} \int_{-1}^0 (y+1) \frac{\mathrm{e}^{-(X+m(r)-y)^2/2r}}{\sqrt{2\pi r}}\,\mathrm{d}y. \tag{4.13}$$

Expanding the square in the Gaussian density, (4.13) is at most

$$\begin{aligned} &\rho \cdot X \cdot \exp\Big(-\sqrt{2}X - \frac{X^2}{2r} + \frac{3X \log r}{2\sqrt{2}r}\Big) \\ &\quad \times \int_{-1}^0 (y+1)\underbrace{\mathrm{e}^{Xy/r + \sqrt{2}y + y(3/(2\sqrt{2}))(\log r)/r}\mathrm{e}^{-y^2/2r}}_{\le 1}\,\mathrm{d}y \\ &\quad \le \rho \cdot X \cdot \exp\Big(-\sqrt{2}X - \frac{X^2}{2r} + \frac{3}{2\sqrt{2}} X \frac{\log r}{r}\Big), \end{aligned} \tag{4.14}$$

settling the proof of the corollary. $\square$

PROOF OF PROPOSITION 8. In view of (4.2), it is plain that for any finite set $I \subset \mathbb{N}$

$$\max_{i\in I}\Big\{\tilde\eta_i + \max_{k\le n_i(r)}\big[x_k^{(i)}(r) - \sqrt{2}r\big]\Big\} \xrightarrow{r\uparrow\infty} -\infty, \tag{4.15}$$

P-almost surely. But the number of Poissonian points $(\tilde\eta_i; i \in \mathbb{N})$ in the interval $[0,\infty)$ *is* finite, P-almost surely: this follows from the fact that the density $CZ\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x$ is integrable on $x \in [0,\infty)$. Hence, Proposition 8 will follow as soon as we prove that

$$P\Big[\exists_{i\in\mathbb{N}} : \tilde\eta_i + \max_{k\le n_i(r)}\{x_k^{(i)}(r) - \sqrt{2}r\} \ge y \text{ and } \tilde\eta_i \in (-\infty, 0)|Z\Big] \xrightarrow{r\uparrow\infty} 0. \tag{4.16}$$

By the Markov inequality, and using that the BBMs superimposed on the Poissonian points are identically distributed, (4.16) is at most

$$\int_{-\infty}^0 P\Big[\max_{k\le n(r)}\{x_k(r) - \sqrt{2}r\} \ge y - x\Big] CZ\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x. \tag{4.17}$$

We rewrite this in terms of $M(r) \equiv \max_{k \le n(r)}\{x_k(r) - m(r)\}$:

$$\begin{aligned}(4.17) &= \int_{-\infty}^{0} P\left[M(r) \ge y - x + \frac{3}{2\sqrt{2}}\log r\right] CZ\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x \\ &= \left(CZ\mathrm{e}^{-\sqrt{2}y}\right) \cdot \frac{1}{r^{3/2}} \int_{y+3/(2\sqrt{2})\log r}^{\infty} P[M(r) \ge X]\mathrm{e}^{\sqrt{2}X}\,\mathrm{d}X,\end{aligned} \tag{4.18}$$

the last step by change of variable $y - x + \frac{3}{2\sqrt{2}}\log r \to X$.

Let us abbreviate $\rho \equiv CZ\mathrm{e}^{-\sqrt{2}y}$. (Note that y and Z are fixed.) For r large enough,

$$y + \frac{3}{2\sqrt{2}}\log r \ge 1, \tag{4.19}$$

hence we may use (4.14) to get that (4.18) is at most

$$\begin{aligned}&\frac{\rho}{r^{3/2}} \int_{y+(3/(2\sqrt{2}))\log r}^{\infty} X \exp\left(\frac{3}{2\sqrt{2}} X \frac{\log r}{r}\right) \mathrm{e}^{-X^2/(2r)}\,\mathrm{d}X \\ &\quad = \frac{\rho}{r^{3/2}} \underbrace{\exp\left(\frac{9}{16}\frac{(\log r)^2}{r}\right)}_{=1+o(1), r\uparrow\infty} \int_{y}^{\infty} \left\{Y + \frac{3}{2\sqrt{2}}\log r\right\} \mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y,\end{aligned} \tag{4.20}$$

by change of variable $X - \frac{3}{2\sqrt{2}}\log r \to Y$.

It thus remains to control the term

$$\begin{aligned}&\frac{1}{r^{3/2}} \int_{y}^{\infty} \left\{Y + \frac{3}{2\sqrt{2}}\log r\right\} \mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y \\ &\quad = \frac{1}{r^{3/2}} \int_{y}^{\infty} Y\mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y + \frac{3}{2\sqrt{2}} \cdot \frac{\log r}{r^{3/2}} \int_{y}^{\infty} \mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y.\end{aligned} \tag{4.21}$$

As for the first term on the right-hand side of (4.21):

$$\frac{1}{r^{3/2}} \int_{y}^{\infty} Y\mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y = \frac{1}{\sqrt{r}} \int_{y/\sqrt{r}}^{\infty} x\mathrm{e}^{-x^2/2}\,\mathrm{d}x \to 0 \qquad \text{as } r \uparrow \infty. \tag{4.22}$$

