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Glauber Dynamics of the Random Energy Model

I. Metastable Motion on the Extreme States*

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Abstract: We investigate the long-time behavior of the Glauber dynamics for the random energy model below the critical temperature. We give very precise estimates on the motion of the process to and between the states of extremal energies. We show that when disregarding time, the consecutive steps of the process on these states are governed by a Markov chain that jumps uniformly on all possible states. The mean times of these jumps are also computed very precisely and are seen to be asymptotically independent of the terminal point. A first indicator of aging is the observation that the mean time of arrival in the set of states that have waiting times of order T is itself of order T. The estimates proven in this paper will furnish crucial input for a follow-up paper where aging is analysed in full detail.

1. Introduction and Background

1.1. Introduction. The concept of "aging" has become one of the main paradigms in the theory of the dynamics of disordered systems[1]. Roughly speaking, this term refers to a particular way in which dynamic properties of a system change with time when relaxing towards equilibrium: the time scale at which the process evolves slows down in proportion to the elapsed time, the system "ages". It is in fact believed that most disordered systems, or at least those qualified as "glassy systems" do exhibit this phenomenon. While this is so, almost no results concerning aging in "real" spin systems do exist. In fact most existing results, even on the heuristic level, concern two types of dynamics: 1) Langevin dynamics in spherical models such as the spherical SK model [BDG, CD], or the spherical p-spin SK model [BCKM]. 2) Trap models [B, BD, BCKM] that are

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[1] The cond-mat archives in Trieste contain 263 papers containing this term in their abstracts, and 124 containing it even in the title. See also [Be] for a recent mathematical review.

inspired by the structure of equilibrium states found in (mostly non-rigorous) analysis of mean field spin-glasses. These dynamics are, however, introduced ad hoc without any attempt to justify and *derive* them from an underlying Glauber dynamics on the microscopic degrees of freedom.

In the context of the spherical models, a rigorous derivation of the aging phenomenon has been given recently in [BDG]. This model lacks, however, many of the expected features of spin glasses, in particular the existence of a complex energy landscape with many "metastable states". The simplest model showing these features is the random energy model (REM) [D1, D2]. This model is indeed traded as one of the standard examples where aging occurs in the physics literature; the arguments in the physics literature are however, all based on the ad hoc introduction of an *effective model* (the REM-like trap model [B, BD, BM]) inspired by known properties of the equilibrium distribution and some heuristic arguments. The behaviour of the trap models can then be analysed in detail.

In this and the companion paper [BBG] we prove the first rigorous results on the Glauber dynamics of the REM that will justify in a suitable sense the predictions based on the trap model heuristic. We feel that this is an important first step in showing that the abundant literature on this model is of relevance for realistic disordered systems. The key point of our analysis, and in fact a central problem of the entire subject, will be to control the behaviour of a Markov chain on a very high-dimensional set on a relatively small, but still asymptotically infinite subset of its "most recurrent" or "most stable" states on appropriate time scales, and to describe the ensuing effective dynamics. While we will have to use many of the particular features of the model we consider here, we feel that the general methodology developed in this paper will be of use in many other contexts of the dynamics of complex systems.

The REM. We recall that the REM [D1, D2] is defined as follows. A spin configuration σ is a vertex of the hypercube $\mathcal{S}_N \equiv \{-1, 1\}^N$. On an abstract probability space $(\Omega, \mathcal{F}, P)$ we define the family of i.i.d. standard normal random variables $\{X_\sigma\}_{\sigma\in\mathcal{S}_N}$. We set $E_\sigma \equiv [X_\sigma]_+ \equiv (X_\sigma \vee 0)$. We define a random (Gibbs) probability measure on $\mathcal{S}_N$, $\mu_{\beta,N}$, by setting

$$\mu_{\beta,N}(\sigma) \equiv \frac{e^{\beta\sqrt{N}E_\sigma}}{Z_{\beta,N}}, \tag{1.1}$$

where $Z_{\beta,N}$ is the normalizing partition function[2]. It is well-known [D1, D2] that this model exhibits a phase transition at $\beta_c = \sqrt{2\ln 2}$. For $\beta \le \beta_c$, the Gibbs measures is supported, asymptotically as $N \uparrow \infty$, on the set of states σ for which $E_\sigma \sim \sqrt{N}\beta$, and no single configuration has positive mass. For $\beta > \beta_c$, on the other hand, the Gibbs measure gives positive mass to the extreme elements of the order statistics of the family E_σ; i.e. if we order the spin configurations according to the magnitude of their energies s.t.

$$E_{\sigma^{(1)}} \ge E_{\sigma^{(2)}} \ge E_{\sigma^{(3)}} \ge \cdots \ge E_{\sigma^{(2^N)}}, \tag{1.2}$$

then for any finite k, the respective mass $\mu_{\beta,N}\left(\sigma^{(k)}\right)$ will converge, as N tends to infinity, to some positive random variable ν_k; in fact, the entire family of masses $\mu_{\beta,N}\left(\sigma^{(k)}\right)$, $\kappa \in$

[2] The standard model has X_σ instead of E_σ. This modification has no effect on the equilibrium properties of the model, and will be helpful for setting up the dynamics.

$\mathbb{N}$ will converge in a suitable sense to a random process $\{\nu_k\}_{k\in\mathbb{N}}$, called Ruelle's point process [Ru]. We explain this in more detail below.

So far the fact that σ are vertices of a hypercube has played no rôle in our considerations. It will enter only in the definition of the *dynamics of the model*. The dynamics we will consider is a *discrete time Glauber dynamics*. That is we construct a Markov chain $\sigma(t)$ with state space $\mathcal{S}_N$ and discrete time $t \in \mathbb{N}$ by prescribing transition probabilities $p_N(\sigma, \eta) = \mathbb{P}[\sigma(t+1) = \eta|\sigma(t) = \sigma]$ by

$$p_N(\sigma, \eta) = \begin{cases} \frac{1}{N} e^{-\beta\sqrt{N}E_\sigma}, & \text{if } \|\sigma - \eta\|_2 = 2 \\ 1 - e^{-\beta\sqrt{N}E_\sigma}, & \text{if } \sigma = \eta \\ 0, & \text{otherwise} \end{cases} . \tag{1.3}$$

Note that the dynamics is also random, i.e. the law of the Markov chain is a measure valued random variable on Ω that takes values in the space of Markov measures on the path space $\mathcal{S}_N^{\mathbb{N}}$. We will mostly take a pointwise point of view, i.e. we consider the dynamics for a given fixed realization of the disorder parameter $\omega \in \Omega$ (dependence on which we persistently suppress in the notation).

Remark. Let us comment on our choice of the dynamics. First, we chose discrete time rather than continuous time as to be closer to computer simulations. Since we work on a discrete space, there is no difficulty to treat the continuous time and all our results hold also in continuous time. Second, the fact that we chose the rates to depend only on the starting point allows us to avoid having to solve the problem of determining very precisely the barrier heights between any pair of points $\sigma^{(i)}, \sigma^{(j)}$, which is a tremendous geometrical problem to which we have no answer. Clearly our choice favours the emergence of Bouchaud's trap model.

It is easy to see that this dynamics is reversible with respect to the Gibbs measure $\mu_{\beta,N}$. One also sees that it represents a nearest neighbor random walk on the hypercube with traps of random depths (i.e. the probability to make a zero step is rather large when E_σ is large)[3]. The idea suggested by the known behavior of the equilibrium distribution is that these dynamics, for $\beta > \beta_c$, will spend long periods of time in the states $\sigma^{(1)}, \sigma^{(2)}, \ldots$ etc. and will move "quickly" from one of these configurations to the next. Based on this intuition, Bouchaud et al. proposed the "REM-like" trap model: the state space is reduced to M points, representing the M "deepest" traps. Each of the states is assigned a positive random energy E_k which is taken to be exponentially distributed with rate one. The dynamics is now a continuous time Markov chain $Y(t)$ taking values in $S_M \equiv \{1, \ldots, M\}$. If the process is in state k, it waits an exponentially distributed time with mean proportional to $e^{E_k\alpha}$, where $\alpha = \beta/\beta_c$, and then jumps with equal probability in one of the other states $k' \in S_M$. This process is then analyzed using essentially techniques from *renewal* theory. The essential point is that if one starts the process from the uniform distribution, it is possible to show that if one only considers the times, T_i, at which the process changes its state, then the counting process, $c(t)$, that counts the number of these jumps in the time interval $(0, t)$ is a classical renewal (counting) process [KT]; moreover, as $n \uparrow \infty$, this renewal process converges to a renewal process with a *deterministic* law for the renewal time with a heavy-tailed distribution (in the sense that the mean is infinite[4]) whose density is proportional to $t^{-1-1/\alpha}$. It is the emergence

[3] We have chosen this particular dynamics for technical reasons. To study e.g. the Metropolis algorithm would require some extra work, but we expect essentially the same results to hold.

[4] This is clearly due to the fact that the average of the waiting time $e^{\alpha E_i}$ over the disorder is infinite.

of such *non-Markovian* limit processes that is ultimately responsible for all the aging phenomena observed in the abundant literature on this and related models. Mathematically, the analysis of this trap model presents no particular challenge and the analysis presented e.g. in the review [BCKM] is essentially rigorous, or can be made so with minor efforts.

Our purpose is to show, in a mathematically rigorous way, how and to what extent the REM-like trap model can be viewed as an approximation of what happens in the REM itself. Clearly the main difficulty in doing this will be to explain why the rather complicated random walk on the hypercube between the most profound traps can be interpreted as a simple jump process. This question has two aspects:

1) Why does the process jump with the uniform distribution on the extremal states?
2) Why can this process be seen as a Markov process, in particular, why are the times between visits of two extreme points asymptotically exponentially distributed?

While these facts may appear "obvious" to most physicists, the reason why they are not addressed in any serious way in the literature is that a) they are not at all easy to solve and b) they are, strictly speaking, not even true. In fact, we will see in the course of the analysis (including the follow-up paper [BBG]) that such properties can be only established in a very weak asymptotic form, which is, however, just enough to imply that the predictions of Bouchaud's model apply to the long time asymptotics of the process. While this fact will emerge here only through some very careful and tedious computations, it is clearly desirable to develop a more profound understanding of the phenomenon.

In this first paper we will essentially address the question 1). We will show that if we look at the sequence of visits of the process on a selected set of the states of lowest energy, disregarding the times of these visits, the law of the sequence can indeed be described asymptotically by a simple discrete time Markov chain on this set, which jumps from one point to the next with the uniform distribution. We will also consider two more questions. First we will compute the mean entrance time and the entrance law on this set starting from an arbitrary point on the hypercube. Second we will compute the mean transition times between points in this set. It will turn out that these mean transition times do indeed depend, asymptotically, only on the starting point. Thus, modulo the Markovian hypothesis, we come very close to the heuristic picture outlined above. Moreover, we will see that the mean time to reach a set of extremes is proportional to the smallest "waiting times" on that set (if $\beta > \sqrt{2 \ln 2}$), which will be interpreted as a first sign of the occurrence of aging. We will also show that in contrast, if $\beta < \sqrt{2 \ln 2}$, then the mean time to reach any such point is much longer (by an exponentially large factor) than the waiting time in that point, independent of the starting point of measure. This dichotomy is in fact the main dynamical signature of the transition in this model. This resolves a question raised in an earlier attempt by Fontes et al. [FIKP] to analyse the dynamics of the REM using estimates on the spectral gap. This analysis revealed no sign of a phase transition in the behaviour of the spectral gap. Indeed, the spectral gap in this model correponds in both the high and the low temperature case essentially to the maximal mean waiting time in one site, which depends in a regular way on the temperature. For a different approach to the high-temperature dynamics see also the recent paper by Mathieu and Picco [MP, M].

The control of the property 2) and the more refined analysis of the aging phenomenon will be left two a companion paper [BBG], which will strongly rely on the results obtained here.

Our analysis will draw heavily on methods introduced only recently in the analysis of *metastability* in similar Markov chains in [BEGK1, BEGK2]. We note, however, that the situation here is in some respects quite different than in the setting investigated in these papers. In particular, the investigation of metastability concentrated on the situation where the time scales associated to each metastable state were sufficiently far apart so that to each state corresponds a *distinct scale*. Moreover, these long, metastable time-scales were assumed to be well separated from the shorter time scales on which the process may stay away from the set of metastable states. In the present situation, and this is a generic feature distinguishing aging from metastability, we have on the contrary an infinity of states that communicate on the same time scale, and to complicate the issue, there will be no "gap" between the time scales we are interested in and the "faster" times scales that we try to ignore. Thus the present situation violates the conditions of the setting investigated in [BEGK2] in a maximal way.

The remainder of the introduction is organized as follows. In the next subsection we present some background results on the equilibrium properties of the REM. Based on this information, we will discuss in Subsect. 1.3 some aspects of the metastable behaviour of the model, and state precisely the results we alluded to above.

1.2. Equilibrium results for the REM. In this sub-section we give the necessary background on the (mostly well known, see e.g. [Ei, GMP, OP, Ru]) static aspects of the REM, i.e. we give a precise description of the infinite volume asymptotics of the Gibbs measures that will help to understand the heuristics of the model. A complete exposition can be found in [Bo]. The basic result is the following theorem that characterizes the precise behavior of the partition function:

Proposition 1.1 ([BKL]). *Let $\mathcal{P}$ denote the Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$. If $\beta > \sqrt{2\ln 2}$, then*

$$e^{-N[\beta\sqrt{2\ln 2}-\ln 2]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} Z_{\beta,N} \xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) \tag{1.4}$$

and

$$\ln Z_{\beta,N} - \mathbb{E}\ln Z_{\beta,N} \xrightarrow{\mathcal{D}} \ln\int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) - \mathbb{E}\ln\int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz). \tag{1.5}$$

Remark. The right-hand side of (1.4) is the partition function of what is known as Ruelle's version of the random energy model [Ru]. The simple proof of this theorem is given in [BKL]. It relies, of course, on the classical theorem on the convergence of the point process of (properly rescaled) extremes of i.i.d. Gaussian r.v.'s to the Poisson point process $\mathcal{P}$ (see e.g. [LLR]). Namely, if we set

$$u_N(x) \equiv \sqrt{2N\ln 2} + \frac{x}{\sqrt{2N\ln 2}} - \frac{1}{2}\frac{\ln(N\ln 2)+\ln 4\pi}{\sqrt{2N\ln 2}} \tag{1.6}$$

and define the point process

$$\mathcal{P}_N \equiv \sum_{\sigma\in\{-1,1\}^N} \delta_{u_N^{-1}(X_\sigma)} \tag{1.7}$$

it is well-known that $\mathcal{P}_N$ converges in distribution to the Poisson point process, $\mathcal{P}$, with intensity measure $e^{-x}dx$ on the real line. Since the left hand side of (1.4) can be written as

$$\int \mathcal{P}_N(dx)e^{\alpha x}, \tag{1.8}$$

the theorem follows if the convergence (in law, as $N \uparrow \infty$) of this integral can be proven, which is the case if and only if $\alpha > 1$. For this reason the Poisson point process $\mathcal{P}$ will play a central rôle in all of our analysis.

Proposition 1.1 can be extended to obtain a precise description of the Gibbs measures as well. To formulate this result, it will be convenient to compactify the space $\mathcal{S}_N$ by mapping it to the interval $[-1, 1]$ via

$$\mathcal{S}_N \ni \sigma \to r_N(\sigma) \equiv \sum_{i=1}^{N} \sigma_i 2^{-i} \in [-1, 1]. \tag{1.9}$$

Define the pure point measure $\tilde{\mu}_{\beta,N}$ on $[-1, 1]$ by

$$\tilde{\mu}_{\beta,N} \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{r_N(\sigma)}\mu_{\beta,N}(\sigma). \tag{1.10}$$

Let us introduce the Poisson point process $\mathcal{R}$ on the strip $[-1, 1] \times \mathbb{R}$ with intensity measure $\frac{1}{2}dy \times e^{-x}dx$. If (Y_k, X_k) denote the atoms of this process, define a new point process $\mathcal{W}_\alpha$ on $[-1, 1] \times (0, 1]$ whose atoms are (Y_k, w_k), where

$$w_k \equiv \frac{e^{\alpha X_k}}{\int \mathcal{R}(dy, dx)e^{\alpha x}}. \tag{1.11}$$

With this notation we have that

Proposition 1.2 ([Bo]). *If $\beta > \sqrt{2\ln 2}$, with $\alpha = \beta/\sqrt{2\ln 2}$,*

$$\tilde{\mu}_{\beta,N} \stackrel{\mathcal{D}}{\to} \tilde{\mu}_\beta \equiv \int_0^1 \mathcal{W}_\alpha(\cdot, dw)w. \tag{1.12}$$

Proof. Define the point process $\mathcal{R}_N$ on $[-1, 1] \times \mathbb{R}$ by

$$\mathcal{R}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{(r_N(\sigma),u_N(X_\sigma))}. \tag{1.13}$$

A standard result of extreme value theory (see [LLR], Theorem 5.7.2) is easily adapted to yield that

$$\mathcal{R}_N \stackrel{\mathcal{D}}{\to} \mathcal{R}, \quad \text{as } N \uparrow \infty, \tag{1.14}$$

where the convergence is in the sense of weak convergence on the space of sigma-finite measures endowed with the (metrizable) topology of vague convergence. Note that

$$\mu_{\beta,N}(\sigma) = \frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}} = \frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\int \mathcal{R}_N(dy, dx)e^{\alpha x}}. \tag{1.15}$$

We can define the point process

$$\mathcal{W}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{\left(r_N(\sigma), \frac{\exp(\alpha u_N^{-1}(X_\sigma))}{\int \mathcal{R}_N(dy,dx)\exp(\alpha x)}\right)} \tag{1.16}$$

on $[-1, 1] \times (0, 1]$. Then

$$\tilde{\mu}_{\beta,N} = \int \mathcal{W}_N(dy, dw)\delta_y w. \tag{1.17}$$

Of course we would like to show that this quantity converges to the same object with $\mathcal{W}_N$ replaced by $\mathcal{W}$, as $N \uparrow \infty$. The only non-trivial issue to be resolved is to see whether the denominators $\int \mathcal{R}_N(dy, dx) \exp(\alpha x)$ converge. But Theorem 1.1 asserts precisely that this is the case whenever $\alpha > 1$. Standard arguments then imply that first $\mathcal{W}_N \overset{\mathcal{D}}{\to} \mathcal{W}$, and consequently, (1.12). $\square$

Remark. Note that Theorem 1.2 contains in particular the convergence of the Gibbs measure in the product topology on $\mathcal{S}_N$, since cylinders correspond to certain subintervals of $[-1, 1]$.

Let us discuss the properties on the limiting process $\tilde{\mu}_\beta$. It is not hard to see that with probability one, the support of $\tilde{\mu}_\beta$ is the entire interval $[-1, 1]$. On the other hand, its mass is concentrated on a countable set, i.e. the measure is pure point. This is quite easy to see and the details of the argument can be found in [Bo].

1.3. Metastability and statement of the main results. The properties of the invariant distribution explained in the previous section clearly imply that at temperatures below the critical one the dynamical process will spend most of its time on the extreme states. This suggests that the long time behaviour of the dynamics can be read off from observations of the process on visits to these states. More precisely, define the sets, for $E \in \mathbb{R}$,

$$T_N(E) \equiv \left\{\sigma \in \mathcal{S}_N \middle| E_\sigma \geq u_N(E)\right\}, \tag{1.18}$$

where $u_N(E)$ is defined in (1.6). We will call the set $T_N(E)$ “the top”, and frequently suppress indices, writing $T_N(E) = T(E) = T$ whenever no confusion is likely (the single letter T will only be used within proofs and the change in the notation will always be clearly signalled). Moreover, we will use the convention that $M \equiv |T_N(E)|$ denotes the cardinality of the top, and $d \equiv 2^M$. Let us introduce, for $\sigma \in \mathcal{S}_N$, $I \subset \mathcal{S}_N$, the (slightly abusive) notation

$$\tau_I^\sigma \equiv \inf\{n > 0|\sigma(n) \in I, \sigma(0) = \sigma\} \tag{1.19}$$

for the first positive time the process starting in σ reaches the set I, i.e. here and in the following we will write

$$\mathbb{P}[\tau_I^\sigma = k] \equiv \mathbb{P}[\tau_I^\sigma = k|\sigma(0) = \sigma]. \tag{1.20}$$

Let us recall that in [BEGK1, BEGK2] a very similar program was carried out in a situation that we consider generic for systems having “metastable states”. A key characterization of the effective dynamics on such a set $\mathcal{M}$ involves the quantities $\mathbb{P}[\tau_I^x < \tau_x^x]$ (that, in potential theoretic language, are closely related to Newtonian capacities). There, as here, we identified certain subsets $\mathcal{M}$ of the state space, Γ. They are called metastable sets, if they satisfy the properties that

$$\frac{\sup_{x\in\Gamma} \mathbb{P}[\tau^x_{\mathcal{M}} < \tau^x_x]}{\inf_{z\in\mathcal{M}} \mathbb{P}[\tau^z_{\mathcal{M}\backslash z} < \tau^z_z]} \ll 1. \tag{1.21}$$

Equation (1.21) implies a separation of the time-scales of the motion *towards* the set $\mathcal{M}$ ("fast scale") and the motion *within* the set $\mathcal{M}$ ("slow scale"). Under some additional "non-degeneracy" hypothesis, namely that

(i) for all pairs $x, y \in \mathcal{M}$, and any set $I \subset \mathcal{M}\backslash\{x, y\}$ either $\mathbb{P}[\tau^x_I < \tau^x_x] \ll \mu(y)\mathbb{P}[\tau^y_I < \tau^y_y]$ or $\mathbb{P}[\tau^x_I < \tau^x_x] \gg \mu(y)\mathbb{P}[\tau^y_I < \tau^y_y]$, and
(ii) there exists $m_1 \in \mathcal{M}$, s.t. for all $x \in \mathcal{M}\backslash m_1$, $\mu(x) \ll \mu(m_1)$,

it was shown in [BEGK2] that the motion on the set $\mathcal{M}$ can be described as a sequence of exits with asymptotically exponentially distributed times (on distinct scales) towards the more stable states, i.e. the equilibrium. It was also shown that the inverse mean exit times from any point $x \in \mathcal{M}$ are asymptotically equal to the small eigenvalues of the generator of the Markov chain.

In the random energy model we will find ourselves in a situation where all of these hypotheses are not satisfied. When checking condition (1.21) with $\mathcal{M} \equiv T(E)$ we will see that this is not satisfied, and that, rather,

$$\frac{\sup_{\sigma\in\mathcal{S}_N} \mathbb{P}[\tau^\sigma_{T(E)} < \tau^\sigma_\sigma]}{\inf_{\sigma\in\mathcal{M}} \mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} < \tau^\sigma_\sigma]} \to 1, \quad \text{as} \quad E \downarrow -\infty. \tag{1.22}$$

Moreover, all the quantities $\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} < \tau^\sigma_\sigma]$ for $x \in T(E)$ will turn out to be comparable. Thus the situation is completely different than in [BEGK2], and we have to expect a much more complicated behaviour of the process on $T(E)$. Moreover, there is no natural criterion for the choice of a particular value of E, and we will, in fact, see later (in [BBG]) that it is somehow natural to consider limits as $E \downarrow -\infty$. In any case our purpose is the description of the process observed on $T(E)$.

Our first result concerns just the "motion" of the process disregarding time. To that effect we consider the random times

$$\begin{aligned} \theta_0 &\equiv \min\{n > 0 | \sigma(n) \in T(E)\}, \\ \theta_\ell &\equiv \min\{n > \theta_{\ell-1} | \sigma(n) \in T(E)\backslash\sigma(\theta_{\ell-1})\}. \end{aligned} \tag{1.23}$$

Let $\xi^1, \dots, \xi^{|T(E)|}$ be an enumeration of the elements of $T(E)$. Now define (for fixed N and E), the stochastic process Y_ℓ with state space $\{1, \dots, |T(E)|\}$ and discrete time $\ell \in \mathbb{N}$ by

$$Y^{(N)}_\ell = i \Leftrightarrow \sigma(\theta_\ell) = \xi^i. \tag{1.24}$$

It is easy to see that Y_ℓ is a Markov process. Moreover, the transition matrix elements can be expressed as

$$p(i, j) \equiv \mathbb{P}\left[\tau^{\xi^i}_{\xi^j} < \tau^{\xi^i}_{T(E)\backslash\{\xi^i\cup\xi^j\}}\right]. \tag{1.25}$$

Note that this Markov chain has a state space whose size $|T(E)|$ is a random variable. To formulate our first theorem it will be convenient to fix the size by conditioning. Thus set $P_M(\cdot) \equiv P(\cdot\,|\,|T(E)| = M)$.

Theorem 1.3. *Let $\sigma(n)$ denote the Markov chain with transition matrix defined in (1.3) and whose initial distribution is the uniform distribution on $\mathcal{S}_N$*[5]*. Let $Y^{(N)}$ be the Markov process defined by (1.24). Let Y_ℓ denote the Markov chain on $\{1, \dots, M\}$ with transition matrix p_M^* given by*

$$p_M^*(i,j) = \begin{cases} \frac{1}{M-1}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases} \tag{1.26}$$

and initial distribution $p_M^*(i) = 1/M$. Then, for all $M \in \mathbb{N}$,

$$Y^{(N)} \xrightarrow{\mathcal{D}} Y, \quad P_M\text{-a.s.} \tag{1.27}$$

Remark. Note that the statement of the theorem also implies the convergence in law (w.r.t. P) of the probability distribution of $Y^{(N)}$ to that of Y.

The next results concern mean times.

Theorem 1.4. *Assume that $\alpha \equiv \beta/\sqrt{2\ln 2} > 1$. Then there exists a subset $\widetilde{\mathcal{E}} \subset \Omega$ with $P(\widetilde{\mathcal{E}}) = 1$, such that for all $\omega \in \widetilde{\mathcal{E}}$, for all N large enough, the following holds:*

i) For all $\eta \in T(E)$,

$$\mathbb{E}\left(\tau^\eta_{T(E)\setminus\eta}\right) = \frac{1}{1-\frac{1}{|T(E)|}}\left[e^{\beta\sqrt{N}E_\eta} + \mathcal{W}_{\beta,N,T(E)}\right](1+O(1/N)). \tag{1.28}$$

ii) For all $\sigma \notin T(E)$,

$$\begin{aligned} \mathbb{E}\left(\tau^\sigma_{T(E)}\right) &\leq \frac{1}{1-\frac{1}{|T(E)|}}\left[e^{\beta\sqrt{N}E_\sigma} + \mathcal{W}_{\beta,N,T(E)}\right](1+O(1/N)) \\ \mathbb{E}(\tau^\sigma_{T(E)}) &\geq \frac{1}{1-\frac{1}{|T(E)|}}\left[e^{\beta\sqrt{N}E_\sigma} + \frac{1-e^E(\alpha-1)}{1+1/|T(E)|}\mathcal{W}_{\beta,N,T(E)}\right](1+O(1/N)). \end{aligned} \tag{1.29}$$

iii) For all $\eta, \bar\eta \in T(E)$, $\eta \neq \bar\eta$,

$$\left|\mathbb{E}\left(\tau^\eta_{\bar\eta} \mid \tau^\eta_{\bar\eta} \leq \tau^\eta_{T(E)\setminus\eta}\right) - \mathbb{E}\left(\tau^\eta_{T(E)\setminus\eta}\right)\right| \leq \frac{1}{1-\frac{1}{|T(E)|}}\mathcal{W}_{\beta,N,T(E)}O(1/N), \tag{1.30}$$

where

$$\mathcal{W}_{\beta,N,T(E)} \equiv \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{|T(E)|(\alpha-1)}\left(1+\mathcal{V}_{N,E}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right) \tag{1.31}$$

and $\mathcal{V}_{N,E}$ is a random variable of mean zero and variance one.

Theorem 1.4 is complemented by a somewhat converse result in the case $\alpha < 1$:

[5] In fact it is enough, for the result to hold, that the initial distribution gives zero mass to an ϵ-neighborhood of $T(E)$.

Theorem 1.5. *Assume that $\alpha < 1$. Then, with probability one, for all N large enough, for all $\sigma \in \mathcal{S}_N$,*

$$\mathbb{E}\tau^{\sigma}_{T(E)} = \frac{1}{|T(E)|-1} e^{N(\beta^2/2+\ln 2)}(1+O(1/N)) \gg \sup_{\eta\in\mathcal{S}_N} \mathbb{E}\tau^{\eta}_{\mathcal{S}_N\setminus\eta}. \tag{1.32}$$

Remark. Since as $N \uparrow \infty, \mathbb{E}|T(E)| \to e^{-E}$, we see that for $-E$ very large, $\mathcal{W}_{\beta,N,T(E)} \sim e^{\alpha(E+u_N(0))}$. Thus (ii) of Theorem 1.4 implies that if $\alpha > 1$, for all $\sigma \notin T(E)$, the mean time of arrival in the top is proportional to $e^{\alpha(E+u_N(0))}$. On the other hand, there exists $\eta \in T(E)$ such that $\sqrt{N}E_\eta \sim E+u_N(0)+O(e^E)$, so that the slowest times of exit from a state, $\mathbb{E}\tau^{\eta}_{\mathcal{S}_N\setminus\eta} = e^{\beta\sqrt{N}E_\eta}$, in $T(E)$ are just of the same order. This can be expressed by saying that on the average the process takes a time t to reach states that have an exit time t. This is a first, and weak, manifestation of the aging phenomenon that we will investigate in much greater detail in [BBG]. In contrast, if $\alpha < 1$, Theorem 1.5 $\mathbb{E}\tau^{\sigma}_{T(E)} \gg \sup_{\eta\in\mathcal{S}_N} \mathbb{E}\tau^{\eta}_{\mathcal{S}_N\setminus\eta}$, and thus the time spent in top states is irrelevant compared to the time between successive visits of such states. Thus we see a clear distinction between the high and the low temperature phase of the REM on the dynamical level.

Remark. Statement iii) of Theorem 1.4 expresses the fact that the mean times of passage from a state $\eta \in T(E)$ to another state $\bar\eta \in T(E)$ are asymptotically independent of the terminal state $\bar\eta$. This confirms to some extent the heuristic picture of Bouchaud. Indeed, if we added the hypothesis that the process observed on the top is Markovian, then the two preceding theorems would immediately imply that the waiting times must be exponentially distributed with rates independent of the terminal state and given by (1.30). We will see in [BBG] that this, however, cannot be justified.

The remainder of this paper is devoted to proving Theorems 1.3, 1.4, and 1.5. Section 2 will in fact prove a number of results that will not only imply Theorem 1.3, but will also furnish basic input to both Sect. 3 and the follow-up paper [BBG]. Section 3 contains the proof of Theorems 1.4 and 1.5.

2. Probability Estimates

In this section we provide estimates that will immediately allow to prove Theorem 1.3. In fact we will prove much more, anticipating what will be needed in Sect. 3 as well as in the follow-up paper [BBG]. These results are collected in the following proposition.

Proposition 2.1. *Set $M = |T(E)|$, $d = 2^M$ and $\delta(N) \equiv \left(\frac{d}{N}\right)^{1/2} \log N$. There exists a subset $\mathcal{E} \subset \Omega$ with $P(\mathcal{E}) = 1$, such that for all $\omega \in \mathcal{E}$, for all N large enough, the following holds:*

For $\varepsilon > 0$ a constant, define the sets

$$\mathcal{B}_{\sqrt{\varepsilon N}}(\sigma) = \left\{\sigma' \in \mathcal{S}_N \mid \|\sigma'-\sigma\|_2 \le \sqrt{\varepsilon N}\right\}, \quad \sigma \in \mathcal{S}_N \tag{2.1}$$

and

$$W_\varepsilon(I) \equiv \bigcap_{\sigma\in I} \mathcal{B}^c_{\sqrt{\varepsilon N}}(\sigma), \quad I \subseteq \mathcal{S}_N. \tag{2.2}$$

Then,

i) For all $\varepsilon > 0$ *there exists a constant* $c > 0$ *such that, for all* $\eta \in T(E)$ *and all* $\sigma \in W_\varepsilon(T(E))$,

$$\left|\mathbb{P}\left(\tau_\eta^\sigma < \tau_{T(E)\setminus\eta}^\sigma\right) - \frac{1}{M}\right| \le \frac{d}{NM}(1 + c\delta(N)). \tag{2.3}$$

ii) There exists a constant $c > 0$ *such that, for all* $\eta \in T(E)$ *and* $\bar\eta \in T(E)$ *with* $\eta \neq \bar\eta$,

$$\left|e^{\beta\sqrt{N}E_{\bar\eta}}\mathbb{P}\left(\tau_\eta^{\bar\eta} < \tau_{T(E)\setminus\eta}^{\bar\eta}\right) - \frac{1}{M}\right| \le \frac{d}{NM}(1 + c\delta(N)). \tag{2.4}$$

iii) There exists a constant $c > 0$ *such that, for all* $\eta \in T(E)$ *and* $\bar\eta \in T(E)$ *with* $\eta \neq \bar\eta$,

$$\left|\mathbb{P}\left(\tau_\eta^{\bar\eta} < \tau_{T(E)\setminus\{\eta,\bar\eta\}}^{\bar\eta}\right) - \frac{1}{M-1}\right| \le \frac{d}{N(M-1)}(1 + c\delta(N)). \tag{2.5}$$

iv) There exists a constant $c > 0$ *such that, for all* $\eta \in T(E)$,

$$\left|e^{\beta\sqrt{N}E_\eta}\mathbb{P}\left(\tau_{T(E)\setminus\eta}^\eta < \tau_\eta^\eta\right) - \left(1 - \frac{1}{M}\right)\right| \le \left(1 - \frac{1}{M}\right)\frac{d}{N}(1 + c\delta(N)). \tag{2.6}$$

v) There exists a constant $c > 0$ *such that, for all* $\sigma \notin T(E)$,

$$\left(1 - \frac{1}{M}\right)\left(1 - \frac{d}{N}(1 + c\delta(N))\right) \le e^{\beta\sqrt{N}E_\sigma}\mathbb{P}\left(\tau_{T(E)}^\sigma < \tau_\sigma^\sigma\right) \le 1. \tag{2.7}$$

vi) For all $\varepsilon > 0$ *there exists a constant* $c > 0$ *such that, for all* $\sigma \notin T(E)$ *and all* $\bar\sigma \in W_\varepsilon(T(E) \cup \sigma)$,

$$\frac{1}{M+1} + \frac{d}{NM}(1 - c\delta(N)) \le \mathbb{P}\left(\tau_\sigma^{\bar\sigma} \le \tau_{T(E)}^{\bar\sigma}\right) \le \frac{1}{M} + \frac{d}{NM}(1 + c\delta(N)). \tag{2.8}$$

Proof of Theorem 1.3. Assuming the proposition, Theorem 1.3 follows immediately from iii) and i), together with the fact that the mass of the set $\mathcal{S}_N \setminus W_\epsilon(T)$ under the uniform measure on $\mathcal{S}_N$ tends to zero as N tends to infinity. □

Let us briefly highlight the structure of the proof of Proposition 2.1. In Subsect. 2.1 we will show that, for $I \subset \mathcal{S}_N$, the probabilities $\mathbb{P}\left(\tau_\eta^\sigma < \tau_I^\sigma\right)$ can be expressed in terms of a *lumped chain* through a *lumping procedure* that allows to reduce the high dimensional state space $\mathcal{S}_N$ to a much smaller one. In Subsect. 2.2 we analyse the lumped chain and establish the probability estimates which will serve as basic input to the proof of Proposition 2.1. The proof of the proposition is then carried out in Subsect. 2.3.