As for the second term on the right-hand side of (4.21):

$$\begin{aligned}&\frac{3}{2\sqrt{2}} \cdot \frac{\log r}{r^{3/2}} \int_{y}^{\infty} \mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y \\ &\quad = \frac{3}{2\sqrt{2}} \cdot \frac{\log r}{r} \int_{y/\sqrt{r}}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \to 0 \qquad \text{as } r \uparrow \infty.\end{aligned} \tag{4.23}$$

This proves (4.16), settling Proposition 8. $\square$

REMARK 11. The above computations also suggest that a point process which is obtained by superimposing independent BBMs with drift $-\sqrt{2}$ on a PPP with a certain density exists in the limit of large times if and only if such density is, up to a (possibly random) constant, $-x\exp(-\sqrt{2}x)\,\mathrm{d}x$ on the negative axis.

In fact, a closer look at the above considerations reveals that the left-hand side of (4.22) is the leading order of the expected number of points (of the superimposed point process) which fall into the subset $[y,\infty)$. Choosing the density of the Poissonian component as in Conjecture 7, (4.22) would then read $r^{-3/2}\int_y^\infty Y^2 \mathrm{e}^{-Y^2/2r}\,\mathrm{d}Y$, which indeed remains of order 1 in the limit $r \to \infty$.

Note added in revision. There has been considerable activity concerning the extremal process of BBM after this paper was submitted for publication. Brunet and Derrida have shown in [11] that all statistical properties of the rightmost points can be extracted from the traveling wave solutions of the Fisher-KPP equation. The validity of Conjecture 7 has been settled in a paper of ours [2], where it is proved that the extremal process of branching Brownian motion weakly converges in the limit of large times to a Poisson cluster process; shortly after that, Aidekon et al. [1] recovered the same results by means of "spine techniques." The Poissonian structure of the extremal process can also be proved using the property of *superposability* as observed by Maillard [22]. This property of the process was conjectured by Brunet and Derrida in [11] and proved in [2].

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REFERENCES

[1] AIDEKON, E., BERESTYCKI, J., BRUNET, E. and SHI, Z. (2011). The branching Brownian motion seen from its tip. Available at arXiv:1104.3738.

[2] ARGUIN, L. P., BOVIER, A. and KISTLER, N. (2011). The extremal process of branching Brownian motion. Available at arXiv:1103.2322.

[3] ARGUIN, L. P., BOVIER, A. and KISTLER, N. (2011). Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.* **64** 1647–1676. MR2838339

[4] ARONSON, D. G. and WEINBERGER, H. F. (1975). Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial Differential Equations and Related Topics* (*Program, Tulane Univ., New Orleans, La.*, 1974). *Lecture Notes in Math.* **446** 5–49. Springer, Berlin. MR0427837

[5] ARONSON, D. G. and WEINBERGER, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30** 33–76. MR0511740

[6] BOVIER, A. and KURKOVA, I. (2004). Derrida's generalized random energy models. II. Models with continuous hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.* **40** 481–495. MR2070335

[7] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44**. MR0705746

[8] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. MR0494541
[9] BRUNET, E. (2010). Private communication.
[10] BRUNET, E. and DERRIDA, B. (2009). Statistics at the tip of a branching random walk and the delay of traveling waves. *Eurphys. Lett.* **87** 60010.
[11] BRUNET, É. and DERRIDA, B. (2011). A branching random walk seen from the tip. *J. Stat. Phys.* **143** 420–446. MR2799946
[12] CARPENTIER, D. and LE DOUSSAL, P. (2001). Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh-Gordon models. *Phys. Rev. E* **63** 026110.
[13] CHAUVIN, B. and ROUAULT, A. (1990). Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Math. Nachr.* **149** 41–59. MR1124793
[14] DERRIDA, B. and SPOHN, H. (1988). Polymers on disordered trees, spin glasses, and traveling waves. *J. Stat. Phys.* **51** 817–840. MR0971033
[15] FISHER, R. A. (1937). The wave of advance of advantageous genes. *Ann. Eugen.* **7** 355–369.
[16] FYODOROV, Y. V. and BOUCHAUD, J.-P. (2008). Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A* **41** 372001. MR2430565
[17] HARRIS, S. C. (1999). Travelling-waves for the FKPP equation via probabilistic arguments. *Proc. Roy. Soc. Edinburgh Sect. A* **129** 503–517. MR1693633
[18] KABLUCHKO, Z. (2010). Private communication.
[19] KESSLER, D. A., LEVINE, H., RIDGWAY, D. and TSIMRING, L. (1997). Evolution on a smooth landscape. *J. Stat. Phys.* **87** 519–544.
[20] KOLMOGOROV, A., PETROVSKY, I. and PISCOUNOV, N. (1937). Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moscou Universitet, Bull. Math.* **1** 1–25.
[21] LALLEY, S. P. and SELLKE, T. (1987). A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.* **15** 1052–1061. MR0893913
[22] MAILLARD, P. (2010). A characterisation of superposable random measures. Available at arXiv:1102.1888.
[23] MCKEAN, H. P. (1975). Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.* **28** 323–331. MR0400428
[24] MUNIER, S. and PESCHANSKI, R. (2004). Traveling wave fronts and the transition to saturation. *Phys. Rev. D* **69** 034008.

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