2.1. Lumped chains: Definition and properties.

Lumping procedure. We begin with some preparatory notation and definitions. For M an integer, let $\mathcal{S}_{M\times N}$ be the set of all $M \times N$ matrices whose elements belong to $\mathcal{S} = \{-1, 1\}$. A matrix $\xi \in \mathcal{S}_{M\times N}$ will be written either in terms of its matrix elements, row vectors or column vectors according to the following notation. In terms of its matrix elements we will write $\xi = (\xi_i^\mu)_{i=1,\dots,N}^{\mu=1,\dots,M}$, where $\xi_i^\mu \in \mathcal{S}$ is the element lying at the intersection of the μ^{th} row and i^{th} column. The row and column vectors of ξ will be denoted respectively by ξ^μ and ξ_i, and written, in terms of their elements, as:

$$\begin{aligned} \xi^\mu &= (\xi_i^\mu)_{i=1,\dots,N} \in \mathcal{S}_N, \quad \mu \in \{1,\dots,M\}, \\ \xi_i &= (\xi_i^\mu)^{\mu=1,\dots,M} \in \mathcal{S}_M, \quad i \in \{1,\dots,N\}. \end{aligned} \tag{2.9}$$

Observe that, when carrying an index placed as a superscript, the letter ξ refers to an element of the cube $\mathcal{S}_N$ while, when carrying an index placed as a subscript, it refers to an element of the cube $\mathcal{S}_M$.

As is usual, ξ may then be written as the N-tuple formed by its column vectors,

$$\xi = (\xi_1, \dots, \xi_i, \dots, \xi_N) \tag{2.10}$$

or, denoting by ${}^t\xi$ the transpose matrix, as the M-tuple formed by its row vectors,

$${}^t\xi = (\xi^1, \dots, \xi^\mu, \dots, \xi^M). \tag{2.11}$$

Given a subset $I \subset \mathcal{S}_N$ we define a partition of the index set $\Lambda \equiv \{1, \dots, i, \dots, N\}$ in the following way. Let $\xi = (\xi_1, \dots, \xi_i, \dots, \xi_N) \in \mathcal{S}_{|I|\times N}$ be any matrix having the property that

$$I = \left\{\xi^1, \dots, \xi^\mu, \dots, \xi^{|I|}\right\} \tag{2.12}$$

in other words, any matrix having the set I for a set of row vectors. Next, let $\{e_1, \dots, e_k, \dots, e_d\}$ be an arbitrarily chosen labeling of all $d = 2^{|I|}$ elements of $\mathcal{S}_{|I|}$ (this labeling will be kept fixed throughout, whatever the choice of I is). Then ξ induces a partition of Λ into d disjoint (possibly empty) subsets, $\Lambda_k(I)$, obtained by grouping together all indices i having the property that $\xi_i = e_k$:

$$\Lambda = \bigcup_{k=1}^{d} \Lambda_k(I), \quad \Lambda_k(I) = \{i \in \Lambda \mid \xi_i = e_k\}. \tag{2.13}$$

We will write

$$\mathcal{P}_I(\Lambda) = \{\Lambda_k(I),\ 1 \le k \le d\}. \tag{2.14}$$

Remark. Observe that with the notation introduced above, we do not keep track of the particular choice of the matrix ξ we made. The reason is that since any two matrices satisfying (2.12) are obtained from each other by a permutation of their rows, the partitions they induce only differ through the labeling of the sets (2.13). As this labeling will be irrelevant for our purposes we will as a rule forget the underlying matrix. It is understood that in all statements involving $\mathcal{P}_I(\Lambda)$, a choice has been fixed.

Finally, this partition is used to define a many-to-one function, γ_I, that maps the elements of $\mathcal{S}_N$ into d-dimensional vectors,

$$\gamma_I(\sigma) = \left(\gamma_I^1(\sigma), \dots, \gamma_I^k(\sigma), \dots, \gamma_I^d(\sigma)\right), \quad \sigma \in \mathcal{S}_N, \tag{2.15}$$

where, for all $k \in \{1, \dots, d\}$,

$$\gamma_I^k(\sigma) = \frac{1}{|\Lambda_k(I)|} \sum_{i \in \Lambda_k(I)} \sigma_i. \tag{2.16}$$

A few elementary properties of γ_I are listed in the lemma below.

Lemma 2.2.
i) The range of γ_I, $\Gamma_{N,d}(I) \equiv \gamma_I(\mathcal{S}_N)$, is a discrete subset of the d-dimensional cube $[-1,1]^d$ and may be described as follows. Let $\{u_k\}_{k=1}^d$ be the canonical basis of $\mathbb{R}^d$. Then,

$$x \in \Gamma_{N,d}(I) \Longleftrightarrow x = \sum_{k=1}^d \frac{n_k}{|\Lambda_k(I)|} u_k, \tag{2.17}$$

where, for all $1 \le k \le d$, $|n_k| \le |\Lambda_k(I)|$ has the same parity as $|\Lambda_k(I)|$.
ii)

$$|\{\sigma \in \mathcal{S}_N \mid \gamma_I(\sigma) = x\}| = \prod_{k=1}^d \binom{|\Lambda_k(I)|}{|\Lambda_k(I)|\frac{1+x_k}{2}}, \quad \forall x \in \Gamma_{N,d}(I). \tag{2.18}$$

In particular, the restriction of γ_I to I is a one-to-one mapping from I onto $\gamma_I(I)$.
iii) The elements of I are mapped onto corners of $[-1,1]^d$: for all $\sigma \in I$,

$$\gamma_I(\sigma) = (\sigma_{i_1}, \dots, \sigma_{i_k} \dots, \sigma_{i_d}), \qquad \text{for any choice of indices} \quad i_k \in \Lambda_k(I). \tag{2.19}$$

iv) Let $\sigma \in \mathcal{S}_N$ be such that $\inf_{\eta \in I \setminus \sigma} \|\sigma - \eta\|_2 \ge \sqrt{\varepsilon N}$ for some $\varepsilon > 0$. Set $x \equiv \gamma_I(\sigma)$ and $\mathcal{I} \equiv \gamma_I(I)$. Then

$$\inf_{y \in \mathcal{I} \setminus x} \|x - y\|_2 \ge \frac{\varepsilon N}{2\sqrt{d} \max_k |\Lambda_k(I)|}. \tag{2.20}$$

Proof of Lemma 2.2. Assertions i), ii), and iii) result from elementary observations. To prove assertion iv) note that for any $\eta \in I \setminus \sigma$, setting $y \equiv \gamma_I(\eta)$ and using 2.19, we have:

$$\begin{aligned}\varepsilon N &\le \sum_{i=1}^N (\sigma_i - \eta_i)^2 = \sum_{k=1}^d \sum_{i \in \Lambda_k} (\sigma_i - y_k)^2 = 2 \sum_{k=1}^d |\Lambda_k(I)|(1 - y_k x_k) \\ &\le 2 \max_k |\Lambda_k(I)|(y, y - x),\end{aligned} \tag{2.21}$$

where we used in the last line that $1 - y_k x_k = y_k(y_k - x_k)$. But $(y, y - x) \le \|y\|_2 \|y - x\|_2 = \sqrt{d}\|y - x\|_2$, so that

$$\|x - y\|_2 \ge \frac{\varepsilon N}{2\sqrt{d} \max_k |\Lambda_k(I)|} \tag{2.22}$$

which, together with assertion ii) yields (2.20). $\square$

The I-lumped chain. In the sequel we will denote by $\{\sigma_N^\circ(t)\}_{t\in\mathbb{N}}$ the ordinary random walk (ORW) associated to $\{\sigma_N(t)\}_{t\in\mathbb{N}}$, that is, the walk evolving on the edges of $\mathcal{G}_N$ according to the transition probabilities

$$p_N^\circ(\sigma,\sigma') = \begin{cases} \frac{1}{N}, & \text{if} \quad \|\sigma-\sigma'\|_2 = \sqrt{2} \\ 0, & \text{otherwise} \end{cases}. \tag{2.23}$$

All objects referring to the ORW will be distinguished from those referring to the chain $\{\sigma_N(t)\}$ by the superscript $^\circ$. Note in particular that $\{\sigma_N^\circ(t)\}$ is reversible w.r.t. the measure

$$\mu_N^\circ(\sigma) = 2^{-N}, \quad \sigma \in \mathcal{S}_N. \tag{2.24}$$

We will denote by $\{X_{I,N}(t)\}_{t\in\mathbb{N}}$ and call the *I-lumped chain* or the *lumped chain induced by I*, the chain defined through

$$X_{I,N}(t) \equiv \gamma_I(\sigma_N^\circ(t)), \quad \forall t \in \mathbb{N}. \tag{2.25}$$

To $\Gamma_{N,d}(I)$ we associate an undirected graph, $\mathcal{G}(\Gamma_{N,d}(I)) = (V(\Gamma_{N,d}(I)), E(\Gamma_{N,d}(I)))$, with set of vertices $V(\Gamma_{N,d}(I)) = \Gamma_{N,d}(I)$ and the set of edges:

$$E(\Gamma_{N,d}(I)) = \left\{(x,x') \in \Gamma_{N,d}(I) \mid \exists_{k\in\{1,\dots,d\}}, \exists_{s\in\{-1,1\}} \ : \ x'-x = s\frac{2}{|\Lambda_k(I)|}u_k\right\}. \tag{2.26}$$

The properties of $\{X_{I,N}(t)\}$ are summarized in the lemma below.

Lemma 2.3. *Given any subset $I \in \mathcal{S}_N$:*

i) The process $\{X_{I,N}(t)\}$ is Markovian no matter how the initial distribution π° of $\{\sigma_N^\circ(t)\}$ is chosen.

ii) Set $\mathbb{Q}_N^\circ = \mu_N^\circ \circ \gamma_I^{-1}$. Then $\mathbb{Q}_N^\circ$ is the unique reversible invariant measure for the chain $\{X_{I,N}(t)\}$. In explicit form, the density of $\mathbb{Q}_N^\circ$ reads:

$$\mathbb{Q}_N^\circ(x) = \frac{1}{2^N}|\{\sigma \in \mathcal{S}_N \mid \gamma_I(\sigma) = x\}|, \quad \forall x \in \Gamma_{N,d}(I). \tag{2.27}$$

iii) The transition function $r_N^\circ(\,.\,,\,.\,)$ of $\{X_{I,N}(t)\}$ does not depend on the choice of π° and is given by:

$$r_N^\circ(x,x') = \begin{cases} \frac{|\Lambda_k(I)|}{N}\frac{1-sx_k}{2} & \textit{if} \quad (x,x') \in E(\Gamma_{N,d}(I)) \ \textit{and} \ x'-x = s\frac{2}{|\Lambda_k(I)|}u_k \\ 0, & \textit{otherwise} \end{cases}. \tag{2.28}$$

Proof. The proof of this lemma is a direct application of the results of Burke and Rosenblatt [BR] on Markovian functions of Markov Chains. □

Comparison lemmata. In order to make use of the above set-up we first need to establish how the Markov chain $\sigma(t)$ relates to the ORW. This is done in the next lemma.

Lemma 2.4. *Let $I \subset \mathcal{S}_N$. Then,*

i) for all $\sigma \notin I$ and $\eta \notin I \cup \sigma$

$$\mathbb{P}\left(\tau_\eta^\sigma < \tau_I^\sigma\right) = \mathbb{P}^\circ\left(\tau_\eta^\sigma < \tau_I^\sigma\right), \tag{2.29}$$

ii) for all $\sigma \in I$ and $\eta \notin I$

$$\mathbb{P}\left(\tau_\eta^\sigma < \tau_I^\sigma\right) = e^{-\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ\left(\tau_\eta^\sigma < \tau_I^\sigma\right). \tag{2.30}$$

It finally remains to establish how the quantities $\mathbb{P}^\circ\left(\tau_\eta^\sigma < \tau_I^\sigma\right)$ can be expressed in terms of a lumped chain.

Lemma 2.5. *Let $I, J, K \subset \mathcal{S}_N$ be such that $I \cap J = \emptyset$ and $I \cup J \subseteq K$. Then, denoting by $\mathbb{R}^\circ$ the law of the K-lumped chain,*

$$\mathbb{P}^\circ\left(\tau_I^\sigma \leq \tau_J^\sigma\right) = \mathbb{R}^\circ\left(\tau_{\gamma_K(I)}^{\gamma_K(\sigma)} \leq \tau_{\gamma_K(J)}^{\gamma_K(\sigma)}\right), \qquad \textit{for all} \quad \sigma \notin I. \tag{2.31}$$

Remark. Note that K in the above lemma does not necessarily contain σ if $\sigma \notin J$.

We skip the proofs of Lemma 2.4 and 2.5 as they are nothing but elementary exercises.

2.2. Main ingredients of the proof of Proposition 2.1. Observe that the entropy produced by the lumping procedure gives rise through (2.27) to a potential, $F_N(x) \equiv -\frac{1}{N}\ln \mathbb{Q}_N^\circ(x)$. It moreover follows from assertions ii) and iii) of Lemma 2.2 that this potential is convex and takes on its global maximum at the corners of the cube $[-1,1]^d$. This allows us to draw on the results of [BEGK1] where such processes were investigated.

Throughout this section I denotes an arbitrary (non empty) subset of $\mathcal{S}_N$ whose size, $|I|$, does not depend on N. Given $0 < \epsilon < 1$ let $\mathcal{K}(I)$ and $\mathcal{K}(I)^c$ be the sets defined through:

$$\begin{aligned} \mathcal{K}(I) &\equiv \mathcal{K}_\epsilon(I) \equiv \left\{k \in \{1,\dots,d\} \mid |\Lambda_k(I)| \geq \epsilon\tfrac{N}{d}\right\}, \\ \mathcal{K}(I)^c &\equiv \mathcal{K}_\epsilon(I)^c \equiv \{1,\dots,d\} \setminus \mathcal{K}_\epsilon(I). \end{aligned} \tag{2.32}$$

Set $\kappa = |\mathcal{K}(I)|$. Of course $\kappa \geq 1$ since supposing $\kappa = 0$, (2.32) implies that $\sum_{k=1}^d |\Lambda_k(I)| < \epsilon N < N$, contradicting (2.13). Let $\pi : \mathbb{R}^d \to \mathbb{R}^\kappa$ be the projection that maps $x = (x_1,\dots,x_d)$ into $\pi x = (x_{i_1},\dots,x_{i_\kappa})$ where, for all $1 \leq j \leq \kappa, i_j \in \mathcal{K}(I)$. Finally, set

$$N_* = \min_{k\in\mathcal{K}(I)} |\Lambda_k|. \tag{2.33}$$

With this notation we have:

Lemma 2.6. *There exists a constant $c > 0$ such that, for all N large enough,*

$$\mathbb{R}^\circ\left(\tau_0^x < \tau_x^x\right) \geq \left(1 - \frac{1}{N}\sum_{\mu\in\mathcal{K}(I)^c}|\Lambda_\mu|\right)\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right), \qquad \textit{for all} \quad x \in \gamma_I(I). \tag{2.34}$$

Lemma 2.7. *Let $x \in \gamma_I(I)$ and $y \in \gamma_I(\mathcal{S}_N)$ be such that $\|\pi x - \pi y\|_2 \geq \delta$ for some constant $\frac{1}{2} > \delta > 0$. Then there exist a constant $h(\delta,\kappa) > 0$ such that, for all N large enough,*

$$\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) \leq e^{-N_* h(\delta,\kappa)}. \tag{2.35}$$

As an important consequence of the previous two lemmata we may immediately state:

Lemma 2.8. *Let $x \in \gamma_I(I)$ and $J \subseteq \gamma_I(I)$ be such that for all $y, y' \in J \cup x$, $\|\pi y' - \pi y\|_2 \geq \delta$ for some $\delta > 0$. Then, for all N large enough,*

$$\frac{\vartheta}{|J|} \leq \mathbb{R}^\circ\left(\tau_x^0 \leq \tau_J^0\right) \leq \frac{1}{\vartheta|J|}, \qquad \textit{for all} \quad J \subseteq \gamma_I(I),\ x \in \gamma_I(I), \tag{2.36}$$

where

$$\vartheta = \left(1 - \frac{1}{N}\sum_{\mu\in\mathcal{K}(I)^c}|\Lambda_\mu|\right)\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right). \tag{2.37}$$

In particular, if $\mathcal{K}(I)^c = \emptyset$,

$$\left|\mathbb{R}^\circ\left(\tau_x^0 \leq \tau_J^0\right) - \frac{1}{|J|}\right| \leq \frac{1}{|J|N_*}\left(1 + \frac{c}{N_*}\right) \tag{2.38}$$

for some constant $c > 0$.

Proof of Lemma 2.6. An L-steps path ω on $\Gamma_{N,d}(I)$, beginning at x and ending at y is defined as a sequence of L sites $\omega = (\omega_0, \omega_1, \dots, \omega_L)$, with $\omega_0 = x$, $\omega_L = y$, and $\omega_l = (\omega_l^k)_{k=1,\dots,d} \in V(\Gamma_{N,d}(I))$ for all $1 \leq l \leq L$, that satisfies:

$$(\omega_l, \omega_{l-1}) \in E(\Gamma_{N,d}(I)), \qquad \text{for all} \quad l = 1, \dots, L. \tag{2.39}$$

(We may also write $|\omega| = L$ to denote the length of ω.)

Recall from Lemma 2.2 that if $x \in \gamma_I(I)$, then a fortiori $x \in \{-1, 1\}^d$. Without loss of generality we may thus choose x in (2.34) as the point $x = (x_k)_{k=1}^d$, $x_k = 1$ for all $1 \leq k \leq d$. There is no loss of generality either in taking $\mathcal{K}(I)$ in (2.32) to be the set $\mathcal{K}(I) = \{1, \dots, \kappa\}$ and in assuming $|\Lambda_k(I)|$ to be even for all $k \in \mathcal{K}(I)$. With this we introduce κ one-dimensional paths in $\Gamma_{N,d}(I)$, each being of length $L \equiv \kappa N_*/2$, and connecting x to the endpoint y defined by

$$y = (y_k)_{k=1}^d, \quad y_k = \begin{cases} 1 - \frac{N_*}{|\Lambda_k|}, & \text{if} \quad k \in \mathcal{K}(I) \\ 1, & \text{if} \quad k \in \mathcal{K}(I)^c \end{cases}. \tag{2.40}$$

Definition 2.9. *For each* $1 \le \mu \le \kappa$, *let* $\omega(\mu) = (\omega_0(\mu), \dots, \omega_n(\mu), \dots, \omega_L(\mu))$, $\omega_n(\mu) = (\omega_n^k(\mu))_{k=1}^d$, *be the path in* $\Gamma_{N,d}(I)$ *defined through*

$$\omega_n^k(\mu) = \begin{cases} \omega_n^{(k+\mu-2)\, mod \kappa+1}, & \text{if} \quad k \in \mathcal{K}(I) \\ 1 & \text{if} \quad k \in \mathcal{K}(I)^c \end{cases}, \tag{2.41}$$

where $\omega = (\omega_0, \dots, \omega_n, \dots, \omega_L)$, $\omega_n = (\omega_n^k)_{k=1}^d$, *is defined by*

$$\omega_0 = x \tag{2.42}$$

and, for $1 \le n \le L$,

$$\omega_n^k = \begin{cases} 1 - \frac{2}{|\Lambda_k(I)|}\left[\frac{n-1}{\kappa}\right] - \frac{2}{|\Lambda_k(I)|}, & \text{if} \quad k \in \mathcal{K}(I) \quad \text{and} \quad k \le n - \kappa\left[\frac{n-1}{\kappa}\right] \\ 1 - \frac{2}{|\Lambda_k(I)|}\left[\frac{n-1}{\kappa}\right], & \text{if} \quad k \in \mathcal{K}(I) \quad \text{and} \quad k > n - \kappa\left[\frac{n-1}{\kappa}\right] \\ 1 & \text{if} \quad k \in \mathcal{K}(I)^c \end{cases} \cdot \tag{2.43}$$

Here $[x]$, $x \in \mathbb{R}$, *denotes the integer part of* x. *(The paths* $\omega(\mu)$ *are in fact paths on the subgraph* $\{z \in \Gamma_{N,d}(I) \mid z_k = 1\, \forall k \in \mathcal{K}(I)^c\}$.*)*

Let D be the subgraph of $\mathcal{G}(\Gamma_{N,d}(I))$ with a set of vertices $V(D) = \{x' \in \Gamma_{N,d}(I) \mid \|x'\|_2 \le \|y\|_2\}$ and a set of edges $E(D) = \{(x', x'') \in E(\Gamma_{N,d}(I)) \mid x', x'' \in V(D)\}$. Denoting by Δ_μ the subgraph of $\mathcal{G}(\Gamma_{N,d}(I))$ "generated" by the path $\omega(\mu)$, i.e., with a set of vertices $V(\Delta_\mu) = \{x' \in \Gamma_{N,d}(I) \mid \exists_{0 \le n \le L} : x' = \omega_n(\mu)\}$, we set

$$\Delta = D \cup \bigcup_{\mu=1}^{\kappa} \Delta_\mu. \tag{2.44}$$

Since both x and 0 belong to Δ it follows from Lemma (4.1) of the appendix that

$$\begin{aligned} \mathbb{R}^\circ\left(\tau_0^x < \tau_x^x\right) &\ge \widetilde{\mathbb{R}}_\Delta^\circ\left(\tau_0^x < \tau_x^x\right) \\ &= \widetilde{\mathbb{R}}_\Delta^\circ\left(\tau_y^x < \tau_x^x\right)\widetilde{\mathbb{R}}_\Delta^\circ\left(\tau_0^y < \tau_x^y\right), \end{aligned} \tag{2.45}$$

where the last equality is nothing but the Markov property. Again, the collection Δ_μ, $1 \le \mu \le \kappa$, being easily seen to verify conditions (4.2) and (4.3) of Lemma 4.1 (w.r.t. the event $\left\{\tau_y^x < \tau_x^x\right\}$), we have, applying the latter lemma twice in a row,

$$\widetilde{\mathbb{R}}_\Delta^\circ\left(\tau_y^x < \tau_x^x\right) \ge \widetilde{\mathbb{R}}_{\cup_{\mu=1}^{\kappa}\Delta_\mu}^\circ\left(\tau_y^x < \tau_x^x\right) \ge \sum_{\mu=1}^{\kappa}\widetilde{\mathbb{R}}_{\Delta_\mu}^\circ\left(\tau_{\omega_L(\mu)}^{\omega_0(\mu)} < \tau_{\omega_0(\mu)}^{\omega_0(\mu)}\right) \tag{2.46}$$

and combining (2.45) and (2.46), we have,

$$\mathbb{R}^\circ\left(\tau_0^x < \tau_x^x\right) \ge \widetilde{\mathbb{R}}_\Delta^\circ\left(\tau_0^y < \tau_x^y\right)\sum_{\mu=1}^{\kappa}\widetilde{\mathbb{R}}_{\Delta_\mu}^\circ\left(\tau_{\omega_L(\mu)}^{\omega_0(\mu)} < \tau_{\omega_0(\mu)}^{\omega_0(\mu)}\right). \tag{2.47}$$

The bound (2.45) is of course meaningless if it so happens that $y = 0$. In this special case we only use (2.46) to write

$$\mathbb{R}^\circ\left(\tau_0^x < \tau_x^x\right) \ge \sum_{\mu=1}^{\kappa}\widetilde{\mathbb{R}}_{\Delta_\mu}^\circ\left(\tau_{\omega_L(\mu)}^{\omega_0(\mu)} < \tau_{\omega_0(\mu)}^{\omega_0(\mu)}\right). \tag{2.48}$$

Thus, in view of (2.47) and (2.48), Lemma 2.6 will be proven if we can establish that:

Lemma 2.10. *Under the assumptions of Lemma 2.6:*

i) There exists a constant $c > 0$ such that, for large enough N, for each $\mu \in \mathcal{K}(I)$ and with N_* defined as in (2.33),

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_\mu}\left(\tau^{\omega_0(\mu)}_{\omega_L(\mu)} < \tau^{\omega_0(\mu)}_{\omega_0(\mu)}\right) \geq \frac{|\Lambda_\mu|}{N}\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right). \tag{2.49}$$

ii) Assume that $y \neq 0$. There exists a constant $c > 0$ such that, for all N large enough,

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta}\left(\tau^y_0 < \tau^y_x\right) \geq 1 - cdN^{3/2}2^{-\epsilon N/d}. \tag{2.50}$$

Proof of Lemma 2.10 i). To simplify the presentation we will only treat the case $\mu = 1$, that is, with the notation of Definition 2.9, establish that

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) \geq \frac{|\Lambda_1|}{N}\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right). \tag{2.51}$$

It is well known that (see e.g. [Sp] or [BEGK1], Lemma 2.5)

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) = \left[\sum_{n=1}^{L} \frac{\widetilde{\mathbb{Q}}^{\circ}_{\Delta_\mu}(\omega_0)}{\widetilde{\mathbb{Q}}^{\circ}_{\Delta_\mu}(\omega_n)} \frac{1}{\widetilde{r}^{\circ}_{\Delta_\mu}(\omega_n, \omega_{n-1})}\right]^{-1} \tag{2.52}$$

which we may also write, using reversibility together with the definitions of $\widetilde{r}^{\circ}_{\Delta_\mu}$ and $\widetilde{\mathbb{Q}}^{\circ}_{\Delta_\mu}$ (see Appendix A),

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) = \left[\sum_{n=0}^{L-1} \frac{\mathbb{Q}_N(\omega_0)}{\mathbb{Q}_N(\omega_n)} \frac{1}{r_N(\omega_n, \omega_{n+1})}\right]^{-1} = \left[\sum_{m=0}^{N_*/2-1} \sum_{l=1}^{\kappa} A_{m,l}\right]^{-1}, \tag{2.53}$$

where

$$A_{m,l} \equiv \frac{\mathbb{Q}_N(\omega_0)}{\mathbb{Q}_N(\omega_{m\kappa+l-1})} \frac{1}{r_N(\omega_{m\kappa+l-1}, \omega_{m\kappa+l})}. \tag{2.54}$$

By (2.27) and (2.18),

$$A^{-1}_{m,l} = \frac{|\Lambda_l| - m}{N} \prod_{k=1}^{\kappa} \binom{|\Lambda_k|}{|\Lambda_k| \frac{1-\omega^k_{m\kappa+l-1}}{2}}, \tag{2.55}$$

and by (2.43)

$$\begin{aligned} A^{-1}_{m,l} &= \frac{|\Lambda_l| - m}{N} \prod_{k>l-1} \binom{|\Lambda_k|}{m} \prod_{k\leq l-1} \binom{|\Lambda_k|}{m+1} \\ &= \frac{|\Lambda_l| - m}{N} \prod_{k'=1}^{\kappa} \binom{|\Lambda_{k'}|}{m} \prod_{k\leq l-1} \frac{|\Lambda_k| - m}{m+1}, \end{aligned} \tag{2.56}$$

where we use the convention that the second product above is one whenever the index set $k \leq l - 1$ is empty. From now on we distinguish two cases.

1) The case $\kappa = 1$. Here $N_* = |\Lambda_1|$. Inserting (2.56) in (2.53) yields

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) = \frac{|\Lambda_1|}{N}\left[\sum_{m=0}^{N_*/2-1} C_m\right]^{-1}, \tag{2.57}$$

where

$$C_m \equiv \binom{|\Lambda_1|}{m}^{-1}\frac{|\Lambda_1|}{|\Lambda_1|-m} = \binom{|\Lambda_1|-1}{m}^{-1}. \tag{2.58}$$

Then, using the bound

$$\binom{|\Lambda_1|}{m}^{-1} \leq \frac{6}{(|\Lambda_1|-1)^3}, \quad 3 \leq m \leq N_*/2-1 \tag{2.59}$$

easily yields

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) \geq \frac{|\Lambda_1|}{N}\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right) \tag{2.60}$$

for some constant $c > 0$.
2) The case $\kappa > 1$. Inserting (2.56) in (2.53) yields

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) = \left[\sum_{m=0}^{N_*/2-1}\left[\prod_{k'=1}^{\kappa}\binom{|\Lambda_{k'}|}{m}\right]^{-1} B_m\right]^{-1}, \tag{2.61}$$

where

$$\begin{aligned} B_m &\equiv \sum_{l=1}^{\kappa}\frac{N}{|\Lambda_l|-m}\prod_{k\leq l-1}\frac{m+1}{|\Lambda_k|-m} \\ &= \frac{N}{|\Lambda_1|-m}\left[1 + \sum_{l=2}^{\kappa}\prod_{l'=2}^{l}\frac{m+1}{|\Lambda_{l'}|-m}\right] \\ &\leq \frac{N}{|\Lambda_1|-m}\left[1 + \sum_{l=2}^{\kappa}\left(\frac{m+1}{N_*-m}\right)^{l-1}\right]. \end{aligned} \tag{2.62}$$

Since $(m+1)/(N_*-m) < 1$ for all $0 \leq m \leq N_*/2-1$,

$$B_m \leq \frac{N(N_*-m)}{(|\Lambda_1|-m)(N_*-2m-1)}. \tag{2.63}$$

Inserting (2.63) in (2.61),

$$\widetilde{\mathbb{R}}^{\circ}_{\Delta_1}\left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0}\right) \geq \frac{|\Lambda_1|}{N}\left[\sum_{m=0}^{N_*/2-1} C_m\right]^{-1}, \tag{2.64}$$

where

$$C_m = \left[\prod_{k'=1}^{\kappa} \binom{|\Lambda_{k'}|}{m} \right]^{-1} \frac{(N_* - m)|\Lambda_1|}{(N_* - 2m - 1)(|\Lambda_1| - m)}. \tag{2.65}$$

Finally, a few simple computations yield the bounds

$$\begin{aligned} C_0 &= 1 + \tfrac{1}{N_* - 1}, \\ C_1 &\leq N_*^{-\kappa}(1 + 5N_*^{-1}), \\ C_m &\leq 2^{\kappa-2} N_*^{-2\kappa+1}, \quad 2 \leq m \leq N_*/2 - 1, \end{aligned} \tag{2.66}$$

from which we easily get

$$\widetilde{\mathbb{R}}^\circ_{\Delta_1} \left(\tau^{\omega_0}_{\omega_L} < \tau^{\omega_0}_{\omega_0} \right) \geq \frac{|\Lambda_1|}{N} \left(1 - \frac{1}{N_*} - \frac{c}{N_*^\kappa} \right) \tag{2.67}$$

for some constant $c > 0$. As (2.60) together with (2.67) give (2.51), the first assertion of Lemma 2.10 is proven. $\square$

Proof of Lemma 2.10 ii). We first write

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_0 < \tau^y_x \right) = 1 - \widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_0 \right) \tag{2.68}$$

and use the renewal identity (see e.g. Corollary 1.9 in [BEGK1]) to get

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_0 \right) = \frac{\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_{y\cup 0} \right)}{\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_{x\cup 0} < \tau^y_y \right)}. \tag{2.69}$$

By reversibility the numerator of (2.69) may be rewritten as

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_{y\cup 0} \right) = \frac{\widetilde{\mathbb{Q}}^\circ_{\Delta}(x)}{\widetilde{\mathbb{Q}}^\circ_{\Delta}(y)} \widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^x_y < \tau^x_{x\cup 0} \right). \tag{2.70}$$

Thus, remembering that $\widetilde{\mathbb{Q}}^\circ_{\Delta}(x') = \mathbb{Q}^\circ_N(x')/\mathbb{Q}^\circ_N(\Delta)$ we have, by (2.27),

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_{y\cup 0} \right) \leq \frac{\mathbb{Q}^\circ_N(x)}{\mathbb{Q}^\circ_N(y)} = \frac{|\{\sigma \mid \gamma(\sigma) = x\}|}{|\{\sigma \mid \gamma(\sigma) = y\}|} = \frac{1}{|\{\sigma \mid \gamma(\sigma) = y\}|} \tag{2.71}$$

which by (2.18), for y defined in (2.40), gives:

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_{y\cup 0} \right) \leq \left[\prod_{k \in \mathcal{K}(I)} \binom{|\Lambda_k|}{N_*/2} \right]^{-1} \leq \binom{N_*}{N_*/2}^{-1}, \tag{2.72}$$

where we used that there exists at least one index $k \in \mathcal{K}(I)$ with the property that $|\Lambda_k| = N_*$. Since by (2.32) $N \geq N_* \geq \epsilon \frac{N}{d}$, Stirling's formula enables us to conclude that, for large enough N,

$$\widetilde{\mathbb{R}}^\circ_{\Delta} \left(\tau^y_x < \tau^y_{y\cup 0} \right) \leq c\sqrt{N_*} 2^{-N_*} \leq c\sqrt{N} 2^{-\epsilon N/d} \tag{2.73}$$

for some constant $c > 0$.

To bound the probability appearing in the denominator of (2.69) we again resort to the path technique employed in the proof of assertion i). As we need only a rough estimate, this probability will be estimated by means of a single path, $\widetilde{\omega}$. Setting

$$
\begin{aligned}
L &= L_0 + \cdots + L_d, \\
L_k &= \begin{cases} 0, & \text{if} \quad k = 0 \\ \frac{1}{2}(|\Lambda_k(I)| - N_*), & \text{if} \quad k \in \mathcal{K}(I) \\ \frac{|\Lambda_k(I)|}{2}, & \text{if} \quad k \in \mathcal{K}(I)^c \end{cases},
\end{aligned}
\tag{2.74}
$$

$\widetilde{\omega} = (\widetilde{\omega}_0, \ldots, \widetilde{\omega}_L)$ is defined as follows

$$
\widetilde{\omega}_n = \begin{cases} y, & \text{if} \quad n = 0 \\ \widetilde{\omega}_{n-1} - \frac{2}{|\Lambda_k(I)|} u_k, & \text{if} \quad \sum_{l=0}^{k-1} L_l < n \le \sum_{l=0}^{k} L_l, \quad 1 \le k \le d \end{cases}. \tag{2.75}
$$

(Observe that $\widetilde{\omega}_L = 0$.) Denoting by $\widetilde{D}$ the subgraph of $\mathcal{G}(\Gamma_{N,d}(I))$ generated by the path $\widetilde{\omega}$ (i.e., with a set of vertices $V(\widetilde{D}) = \{x' \in \Gamma_{N,d}(I) \mid \exists_{0 \le n \le L} \ : \ x' = \widetilde{\omega}_n\}$), we have

$$
\widetilde{D} \subset D \subset \Delta \tag{2.76}
$$

and thus, by Lemma 6.1, that

$$
\widetilde{\mathbb{R}}^{\circ}_{\Delta}\left(\tau^{y}_{x\cup 0} < \tau^{y}_{y}\right) \ge \widetilde{\mathbb{R}}^{\circ}_{\Delta}\left(\tau^{y}_{0} < \tau^{y}_{y}\right) \ge \widetilde{\mathbb{R}}^{\circ}_{\widetilde{D}}\left(\tau^{y}_{0} < \tau^{y}_{y}\right). \tag{2.77}
$$

To bound the last probability in (2.77) note that, just as in (2.53),

$$
\widetilde{\mathbb{R}}^{\circ}_{\widetilde{D}}\left(\tau^{\widetilde{\omega}_0}_{\widetilde{\omega}_L} < \tau^{\widetilde{\omega}_0}_{\widetilde{\omega}_0}\right) = \left[\sum_{n=0}^{L-1} \frac{\mathbb{Q}_N(\widetilde{\omega}_0)}{\mathbb{Q}_N(\widetilde{\omega}_n)} \frac{1}{r_N(\widetilde{\omega}_n, \widetilde{\omega}_{n+1})}\right]^{-1}. \tag{2.78}
$$

At this stage, simply observe that on the one hand, $\mathbb{Q}_N(\widetilde{\omega}_n)$ increases as n increases from 0 to L, implying that $\mathbb{Q}_N(\widetilde{\omega}_0)/\mathbb{Q}_N(\widetilde{\omega}_n) \le 1$ for all $0 \le n \le L$, while on the other hand, for each $1 \le k \le d$ and all $\sum_{l=0}^{k-1} L_l \le n < \sum_{l=0}^{k} L_l$,

$$
r_N(\widetilde{\omega}_n, \widetilde{\omega}_{n+1}) = \frac{|\Lambda_k(I)|}{2N}(1 + \widetilde{\omega}^k_n) \ge \frac{|\Lambda_k(I)|}{2N}. \tag{2.79}
$$

Therefore

$$
\widetilde{\mathbb{R}}^{\circ}_{\widetilde{D}}\left(\tau^{\widetilde{\omega}_0}_{\widetilde{\omega}_L} < \tau^{\widetilde{\omega}_0}_{\widetilde{\omega}_0}\right) \ge \left[\sum_{k=1}^{d} \sum_{m=0}^{L_k - 1} \frac{|\Lambda_k(I)|}{2N}\right]^{-1} = \left[\sum_{k=1}^{d} L_k \frac{|\Lambda_k(I)|}{2N}\right]^{-1} \ge \frac{1}{Nd}, \tag{2.80}
$$

where the last inequality follows from (2.74). Putting (2.80) back in (2.77) finally gives

$$
\widetilde{\mathbb{R}}^{\circ}_{\Delta}\left(\tau^{y}_{x\cup 0} < \tau^{y}_{y}\right) \ge (Nd)^{-1}. \tag{2.81}
$$

Inserting (2.81) and (2.73) in (2.69) and plugging the resulting bound in (2.68) yields (2.50). The second assertion of Lemma 2.10 being proven, this concludes the proof of Lemma 2.10. □

Inserting the bounds of Lemma 2.10 in 2.47 we obtain

$$\mathbb{R}^\circ\left(\tau_0^x < \tau_x^x\right) \geq \left(1 - \frac{1}{N}\sum_{\mu\in\mathcal{K}(I)^c}|\Lambda_\mu|\right)\left(1 - \frac{1}{N_*} - \frac{c}{N_*^2}\right)\left(1 - c'N2^{-\epsilon N/d}\right)$$
$$\geq \left(1 - \frac{1}{N}\sum_{\mu\in\mathcal{K}(I)^c}|\Lambda_\mu|\right)\left(1 - \frac{1}{N_*} - \frac{c''}{N_*^2}\right), \tag{2.82}$$

where the last inequality holds true for some constant $c'' > 0$, provided that N is large enough. The first assertion of 2.6 is proven. □

Proof of Lemma 2.7. For $\rho \geq 0$ and $x \in \gamma_I(I)$ set

$$\Gamma_{N,d}^{x,\rho}(I) = \left\{y' \in \Gamma_{N,d}(I) \mid \|\pi x - \pi y'\|_2 > \rho\right\}. \tag{2.83}$$

By hypothesis,

$$y \in \Gamma_{N,d}^{x,\delta}(I). \tag{2.84}$$

Observe moreover that either y satisfies

i) $\forall z \in \{-1,1\}^d \cap \Gamma_{N,d}^{x,\delta/2}(I)$, $\|\pi z - \pi y\|_2 > \delta/2$
or else
ii) $\exists z \in \{-1,1\}^d \cap \Gamma_{N,d}^{x,\delta}(I)$ such that $\|\pi z - \pi y\|_2 \leq \delta/2$.

We will first show that in case i), (2.35) is a direct consequence of reversibility. Indeed, as in (2.69),

$$\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) = \frac{\mathbb{R}^\circ\left(\tau_x^y < \tau_{y\cup 0}^y\right)}{\mathbb{R}^\circ\left(\tau_{x\cup 0}^y < \tau_y^y\right)}. \tag{2.85}$$

A straightforward adaptation of the proof of the bound (2.81) to the case at hand shows that the denominator of (2.85) obeys the bound

$$\mathbb{R}^\circ\left(\tau_{x\cup 0}^y < \tau_y^y\right) \geq 1/N \tag{2.86}$$

while by reversibility its numerator may be rewritten as

$$\mathbb{R}^\circ\left(\tau_x^y < \tau_{y\cup 0}^y\right) = \frac{\mathbb{Q}^\circ(x)}{\mathbb{Q}^\circ(y)}\mathbb{R}^\circ\left(\tau_y^x < \tau_{x\cup 0}^x\right). \tag{2.87}$$

Thus, by (2.27),

$$\mathbb{R}^\circ\left(\tau_x^y < \tau_{y\cup 0}^y\right) \leq \frac{\mathbb{Q}_N^\circ(x)}{\mathbb{Q}_N^\circ(y)} = \frac{|\{\sigma \mid \gamma(\sigma) = x\}|}{|\{\sigma \mid \gamma(\sigma) = y\}|} = \frac{1}{|\{\sigma \mid \gamma(\sigma) = y\}|}, \tag{2.88}$$

where the last equality follows from the fact that $x \in \gamma_I(I)$ (see Lemma 2.2). To estimate the last ratio note that condition i) combined with (2.84) implies that

$$\inf_{z'\in\{-1,1\}^\kappa} \|z' - \pi y\|_2 > \delta/2 \tag{2.89}$$

which in turn implies that there exists $k' \in \mathcal{K}(I)$ such that $\inf_{s=\pm 1} |s - y_{k'}| > \delta/2\sqrt{\kappa}$, or in other words, such that $|y_{k'}| < 1 - \delta/2\sqrt{\kappa}$. Thus, making use of (2.18) and Stirling's formula,

$$
\begin{aligned}
|\{\sigma \mid \gamma(\sigma) = y\}|^{-1} &\leq \binom{|\Lambda_{k'}(I)|}{|\Lambda_{k'}(I)|\frac{1+y_{k'}}{2}}^{-1} \\
&\leq c \exp\left\{-|\Lambda_{k'}(I)|\mathcal{I}(y_{k'})\right\} \\
&\leq c \exp\left\{-N_* \inf_{|u|<1-\delta/2\sqrt{\kappa}} \mathcal{I}(u)\right\}, \qquad (2.90)
\end{aligned}
$$

for some constant $c > 0$, with N_* defined as in (2.33), and where

$$
-\mathcal{I}(u) = \left(\frac{1-u}{2}\right) \ln\left(\frac{1-u}{2}\right) + \left(\frac{1+u}{2}\right) \ln\left(\frac{1+u}{2}\right), \quad |u| \leq 1. \qquad (2.91)
$$

Collecting all our bounds we arrive at

$$
\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) \leq cN \exp\left\{-N_* \inf_{|u|<1-\delta/2\sqrt{\kappa}} \mathcal{I}(u)\right\}. \qquad (2.92)
$$

As $\delta > 0$, $\inf_{|u|<1-\delta/2\sqrt{\kappa}} \mathcal{I}(u) > 0$. Choosing $h(\delta, \kappa) = \inf_{|u|<1-\delta/2\sqrt{\kappa}} \mathcal{I}(u)/2$, it then follows from (2.92) that, for all N large enough,

$$
\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) \leq \exp\left\{-N_* h(\delta, \kappa)\right\}. \qquad (2.93)
$$

Thus, under the assumption made in i), (2.35) is proven. Let us turn to the case ii). Observe that for $\delta < 1/2$ there exists a unique point $z \in \{-1, 1\}^\kappa$ such that $\|\pi z - \pi y\|_2 \leq \delta/2$. Calling z^* this point and introducing the discrete hyper-surface $H_{\delta/2}(z^*) = \{z' \in \Gamma_{N,d}(I) \mid \|z^* - \pi z'\|_2 = \delta/2\}$, we have

$$
\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) = \sum_{z' \in H_{\delta/2}(z^*)} \mathbb{R}^\circ\left(\tau_{z'}^y < \tau_{H_{\delta/2}(z^*)}^y\right) \mathbb{R}^\circ\left(\tau_x^{z'} < \tau_0^{z'}\right). \qquad (2.94)
$$

Now all points $z' \in H_{\delta/2}(z^*)$ have the following properties: firstly, as is obvious from the definition of $H_{\delta/2}(z^*)$, $\|\pi z - \pi z'\|_2 > \delta/4$ for all $z \in \{-1, 1\}^d \cap \Gamma_{N,d}^{x,\delta}(I)$, implying that assumption i) is satisfied with δ replaced by $\delta/2$; secondly, since $z^* \in \Gamma_{N,d}^{x,\delta}(I)$ by assumption, then

$$
\delta \leq \|\pi x - \pi z^*\|_2 \leq \|\pi x - \pi z'\|_2 + \|\pi z' - \pi z^*\|_2 \leq \|\pi x - \pi z'\|_2 + \delta/2 \qquad (2.95)
$$

implying that $\|\pi x - \pi z'\|_2 \geq \delta/2$, i.e., that $z' \in \Gamma_{N,d}^{x,\delta/2}(I)$.

As a result, for each $z' \in H_{\delta/2}(z^*)$, the probability $\mathbb{R}^\circ\left(\tau_x^{z'} < \tau_0^{z'}\right)$ obeys the bound (2.92) with δ replaced by $\delta/2$. It therefore follows from (2.94) that

$$
\begin{aligned}
\mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right) &\leq \exp\left\{-N_* h(\delta/2, \kappa)\right\} \sum_{z' \in \partial\mathcal{B}_{\delta/2}(z^*)} \mathbb{R}^\circ\left(\tau_{z'}^y < \tau_{\partial\mathcal{B}_{\delta/2}(z^*)}^y\right) \\
&\leq \exp\left\{-N_* h(\delta/2, \kappa)\right\}. \qquad (2.96)
\end{aligned}
$$

This concludes the proof of Lemma 2.7. □

Proof of Lemma 2.8. Again using renewal as in (2.69),

$$\mathbb{R}^\circ\left(\tau_x^0 \le \tau_J^0\right) = \frac{\mathbb{R}^\circ\left(\tau_x^0 \le \tau_{J\cup 0}^0\right)}{\mathbb{R}^\circ\left(\tau_J^0 < \tau_0^0\right)} = \frac{\mathbb{R}^\circ\left(\tau_x^0 \le \tau_{J\cup 0}^0\right)}{\sum_{y\in J}\mathbb{R}^\circ\left(\tau_y^0 \le \tau_{J\cup 0}^0\right)}, \tag{2.97}$$

so that we are left to bound a term of the form $\mathbb{R}^\circ\left(\tau_y^0 \le \tau_{J\cup 0}^0\right)$, $y \in J$. To do so observe that

$$\mathbb{R}^\circ\left(\tau_y^0 \le \tau_{J\cup 0}^0\right) = \mathbb{R}^\circ\left(\tau_y^0 < \tau_0^0\right) - \mathbb{R}^\circ\left(\tau_{J\setminus y}^0 < \tau_y^0 < \tau_0^0\right) \tag{2.98}$$

and that

$$\mathbb{R}^\circ\left(\tau_{J\setminus y}^0 < \tau_y^0 < \tau_0^0\right) = \sum_{z\in J\setminus y}\mathbb{R}^\circ\left(\tau_z^0 \le \tau_{J\cup 0}^0\right)\mathbb{R}^\circ\left(\tau_y^z < \tau_0^z\right). \tag{2.99}$$

By assumption, the probabilities $\mathbb{R}^\circ\left(\tau_y^z < \tau_0^z\right)$ in the r.h.s. above obey the bound (2.35) of Lemma 2.7. Thus

$$\begin{aligned}\mathbb{R}^\circ\left(\tau_{J\setminus y}^0 < \tau_y^0 < \tau_0^0\right) &\le e^{-Nh(\delta,\kappa)}\sum_{z\in J\setminus y}\mathbb{R}^\circ\left(\tau_z^0 \le \tau_{J\cup 0}^0\right)\\ &\le e^{-Nh(\delta,\kappa)}\mathbb{R}^\circ\left(\tau_J^0 < \tau_0^0\right).\end{aligned} \tag{2.100}$$

From (2.98) and (2.100) we deduce that

$$\mathbb{R}^\circ\left(\tau_y^0 < \tau_0^0\right) - e^{-Nh(\delta,\kappa)}\mathbb{R}^\circ\left(\tau_J^0 < \tau_0^0\right) \le \mathbb{R}^\circ\left(\tau_y^0 \le \tau_{J\cup 0}^0\right) \le \mathbb{R}^\circ\left(\tau_y^0 < \tau_0^0\right) \tag{2.101}$$

and, summing over $y \in J$,

$$\sum_{y\in J}\mathbb{R}^\circ\left(\tau_y^0 \le \tau_0^0\right) - |J|e^{-Nh(\delta,\kappa)}\mathbb{R}^\circ\left(\tau_J^0 < \tau_0^0\right) \le \mathbb{R}^\circ\left(\tau_J^0 \le \tau_0^0\right) \le \sum_{y\in J}\mathbb{R}^\circ\left(\tau_y^0 < \tau_0^0\right). \tag{2.102}$$

Inserting the bounds (2.101) and (2.102) into (2.97), and using that

$$\frac{\mathbb{R}^\circ\left(\tau_J^0 \le \tau_0^0\right)}{\sum_{y\in J}\mathbb{R}^\circ\left(\tau_y^0 < \tau_0^0\right)} \le 1 \tag{2.103}$$

we arrive at:

$$R - e^{-Nh(\delta,\kappa)} \le \mathbb{R}^\circ\left(\tau_x^0 \le \tau_J^0\right) \le R\left(\frac{1}{1-|J|e^{-Nh(\delta,\kappa)}}\right), \tag{2.104}$$

where

$$R \equiv \frac{\mathbb{R}^\circ\left(\tau_x^0 \le \tau_0^0\right)}{\sum_{y\in J}\mathbb{R}^\circ\left(\tau_y^0 \le \tau_0^0\right)}. \tag{2.105}$$

To estimate the above ratio we use first that, by reversibility,

$$R = \frac{\mathbb{Q}_N^\circ(x)\mathbb{R}^\circ\left(\tau_0^x \le \tau_x^x\right)}{\sum_{y\in J}\mathbb{Q}_N^\circ(y)\mathbb{R}^\circ\left(\tau_0^y \le \tau_y^y\right)} \tag{2.106}$$

and next that, by Lemma 2.6,

$$\vartheta\overline{R} \le \mathbb{R}^\circ\left(\tau_x^0 \le \tau_J^0\right) \le \frac{\overline{R}}{\vartheta}, \tag{2.107}$$

where ϑ is defined in (2.37) and

$$\overline{R} \equiv \frac{\mathbb{Q}_N^\circ(x)}{\sum_{y\in J}\mathbb{Q}_N^\circ(y)}. \tag{2.108}$$

Now since $J \subseteq \gamma_I(I)$, and since $\mathbb{Q}_N^\circ(y) = 2^{-N}$ for all $y \in \gamma_I(I)$,

$$\overline{R} = \frac{1}{|J|}. \tag{2.109}$$

Collecting (2.104), (2.107), and (2.109) yields (2.38), concluding the proof of Lemma 2.8. $\square$

2.3. Proof of Proposition 2.1. While the estimates of Sect. 2.2 will furnish all the basic ingredients to the proof of Proposition 2.1, they depend upon the choice of the mapping γ_I through several quantities. To put them to use we still have to identify which mappings γ_I will be needed and establish the properties of all related objects. Taking a look at Proposition 2.1 in the light of Lemma 2.5 tells us at once that we will be concerned with two cases only: the case where the mapping γ_I is induced by the elements of the top (as required for the proof of the first four assertions) or the top augmented by a non-random element of $\mathcal{S}_N$ (which is needed for the proof of the last one). These two cases are analysed below.

Notation. In this section we will systematically write T for $T(E)$.

Lumped chain induced by the Top. Let ${}^t\xi = (\xi^1, \xi^2, \dots, \xi^{|T|})$ be the matrix formed of the elements of the top ordered according to the magnitude of X_σ:

$$T = \{\xi^1, \xi^2, \dots, \xi^{|T|}\}, \quad \text{where} \quad X_{\xi^1} \ge X_{\xi^2} \ge \cdots \ge X_{\xi^{|T|}}. \tag{2.110}$$

Thus ξ is here a random variable on the probability space $(\Omega, \mathcal{F}, P)$. One easily verifies that the conditional distribution of ξ, given that the top contains exactly M points, is the uniform distribution over the set $\widetilde{\mathcal{S}}_{M\times N}$ of M-tuples of mutually distinct points of $\mathcal{S}_N$, i.e.:

$$P\left(\xi = \zeta \mid |T| = M\right) = \begin{cases} \frac{(2^N-M)!}{(2^N)!} & \text{if} \quad \zeta \in \widetilde{\mathcal{S}}_{M\times N} \\ 0, & \text{otherwise} \end{cases}, \tag{2.111}$$

where

$$\widetilde{\mathcal{S}}_{M\times N} \equiv \left\{ \zeta \in \mathcal{S}_{M\times N} \;\middle|\; \zeta^\mu \neq \zeta^\nu \quad \text{for all} \quad 1 \leq \mu, \nu \leq M, \mu \neq \nu \right\}. \tag{2.112}$$

Set

$$\delta(N) \equiv (d/N)^{1/2} \ln N \tag{2.113}$$

(where, as before, $d = 2^M$) and let $\overline{\mathcal{S}}_{M\times N}$ be defined through

$$\overline{\mathcal{S}}_{M\times N} \equiv \left\{ \zeta \in \mathcal{S}_{M\times N} \;\middle|\; |\Lambda_k(\zeta)| = \tfrac{N}{d}(1 + \lambda_k(N)),\ |\lambda_k(N)| < \delta(N),\ 1 \leq k \leq d \right\}. \tag{2.114}$$

The set $\mathcal{E}$ appearing in the statement of Proposition 2.1 may be chosen as

$$\mathcal{E} = \bigcup_{N_0} \bigcap_{N > N_0} \mathcal{E}_N, \tag{2.115}$$

where $\mathcal{E}_N$ is given by

$$\mathcal{E}_N \equiv \left\{ \omega \in \Omega \mid \xi(\omega) \in \overline{\mathcal{S}}_{|T|\times N} \right\}. \tag{2.116}$$

It is easy to see, using the proof of Lemma 4.2 of [G], that:

Lemma 2.11.

$$P(\mathcal{E}) = 1. \tag{2.117}$$

We will need a certain number of geometric properties of the set T, which we collect below.

Lemma 2.12. *For all* $0 \leq \varepsilon < 1/2$*, all* $\omega \in \mathcal{E}_N$*, and large enough* N *the following holds: for all* $\eta \neq \bar{\eta}$*,* $\eta \in T$*,* $\bar{\eta} \in T$*,*

$$\mathcal{B}_{\sqrt{\varepsilon N}}(\eta) \cap \mathcal{B}_{\sqrt{\varepsilon N}}(\bar{\eta}) = \emptyset, \tag{2.118}$$

and

$$\frac{1}{N} \sum_{i=1}^{N} \eta_i \bar{\eta}_i \leq \delta(N). \tag{2.119}$$

Proof. With the notation of (2.110) let $\xi^\mu \neq \xi^\nu$ be any two distinct elements of T. For all $\sigma \in \mathcal{B}_{\sqrt{\varepsilon N}}(\xi^\mu)$ we have,

$$\begin{aligned} \|\sigma - \xi^\nu\|_2 &\geq \|\xi^\nu - \xi^\mu\|_2 - \|\sigma - \xi^\mu\|_2 \\ &\geq \|\xi^\nu - \xi^\mu\|_2 - \sqrt{\varepsilon N} \\ &= \sqrt{2N} \left(1 - \frac{1}{N} \sum_{i=1}^{N} \xi_i^\nu \xi_i^\mu \right)^{1/2} - \sqrt{\varepsilon N}. \end{aligned} \tag{2.120}$$

Using (2.13) we may write

$$\frac{1}{N}\sum_{i=1}^{N}\xi_i^\nu\xi_i^\mu = \frac{1}{N}\sum_{k=1}^{d}|\Lambda_k(\xi)|e_k^\nu e_k^\mu. \tag{2.121}$$

Since $\omega \in \mathcal{E}_N$ by assumption, it follows from (2.114) that

$$\begin{aligned}\tfrac{1}{N}\sum_{i=1}^{N}\xi_i^\nu\xi_i^\mu &= \tfrac{1}{d}\sum_{k=1}^{d}e_k^\nu e_k^\mu + \tfrac{1}{d}\sum_{k=1}^{d}\lambda_k(N)e_k^\nu e_k^\mu\\ &= \tfrac{1}{d}\sum_{k=1}^{d}\lambda_k(N)e_k^\nu e_k^\mu\\ &\le \delta(N),\end{aligned} \tag{2.122}$$

where the second equality follows from Lemma 2.1 of [G]. Thus (2.119) is proven. Inserting (2.122) in (2.120) yields,

$$\|\sigma - \xi^\nu\|_2 \ge \sqrt{2N(1-\delta(N))} - \sqrt{\varepsilon N}, \tag{2.123}$$

which would entail (2.118) if we had

$$\sqrt{2N(1-\delta(N))} - \sqrt{\varepsilon N} > \sqrt{\varepsilon N}. \tag{2.124}$$

Now our assumptions on ε imply that this is the case for all N large enough. The lemma is therefore proven. $\square$

With our choice of $\mathcal{E}_N$ it readily follows from (2.16) that, for $\omega \in \mathcal{E}_N$, choosing e.g. $\epsilon = 1/2$ in definition (2.33),

$$\mathcal{K}(T)^c = \emptyset \tag{2.125}$$

and

$$\begin{aligned}N_* = \min_{k\in\mathcal{K}(T)}|\Lambda_k(T)| &\ge \tfrac{N}{d}\left(1 - \left(\tfrac{d}{N}\right)^{1/2}\ln N\right)\\ \max_{k\in\mathcal{K}(T)}|\Lambda_k(T)| &\le \tfrac{N}{d}\left(1 + \left(\tfrac{d}{N}\right)^{1/2}\ln N\right).\end{aligned} \tag{2.126}$$

Of course $\kappa = d$ and the projection π defined in the line preceding (2.33) simply is the identity. Knowing this we have:

Lemma 2.13. *Assume that $\omega \in \mathcal{E}_N$. Then, for all N large enough,*

i) For all $\sigma \in W_\varepsilon(T)$,

$$\inf_{x\in\gamma_T(T)}\|\pi x - \pi\gamma_T(\sigma)\|_2 \ge \frac{\varepsilon\sqrt{d}}{2}(1-\delta(N)). \tag{2.127}$$

ii) For all $\sigma \in T$,

$$\inf_{x\in\gamma_T(T)\setminus\gamma_T(\sigma)}\|\pi x - \pi\gamma_T(\sigma)\|_2 \ge (1-2\delta(N))\sqrt{d}. \tag{2.128}$$

Proof. As a consequence of (2.126) and assertion iv) of Lemma 2.2 we have, for all $\sigma \in W_\varepsilon(T)$,

$$\begin{aligned}\inf_{x\in\gamma_T(T)} \|\pi x - \pi\gamma_T(\sigma)\|_2 &= \inf_{x\in\gamma_T(T)} \|x - \gamma_T(\sigma)\|_2 \\ &\geq \frac{\varepsilon N}{2\sqrt{d}\max_k |\Lambda_k(T)|} \\ &\geq \frac{\varepsilon\sqrt{d}}{2}(1-\delta(N))^{-1} \end{aligned}\tag{2.129}$$

which yields (2.127). Similarly note that if $\sigma \in T$ then, by Lemma 2.12,

$$\inf_{\eta\in T\setminus\sigma} \|\eta - \sigma\|_2 \geq \sqrt{2N(1-\delta(N))}.\tag{2.130}$$

Just as in (2.129) this property combined with (2.126) and Lemma 2.2, iv) implies that, for all $\sigma \in T$,

$$\inf_{x\in\gamma_T(T)\setminus\gamma_T(\sigma)} \|\pi x - \pi\gamma_T(\sigma)\|_2 \geq \sqrt{d}\frac{1-\delta(N)}{1+\delta(N)},\tag{2.131}$$

which proves (2.128). $\square$

We are now ready to prove the first five assertions of Proposition 2.1.

Notation. The following notation will be used throughout: $\mathcal{T} = \gamma_T(T)$, $y = \gamma_T(\sigma)$, $x = \gamma_T(\eta)$ and $\bar{x} = \gamma_T(\bar{\eta})$. It will moreover be assumed that $\omega \in \mathcal{E}_N$.

Proof of Proposition 2.1, i). Using in turn assertion i) of Lemma 2.4 and Lemma 2.5,

$$\mathbb{P}\left(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma\right) = \mathbb{P}^\circ\left(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma\right) = \mathbb{R}^\circ\left(\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\right).\tag{2.132}$$

Defining

$$\begin{aligned} R_1 &\equiv \mathbb{R}^\circ\left(\{\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\} \cap \{\tau_0^y < \tau_x^y\}\right), \\ R_2 &\equiv \mathbb{R}^\circ\left(\{\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\} \cap \{\tau_x^y < \tau_0^y\}\right), \end{aligned}\tag{2.133}$$

$\mathbb{R}^\circ\left(\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\right)$ may be decomposed as

$$\mathbb{R}^\circ\left(\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\right) = R_1 + R_2.\tag{2.134}$$

Obviously

$$0 \leq R_2 \leq \mathbb{R}^\circ\left(\tau_x^y < \tau_0^y\right),\tag{2.135}$$

while

$$\begin{aligned} R_1 &= \mathbb{R}^\circ\left(\tau_0^y < \tau_x^y < \tau_{\mathcal{T}\setminus x}^y\right) \\ &= \mathbb{R}^\circ\left(\tau_0^y < \tau_{\mathcal{T}}^y\right)\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right) \\ &= \left[1 - \mathbb{R}^\circ\left(\tau_{\mathcal{T}}^y < \tau_0^y\right)\right]\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right) \end{aligned}\tag{2.136}$$

which, together with the bound

$$\mathbb{R}^\circ\left(\tau_{\mathcal{T}}^y < \tau_0^y\right) \le \sum_{x'\in\mathcal{T}} \mathbb{R}^\circ\left(\tau_{x'}^y < \tau_0^y\right) \tag{2.137}$$

yields

$$\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right)\left[1 - M \sup_{x'\in\mathcal{T}} \mathbb{R}^\circ\left(\tau_{x'}^y < \tau_0^y\right)\right] \le R_1 \le \mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right). \tag{2.138}$$

We are thus left to bound the quantities $\sup_{x'\in\mathcal{T}} \mathbb{R}^\circ\left(\tau_{x'}^y < \tau_0^y\right)$ and $\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right)$, which will be done by means of, respectively, Lemma 2.7 and Lemma 2.8: on the one hand, since by assumption $\sigma \in W_\varepsilon(T)$, it follows from (2.127) that δ in Lemma 2.7 may be chosen as $\delta = \frac{\varepsilon\sqrt{d}}{4}$, so that inserting the bound (2.126) in (2.35) yields

$$\mathbb{R}^\circ\left(\tau_{x'}^y < \tau_0^y\right) \le e^{-Nh'(d)}, \qquad \text{for all} \quad x' \in \mathcal{T} \tag{2.139}$$

for some constant $h'(d) > 0$ and large enough N; on the other hand, it follows from (2.128) that δ in Lemma 2.8 may be chosen as $\delta = \sqrt{d}$ so that, in view of (2.125), combining the bounds (2.38) and (2.126), we obtain

$$\left|\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right) - \frac{1}{M}\right| \le \frac{d}{NM}\left(1 + c_0\left(\tfrac{d}{N}\right)^{1/2}\ln N\right) \tag{2.140}$$

for some constant $c_0 > 0$.

Collecting the previous estimates we obtain that, for large enough N,

$$\left|\mathbb{R}^\circ\left(\tau_x^y < \tau_{\mathcal{T}\setminus x}^y\right) - \frac{1}{M}\right| \le \frac{d}{NM}\left(1 + c_1\left(\tfrac{d}{N}\right)^{1/2}\ln N\right) \tag{2.141}$$

for some constant $c_1 > 0$. Inserting (2.141) in (2.132) yields the claim of assertion i). □

Proof of Proposition 2.1, ii). The proof of this second assertion closely follows that of assertion i). Keeping in mind the notation $\mathcal{T} = \gamma_T(T)$, $x = \gamma_T(\eta)$ and $\bar{x} = \gamma_T(\bar{\eta})$ we may write, using in turn assertion ii) of Lemma 2.4 and Lemma 2.5,

$$\mathbb{P}\left(\tau_\eta^{\bar{\eta}} < \tau_{T\setminus\eta}^{\bar{\eta}}\right) = e^{-\beta\sqrt{N}E_{\bar{\eta}}}\mathbb{P}^\circ\left(\tau_\eta^{\bar{\eta}} < \tau_{T\setminus\eta}^{\bar{\eta}}\right) = e^{-\beta\sqrt{N}E_{\bar{\eta}}}\mathbb{R}^\circ\left(\tau_x^{\bar{x}} < \tau_{\mathcal{T}\setminus x}^{\bar{x}}\right). \tag{2.142}$$

We then decompose $\mathbb{R}^\circ\left(\tau_x^{\bar{x}} < \tau_{\mathcal{T}\setminus x}^{\bar{x}}\right)$ as in (2.134), and bound R_2 as in (2.135). As for R_1 we write, just as in (2.136),

$$R_1 = \left[1 - \mathbb{R}^\circ\left(\tau_{\mathcal{T}}^{\bar{x}} < \tau_0^{\bar{x}}\right)\right]\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right), \tag{2.143}$$

but this time use (2.137) to deduce that

$$\begin{aligned} 1 - \mathbb{R}^\circ\left(\tau_{\mathcal{T}}^{\bar{x}} < \tau_0^{\bar{x}}\right) &\ge 1 - \sum_{x'\in\mathcal{T}} \mathbb{R}^\circ\left(\tau_{x'}^{\bar{x}} < \tau_0^{\bar{x}}\right) \\ &= \mathbb{R}^\circ\left(\tau_0^{\bar{x}} < \tau_{\bar{x}}^{\bar{x}}\right) - \sum_{x'\in\mathcal{T}\setminus\bar{x}} \mathbb{R}^\circ\left(\tau_{x'}^{\bar{x}} < \tau_0^{\bar{x}}\right) \\ &\ge \mathbb{R}^\circ\left(\tau_0^{\bar{x}} < \tau_{\bar{x}}^{\bar{x}}\right) - (M-1)\sup_{x'\in\mathcal{T}\setminus\bar{x}} \mathbb{R}^\circ\left(\tau_{x'}^{\bar{x}} < \tau_0^{\bar{x}}\right). \end{aligned} \tag{2.144}$$

Therefore

$$\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right)\ \left[\mathbb{R}^\circ\left(\tau_0^{\bar x} < \tau_{\bar x}^{\bar x}\right) - (M-1)\sup_{x'\in\mathcal{T}\setminus\bar x}\mathbb{R}^\circ\left(\tau_{x'}^{\bar x} < \tau_0^{\bar x}\right)\right]$$
$$\leq R_1 \leq \mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right). \tag{2.145}$$

Having already estimated the probability $\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right)$ in (2.140), we are left to treat the terms $\mathbb{R}^\circ\left(\tau_0^{\bar x} < \tau_{\bar x}^{\bar x}\right)$ and $\sup_{x'\in\mathcal{T}\setminus\bar x}\mathbb{R}^\circ\left(\tau_{x'}^{\bar x} < \tau_0^{\bar x}\right)$. The probabilities $\mathbb{R}^\circ\left(\tau_{x'}^{\bar x} < \tau_0^{\bar x}\right)$ entering the latter term are easily dealt with by means of Lemma 2.7: note that for all $x'\in\mathcal{T}\setminus x$ and $\bar x\in\mathcal{T}$, it follows from (2.128) that δ in Lemma 2.7 may be chosen as $\delta=\sqrt{d}$, so that inserting the bound (2.126) in (2.35) yields

$$\mathbb{R}^\circ\left(\tau_{x'}^{\bar x} < \tau_0^{\bar x}\right) \leq e^{-Nh'(d)}, \qquad \text{for all} \quad x'\in\mathcal{T}\setminus\bar x \tag{2.146}$$

for some constant $h'(d) > 0$ and large enough N. To bound $\mathbb{R}^\circ\left(\tau_0^{\bar x} < \tau_{\bar x}^{\bar x}\right)$ we simply use that by Lemma 2.6, in view of (2.125) and (2.126), there exists a constant $c_2 > 0$ such that

$$\mathbb{R}^\circ\left(\tau_0^{\bar x} < \tau_{\bar x}^{\bar x}\right) \geq 1 - \frac{d}{NM}\left(1 + c_2\left(\tfrac{d}{N}\right)^{1/2}\ln N\right). \tag{2.147}$$

Gathering our bounds, we finally obtain

$$\left|\mathbb{R}^\circ\left(\tau_x^{\bar x} < \tau_{\mathcal{T}\setminus x}^{\bar x}\right) - \frac{1}{M}\right| \leq \frac{d}{NM}\left(1 + c_3\left(\tfrac{d}{N}\right)^{1/2}\ln N\right) \tag{2.148}$$

for some constant $c_3 > 0$. Inserting (2.148) in (2.142) concludes the proof of assertion ii). □

Proof of Proposition 2.1, iii). Again, the proof of this third assertion is very similar to that of assertion i). Using in turn assertion i) of Lemma 2.4 and Lemma 2.5,

$$\mathbb{P}\left(\tau_\eta^{\bar\eta} < \tau_{\mathcal{T}\setminus\{\eta,\bar\eta\}}^{\bar\eta}\right) = \mathbb{P}^\circ\left(\tau_\eta^{\bar\eta} < \tau_{\mathcal{T}\setminus\{\eta,\bar\eta\}}^{\bar\eta}\right) = \mathbb{R}^\circ\left(\tau_x^{\bar x} < \tau_{\mathcal{T}\setminus\{x,\bar x\}}^{\bar x}\right). \tag{2.149}$$

Defining

$$\begin{aligned} R_1 &\equiv \mathbb{R}^\circ\left(\{\tau_x^{\bar x} < \tau_{\mathcal{T}\setminus\{x,\bar x\}}^{\bar x}\}\cap\{\tau_0^{\bar x} < \tau_x^{\bar x}\}\right)\\ R_2 &\equiv \mathbb{R}^\circ\left(\{\tau_x^{\bar x} < \tau_{\mathcal{T}\setminus\{x,\bar x\}}^{\bar x}\}\cap\{\tau_x^{\bar x} < \tau_0^{\bar x}\}\right) \end{aligned} \tag{2.150}$$

we have

$$\mathbb{R}^\circ\left(\tau_x^{\bar x} < \tau_{\mathcal{T}\setminus\{x,\bar x\}}^{\bar x}\right) = R_1 + R_2. \tag{2.151}$$

Next, just as in (2.135) we write

$$0 \leq R_2 \leq \mathbb{R}^\circ\left(\tau_x^{\bar x} < \tau_0^{\bar x}\right) \tag{2.152}$$

while proceeding as in (2.136) and (2.137) to treat the term R_1 yields, in analogy with (2.138),

$$\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus\{x,\bar{x}\}}^0\right)\left[1-(M-1)\sup_{x'\in\mathcal{T}\setminus\bar{x}}\mathbb{R}^\circ\left(\tau_{x'}^{\bar{x}} < \tau_0^{\bar{x}}\right)\right] \leq R_1 \leq \mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus\{x,\bar{x}\}}^0\right). \tag{2.153}$$

Since the probabilities $\mathbb{R}^\circ\left(\tau_{x'}^{\bar{x}} < \tau_0^{\bar{x}}\right)$, $x' \in \mathcal{T}\setminus\bar{x}$, appearing in (2.152) and (2.153) have already been bounded in (2.146), we are left to estimate $\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus\{x,\bar{x}\}}^0\right)$. To do this we proceed exactly as in the proof of (2.140) (the only difference being that the set J in Lemma 2.8 is here given by $J = \mathcal{T}\setminus\{x,\bar{x}\}$ so that $|J| = M-1$) and obtain

$$\left|\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus\{x,\bar{x}\}}^0\right) - \frac{1}{M-1}\right| \leq \frac{d}{N(M-1)}\left(1 + c_0\left(\tfrac{d}{N}\right)^{1/2}\ln N\right) \tag{2.154}$$

for some constant $c_0 > 0$. Collecting our bounds yields the claim of assertion iii). □

Proof of Proposition 2.1, iv). This assertion is nothing but a direct consequence of assertion ii) since

$$\mathbb{P}\left(\tau_{T\setminus\eta}^\eta < \tau_\eta^\eta\right) = \sum_{\bar{\eta}\in T\setminus\eta}\mathbb{P}\left(\tau_{\bar{\eta}}^\eta < \tau_{T\setminus\bar{\eta}}^\eta\right). \tag{2.155}$$

Thus (2.6) is proven. For later use (see the proof of assertion v)) let us however give a full derivation of the lower bound in (2.6): again, with the same notation as in the proofs of the first two assertions, using in turn assertion ii) of Lemma 2.4 and Lemma 2.5, it follows from (2.155) that

$$\mathbb{P}\left(\tau_{T\setminus\eta}^\eta < \tau_\eta^\eta\right) = e^{-\beta\sqrt{N}E_\eta}\mathbb{P}^\circ\left(\tau_{T\setminus\eta}^\eta < \tau_\eta^\eta\right) = e^{-\beta\sqrt{N}E_\eta}\mathbb{R}^\circ\left(\tau_{\mathcal{T}\setminus x}^x < \tau_x^x\right). \tag{2.156}$$

Defining

$$\begin{aligned} R_1 &\equiv \mathbb{R}^\circ\left(\{\tau_{\mathcal{T}\setminus x}^x < \tau_x^x\}\cap\{\tau_0^x < \tau_{\mathcal{T}\setminus x}^x\}\right), \\ R_2 &\equiv \mathbb{R}^\circ\left(\{\tau_{\mathcal{T}\setminus x}^x < \tau_x^x\}\cap\{\tau_{\mathcal{T}\setminus x}^x < \tau_0^x\}\right), \end{aligned} \tag{2.157}$$

we have

$$1 \geq \mathbb{R}^\circ\left(\tau_{\mathcal{T}\setminus x}^x < \tau_x^x\right) = R_1 + R_2 \geq R_1, \tag{2.158}$$

and since

$$\begin{aligned} R_1 &= \mathbb{R}^\circ\left(\tau_0^x < \tau_{\mathcal{T}\setminus x}^x < \tau_x^x\right) \\ &= \mathbb{R}^\circ\left(\tau_0^x < \tau_{\mathcal{T}}^x\right)\mathbb{R}^\circ\left(\tau_{\mathcal{T}\setminus x}^0 < \tau_x^0\right) \\ &= \left[1-\mathbb{R}^\circ\left(\tau_{\mathcal{T}}^x < \tau_0^x\right)\right]\left[1-\mathbb{R}^\circ\left(\tau_x^0 < \tau_{\mathcal{T}\setminus x}^0\right)\right] \end{aligned} \tag{2.159}$$

we obtain, proceeding as in (2.144) to bound $1-\mathbb{R}^\circ\left(\tau^x_{\mathcal{T}}<\tau^x_0\right)$,

$$R_1 \geq \left[\mathbb{R}^\circ\left(\tau^x_0<\tau^x_x\right)-(M-1)\sup_{x'\in\mathcal{T}\setminus x}\mathbb{R}^\circ\left(\tau^x_{x'}<\tau^x_0\right)\right]\left[1-\mathbb{R}^\circ\left(\tau^0_x<\tau^0_{\mathcal{T}\setminus x}\right)\right]. \tag{2.160}$$

Now all the probabilities entering the above expression have already been estimated (see respectively (2.147), (2.146) and (2.140) for the estimates on $\mathbb{R}^\circ\left(\tau^x_0<\tau^x_x\right)$, $\mathbb{R}^\circ\left(\tau^x_{x'}<\tau^x_0\right)$, $x'\in\mathcal{T}\setminus x$, and $\mathbb{R}^\circ\left(\tau^0_x<\tau^0_{\mathcal{T}\setminus x}\right)$). Plugging these estimates in (2.160) we obtain

$$\mathbb{R}^\circ\left(\tau^x_{\mathcal{T}\setminus x}<\tau^x_x\right)\geq\left(1-\tfrac{1}{M}\right)\left(1-\tfrac{d}{N}\left(1+c_4\left(\tfrac{d}{N}\right)^{1/2}\ln N\right)\right) \tag{2.161}$$

for some constant $c_4>0$. Inserting (2.161) in (2.156) proves the lower bound of (2.6). $\square$

As is by now routine, the proof of assertion v) of Proposition 2.1 begins as in (2.156): we first invoke Lemma 2.4 to write

$$\mathbb{P}\left(\tau^\sigma_T<\tau^\sigma_\sigma\right)=e^{-\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ\left(\tau^\sigma_T<\tau^\sigma_\sigma\right) \tag{2.162}$$

and next use Lemma 2.5 to express the last probability above in terms of a lumped chain:

$$\mathbb{P}^\circ\left(\tau^\sigma_T<\tau^\sigma_\sigma\right)=\mathbb{R}^\circ\left(\tau^{\gamma_{T\cup\sigma}(\sigma)}_{\gamma_{T\cup\sigma}(T)}<\tau^{\gamma_{T\cup\sigma}(\sigma)}_{\gamma_{T\cup\sigma}(\sigma)}\right). \tag{2.163}$$

Similarly, to prove assertion vi) we begin by writing:

$$\mathbb{P}\left(\tau^{\bar\sigma}_\sigma\leq\tau^{\bar\sigma}_T\right)=\mathbb{R}^\circ\left(\tau^{\gamma_{T\cup\sigma}(\bar\sigma)}_{\gamma_{T\cup\sigma}(\sigma)}<\tau^{\gamma_{T\cup\sigma}(\bar\sigma)}_{\gamma_{T\cup\sigma}(T)}\right). \tag{2.164}$$

At this point however we see that contrary to the cases encountered so far the mapping γ involved in the last two identities is not constructed from the top alone, but the top augmented by a non-random point σ. To proceed any further we thus need to investigate its properties.

Lumped chain induced by the Top and a non-random point. In order to study the mapping $\gamma_{T\cup\sigma}$ we must go back to its definition (see Sect. 2.1). Most of the results we will establish below rely on the simple observation that the partition $\mathcal{P}_{T\cup\sigma}(\Lambda)$ induced by $T\cup\sigma$ may be constructed by first constructing the partition $\mathcal{P}_T(\Lambda)$ induced by the top alone and next, partitioning each of the elements of $\mathcal{P}_T(\Lambda)$ according to the sign of σ_i. More precisely:

Lemma 2.14. *Set $d'=2^{M+1}$, $d=2^M$. There is a one-to-one correspondence between the elements of the partition $\mathcal{P}_{T\cup\sigma}(\Lambda)$,*

$$\Lambda_{k'}(T\cup\sigma),\quad k'\in\{1,\dots,d'\} \tag{2.165}$$

and the sets

$$\Lambda^s_k(T)=\{i\in\Lambda_k(T)\mid\sigma_i=s\},\quad(s,k)\in\{-1,1\}\times\in\{1,\dots,d\}. \tag{2.166}$$

Proof. Let $\{e'_1, \dots, e'_{k'}, \dots, e'_{d'}\}$ and $\{e_1, \dots, e_k, \dots, e_d\}$ be arbitrarily chosen labellings of, respectively, all d' elements of $\mathcal{S}_{M+1}$ and all d elements of $\mathcal{S}_M$. For $u = (u_1, \dots, u_{M+1}) \in \mathbb{R}^{M+1}$ write $u = (\underline{u}, \overline{u})$, with $\underline{u} = (u_1, \dots, u_M)$ and $\overline{u} = u_{M+1}$. Then, clearly,

$$\{e'_{k'}\}_{k' \in \{1,\dots,d'\}} = \{(e_k, s)\}_{(s,k) \in \{-1,1\} \times \in \{1,\dots,d\}} \tag{2.167}$$

and that the relation

$$e'_{k'} = (e_k, s) \tag{2.168}$$

induces a one-to-one correspondance between the indices $k' \in \{1, \dots, d'\}$ and the pairs $(s, k) \in \{-1, 1\} \times \in \{1, \dots, d\}$.

Let now $\xi' \in \mathcal{S}_{(|T|+1)\times N}$ and $\xi \in \mathcal{S}_{(|T|)\times N}$ be two matrices satisfying property (2.12) with, respectively, $I = T \cup \sigma$ and $I = T$, and chosen such that:

$$\xi'^{\mu} = \begin{cases} \xi^{\mu} & \text{if} \quad \mu \in \{1, \dots, M\} \\ \sigma, & \text{if} \quad \mu = M+1 \end{cases}. \tag{2.169}$$

It then follows from Definition (2.13) that, whenever (2.168) holds,

$$\begin{aligned} \Lambda_{k'}(T \cup \sigma) &\equiv \{i \in \Lambda \mid \xi'_i = e'_{k'}\} \\ &= \{i \in \Lambda \mid \underline{\xi}'_i = \underline{e}'_{k'}, \overline{\xi}'_i = \overline{e}'_{k'}\} \\ &= \{i \in \Lambda \mid \xi_i = e_k, \sigma_i = s\} \\ &= \{i \in \Lambda_k(T) \mid \sigma_i = s\} \\ &= \Lambda^s_k(T). \end{aligned} \tag{2.170}$$

The lemma is therefore proven. □

Lemma 2.15. *Let $\mathcal{K}(T \cup \sigma) \equiv \mathcal{K}_\epsilon(T \cup \sigma)$ and $\delta(N)$ be defined as in (2.32), resp. (2.113). Choose $\epsilon = 1 - 2\delta(N)$. Then, for all $\omega \in \mathcal{E}_N$,*

$$\frac{d'}{2} \le |\mathcal{K}(T \cup \sigma)| \le d' \tag{2.171}$$

and

$$\begin{aligned} N_* \equiv \min_{k \in \mathcal{K}(T\cup\sigma)} |\Lambda_k(T \cup \sigma)| &\ge (1 - 2\delta(N))\frac{N}{d'} \\ \max_{k \in \mathcal{K}(T\cup\sigma)} |\Lambda_k(T \cup \sigma)| &\le 2(1 + \delta(N))\frac{N}{d'}. \end{aligned} \tag{2.172}$$

Proof. It obviously follows from Definition (2.166) that

$$|\Lambda_k(T)| = |\Lambda^+_k(T)| + |\Lambda^-_k(T)|. \tag{2.173}$$

For fixed $k \in \{1, \dots, d\}$, assume that there exists $s \in \{-1, 1\}$ such that $|\Lambda^s_k(T)| < \epsilon\frac{N}{d'}$. It then follows from (2.173) that

$$|\Lambda^{-s}_k(T)| \ge |\Lambda_k(T)| - \epsilon\frac{N}{d'} \ge (1 - \delta(N))\frac{N}{d} - \epsilon\frac{N}{2d} \ge \epsilon\frac{N}{d'}, \tag{2.174}$$

where the second inequality follows from (2.126) and the fact that $\omega \in \mathcal{E}_N$, while the last line results from our choice of ϵ. Thus for each $k \in \{1, \dots, d\}$ there exists at least one index $s \in \{-1, 1\}$ such that $|\Lambda_k^s(T)| \geq \epsilon \frac{N}{d'}$. This together with Lemma 2.14 yields the lower bound of (2.171). The upper bound beeing immediate, (2.171) is proven.

Let us turn to (2.172). The first inequality simply follows from the definition of $\mathcal{K}(T \cup \sigma)$ and our choice ϵ. To prove the second inequality we first use that by (2.173), for each pair $(s, k) \in \{-1, 1\} \times \in \{1, \ldots, d\}$,

$$|\Lambda_k^s(T)| \leq |\Lambda_k(T)| \leq (1 + \delta(N))\frac{N}{d} = 2(1 + \delta(N))\frac{N}{d'}, \tag{2.175}$$

where the second inequality follows from (2.126), and next conclude by means of Lemma 2.14. The lemma is proven. □

To state the next lemma we need some extra notation. Set $\kappa' = |\mathcal{K}(T \cup \sigma)|$ and let $\pi' : \mathbb{R}^{d'} \to \mathbb{R}^{\kappa'}$ be the projection that maps $x = (x_{k'})_{k' \in \{1,\ldots,d'\}}$ into $\pi' x = (x_{k'})_{k' \in \mathcal{K}(T \cup \sigma)}$. For each $k \in \{1, \ldots, d\}$, let $s_* \in \{-1, 1\}$ be defined through

$$|\Lambda_k^{s_*}(T)| \geq |\Lambda_k^{-s_*}(T)| \tag{2.176}$$

and set

$$\mathcal{D} = \left\{k' \in \{1, \ldots, d'\} \mid \Lambda_{k'}(T \cup \sigma) = \Lambda_k^{s_*}(T),\ k \in \{1, \ldots, d\}\right\}. \tag{2.177}$$

Finally, let $\pi^* : \mathbb{R}^{d'} \to \mathbb{R}^d$ be the projection that maps $x = (x_{k'})_{k' \in \{1,\ldots,d'\}}$ into $\pi^* x = (x_{k'})_{k' \in \mathcal{D}}$.

Lemma 2.16. *For all $\sigma \in T^c$ the following holds true:*

i) For all $\eta \in T$,

$$\pi^* \gamma_{T \cup \sigma}(\eta) = \gamma_T(\eta). \tag{2.178}$$

For $0 \leq \varepsilon < \frac{1}{2}$, define

$$\mathcal{A}_\varepsilon(\sigma) = \left\{\eta \in T \;\middle|\; \|\pi^* \gamma_{T \cup \sigma}(\eta) - \pi^* \gamma_{T \cup \sigma}(\sigma)\|_2 \leq \varepsilon \sqrt{d}\right\}. \tag{2.179}$$

Then,

ii) either $\mathcal{A}_\varepsilon(\sigma) = \emptyset$ or else, $|\mathcal{A}_\varepsilon(\sigma)| = 1$.

iii) For all $\eta \in T \setminus \mathcal{A}_\varepsilon(\sigma)$,

$$\|\pi' \gamma_{T \cup \sigma}(\eta) - \pi' \gamma_{T \cup \sigma}(\sigma)\|_2 \geq \varepsilon \sqrt{d} \tag{2.180}$$

and, for all $\eta \in T$,

$$\inf_{\overline{\eta} \in T \setminus \eta} \|\pi' \gamma_T(\overline{\eta}) - \pi' \gamma_T(\eta)\|_2 \geq (1 - 2\delta(N))\sqrt{d}. \tag{2.181}$$

Proof. We first prove assertion i). By Lemma 2.14, to each $k' \in \{1, \ldots, d'\}$ there corresponds a unique pair $(s, k) \in \{-1, 1\} \times \in \{1, \ldots, d\}$ verifying

$$\Lambda_{k'}(T \cup \sigma) = \Lambda_k^s(T). \tag{2.182}$$

Fix $k' \in \{1, \ldots, d'\}$. It follows from Definition (2.16) and (2.182) that

$$\gamma_{T \cup \sigma}^{k'}(\eta) = \frac{1}{|\Lambda_k^s(T)|} \sum_{i \in \Lambda_k^s(T)} \eta_i. \tag{2.183}$$

Now by (2.166), we have,

$$\Lambda_k^s(T) \subseteq \Lambda_k(T) \quad \text{for each} \quad s \in \{-1, 1\}. \tag{2.184}$$

But assertion iii) of Lemma 2.2 states that

$$\eta_i = \gamma_T^k(\eta), \quad \text{for all} \quad i \in \Lambda_k(T). \tag{2.185}$$

Hence, combining (2.185) and (2.183) we get

$$\gamma_{T\cup\sigma}^{k'}(\eta) = \gamma_T^k(\eta) \quad \text{for each} \quad s \in \{-1, 1\}. \tag{2.186}$$

Since (2.186) holds for each $s \in \{-1, 1\}$ it holds true for $s = s_*$. We therefore have proven that for each $k' \in \{1, \ldots, d'\}$ and each $k \in \{1, \ldots, d\}$ related through $\Lambda_{k'}(T \cup \sigma) = \Lambda_k^{s_*}(T)$, $\gamma_{T\cup\sigma}^{k'}(\eta) = \gamma_T^k(\eta)$. But this, in view of (2.177), implies that $\pi^*\gamma_{T\cup\sigma}(\eta) = \gamma_T(\eta)$, concluding the proof of assertion i).

We now turn to the proof of assertion ii). Note that by (2.178), $\mathcal{A}_\varepsilon(\sigma)$ may be written as

$$\mathcal{A}_\varepsilon(\sigma) = \left\{\eta \in T \;\middle|\; \|\gamma_T(\eta) - \pi^*\gamma_{T\cup\sigma}(\sigma)\|_2 \le \varepsilon\sqrt{d}\right\}. \tag{2.187}$$

Assume that $\mathcal{A}_\varepsilon(\sigma) \neq \emptyset$. Then there exists $\eta \in T$ such that $\|\gamma_T(\eta) - \pi^*\gamma_{T\cup\sigma}(\sigma)\|_2 \le \varepsilon\sqrt{d}$. Thus,

$$\begin{aligned}\inf_{\overline{\eta}\in T\setminus\eta} \|\gamma_T(\overline{\eta}) - \pi^*\gamma_{T\cup\sigma}(\sigma)\|_2 &\ge \inf_{\overline{\eta}\in T\setminus\eta} \|\gamma_T(\overline{\eta}) - \gamma_T(\eta)\|_2 - \varepsilon\sqrt{d} \\ &\ge (1 - 2\delta(N))\sqrt{d} - \varepsilon\sqrt{d},\end{aligned} \tag{2.188}$$

where the last inequality follows from Lemma 2.13. Since for all $0 \le \varepsilon < \frac{1}{2}$, $1 - 2\delta(N) - \varepsilon > \varepsilon$, provided that N is sufficiently large, we conclude that $|\mathcal{A}_\varepsilon(\sigma)| = 1$. The claim of assertion ii) is thus proven and it remains to prove iii).

To do so note that proceeding just as in the proof of (2.171), we easily see that $\mathcal{D} \subseteq \mathcal{K}(T \cup \sigma)$. Hence, for all $y, y' \in \mathbb{R}^{d'}$,

$$\|\pi' y' - \pi' y\|_2 \ge \|\pi^* y' - \pi^* y\|_2. \tag{2.189}$$

Now (2.180) is an immediate consequence of (2.189) and the definition of $\mathcal{A}_\varepsilon(\sigma)$ while (2.181) results from the combination of (2.189) and (2.128) of Lemma 2.13. Assertion iii) being proven, the proof of the lemma is done. □

We are now ready to prove the last two assertions of Proposition 2.1. The following notation will be used throughout: $I \equiv T \cup \sigma$, $\mathcal{I} \equiv \gamma_I(I)$, $y \equiv \gamma_I(\sigma)$, and $\bar{y} \equiv \gamma_I(\bar{\sigma})$. It will moreover be assumed that $\omega \in \mathcal{E}_N$.

Proof of Proposition 2.1, v and vi). With the notation introduced above, (2.163) and (2.164) read, respectively,

$$\mathbb{P}^\circ\left(\tau_T^\sigma < \tau_\sigma^\sigma\right) = \mathbb{R}^\circ\left(\tau_{\mathcal{I}\setminus y}^y < \tau_y^y\right) \tag{2.190}$$

and

$$\mathbb{P}\left(\tau_\sigma^{\bar{\sigma}} \le \tau_T^{\bar{\sigma}}\right) = \mathbb{R}^\circ\left(\tau_y^{\bar{y}} < \tau_{\mathcal{I}\setminus y}^{\bar{y}}\right). \tag{2.191}$$

We may now distinguish two cases since, according to assertion ii) of Lemma 2.16, either σ is such that, case 1), $\mathcal{A}_\varepsilon(\sigma) = \emptyset$, or else case 2), $\mathcal{A}_\varepsilon(\sigma) = \{\eta\}$ for some $\eta \in T$.

In case 1), a simple adaptation of the proof of the lower bound (2.161) of assertion iii) yields

$$\mathbb{R}^\circ\left(\tau^y_{\mathcal{I}\setminus y} < \tau^y_y\right) \geq \left(1 - \tfrac{1}{M+1}\right)\left(1 - \tfrac{d}{N}\left(1 + c_5(1+\delta(N))\right)\right) \tag{2.192}$$

for some constant $c_5 > 0$. Similarly, retracing the proof of the upper bound of assertion i), we readily obtain that

$$\left|\mathbb{R}^\circ\left(\tau^{\bar{y}}_y < \tau^{\bar{y}}_{\mathcal{I}\setminus y}\right) - \tfrac{1}{M+1}\right| \leq \tfrac{d}{NM}\left(1 + c_6(1+\delta(N))\right) \tag{2.193}$$

for some constant $c_6 > 0$.

Case 2) will also be brought back to well known situations once observed that, setting $x \equiv \gamma_I(\eta)$,

$$\mathbb{R}^\circ\left(\tau^y_{\mathcal{I}\setminus y} < \tau^y_y\right) \geq \mathbb{R}^\circ\left(\tau^y_{\mathcal{I}\setminus\{y,x\}} < \tau^y_y\right) \tag{2.194}$$

while

$$\mathbb{R}^\circ\left(\tau^{\bar{y}}_y < \tau^{\bar{y}}_{\mathcal{I}\setminus y}\right) \leq \mathbb{R}^\circ\left(\tau^{\bar{y}}_y < \tau^{\bar{y}}_{\mathcal{I}\setminus\{y,x\}}\right). \tag{2.195}$$

Then, proceeding as in the proof of (2.192) we obtain that

$$\mathbb{R}^\circ\left(\tau^y_{\mathcal{I}\setminus\{y,x\}} < \tau^y_y\right) \geq \left(1 - \tfrac{1}{M}\right)\left(1 - \tfrac{d}{N}\left(1 + c_7(1+\delta(N))\right)\right) \tag{2.196}$$

for some constant $c_7 > 0$, while going back over the proof of (2.193) yields

$$\left|\mathbb{R}^\circ\left(\tau^{\bar{y}}_y < \tau^{\bar{y}}_{\mathcal{I}\setminus\{y,x\}}\right) - \frac{1}{M}\right| \leq \tfrac{d}{NM}\left(1 + c_8(1+\delta(N))\right) \tag{2.197}$$

for some constant $c_8 > 0$.

The lower bound in (2.7) then follows from (2.162) together with (2.190), (2.192), (2.194), and (2.196); the coresponding upper bound being immediate, assertion v) is proven. Finally, collecting (2.191), (2.193), (2.195), and (2.197) proves (2.8) of assertion vi). This completes the proof of Proposition 2.1. □

3. Expected Times

In this section we prove Theorems 1.4 and 1.5. Let $\mathbb{E}_{T(E)}(\,.\,)$ and $\mathbb{V}_{T(E)}(\,.\,)$ denote the expectation and the variance with respect to the conditional distribution $P\,(\,.\mid T(E)(\omega) = T(E))$. Define

$$Z_{\beta,N}(T^c(E)) \equiv \sum_{\sigma\in T(E)^c} e^{\beta\sqrt{N}E_\sigma}, \tag{3.1}$$

$$\mathcal{V}_{N,E} \equiv \mathcal{V}_{\beta,N,T(E)} \equiv \frac{Z_{\beta,N}(T^c(E)) - \mathbb{E}_{T(E)}(Z_{\beta,N}(T^c(E)))}{\mathbb{V}_{T(E)}(Z_{\beta,N}(T^c(E)))}. \tag{3.2}$$

Recall that

$$\mathcal{W}_{\beta,N,T(E)} \equiv \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{M(\alpha-1)}\left(1 + \mathcal{V}_{\beta,N,T(E)}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right), \tag{3.3}$$

where, as in Sect. 2, $M = |T(E)|$.

Remark. A remark is in order concerning the random variables defined in (3.1) to (3.3). The behavior of $Z_{\beta,N}(T^c(E))$ will be studied in Lemma 3.3. It will in particular be established that $Z_{\beta,N}(T^c(E)) = M\mathcal{W}_{\beta,N,T(E)}(1+O(1/N))$ (see (3.27)). This of course implies that $\mathcal{W}_{\beta,N,T(E)}$ is a positive random variable. Note also that by definition $\mathcal{V}_{\beta,N,T(E)}$ has mean zero and variance one, and that all its moments are finite.

Notation. From now on we will systematically write T for $T(E)$ and drop the indices β, N, and $T(E)$ in all the symbols appearing in (3.1), (3.2) and (3.3).

The cornerstone of the proof of Proposition 1.4 is a classical identity from potential theory (see e.g. [So] or Corollary (3.3) of [BEGK2]) that expresses the expectation of conditioned transition times in terms of the invariant measure and transition probabilities. Namely, it states that for all subsets $I, J \subseteq \mathcal{S}_N$, and all $\sigma \in \mathcal{S}_N$ such that $\sigma \notin I \cup J$,

$$\begin{aligned}\mathbb{E}\left(\tau_I^\sigma \mid \tau_I^\sigma \leq \tau_J^\sigma\right) &= \frac{1}{\mu_{\beta,N}(\sigma)\mathbb{P}(\tau_{I\cup J}^\sigma < \tau_\sigma^\sigma)}\\ &\times\left[\mu_{\beta,N}(\sigma) + \sum_{\sigma'\in(I\cup J\cup\sigma)^c}\mu_{\beta,N}(\sigma')\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_{I\cup J}^{\sigma'})\frac{\mathbb{P}(\tau_I^{\sigma'} \leq \tau_J^{\sigma'})}{\mathbb{P}(\tau_I^\sigma \leq \tau_J^\sigma)}\right]. \end{aligned}\tag{3.4}$$

Equation (3.4) generalizes the following expression for the expected value of unconditioned transition times: for all subset $I \subseteq \mathcal{S}_N$ and all $\sigma \in \mathcal{S}_N$ such that $\sigma \notin I$,

$$\mathbb{E}(\tau_I^\sigma) = \frac{1}{\mu_{\beta,N}(\sigma)\mathbb{P}(\tau_I^\sigma < \tau_\sigma^\sigma)}\left[\mu_{\beta,N}(\sigma) + \sum_{\sigma'\in(I\cup\sigma)^c}\mu_{\beta,N}(\sigma')\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_I^{\sigma'})\right]. \tag{3.5}$$

Therefore, by definition of $\mu_{\beta,N}$, (3.4) reads

$$\begin{aligned}\mathbb{E}(\tau_I^\sigma \mid \tau_I^\sigma \leq \tau_J^\sigma) &= \frac{1}{e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_{I\cup J}^\sigma < \tau_\sigma^\sigma)}\\ &\times\left[e^{\beta\sqrt{N}E_\sigma} + \sum_{\sigma'\in(I\cup J\cup\sigma)^c}e^{\beta\sqrt{N}E_{\sigma'}}\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_{I\cup J}^{\sigma'})\frac{\mathbb{P}(\tau_I^{\sigma'} \leq \tau_J^{\sigma'})}{\mathbb{P}(\tau_I^\sigma \leq \tau_J^\sigma)}\right]\end{aligned}\tag{3.6}$$

and similarly,

$$\mathbb{E}(\tau_I^\sigma) = \frac{1}{e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_I^\sigma < \tau_\sigma^\sigma)}\left[e^{\beta\sqrt{N}E_\sigma} + \sum_{\sigma'\in(I\cup\sigma)^c}e^{\beta\sqrt{N}E_{\sigma'}}\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_I^{\sigma'})\right]. \tag{3.7}$$

Applying (3.6) and (3.7) to the quantities $\mathbb{E}(\tau_{\bar\eta}^\eta \mid \tau_{\bar\eta}^\eta \leq \tau_{T\setminus\eta}^\eta)$, $\mathbb{E}(\tau_{T\setminus\eta}^\eta)$ and $\mathbb{E}(\tau_T^\sigma)$, and inserting the probability estimates of Proposition 2.1 in the resulting expressions, the proof of Proposition 1.4 essentially reduces to studying the behavior of the random variable $Z(T^c) = \sum_{\sigma'\in T^c}e^{\beta\sqrt{N}E_{\sigma'}}$. We start by proving the first assertion of the proposition.

Proof of assertion i) of Theorem 1.4. We will assume throughout that the assumptions of Proposition 2.1 are satisfied. It follows from (3.7) that, for all $\eta \in T$,

$$\mathbb{E}(\tau_{T\setminus\eta}^{\eta}) = \frac{1}{e^{\beta\sqrt{N}E_\eta}\mathbb{P}(\tau_{T\setminus\eta}^{\eta} < \tau_\eta^\eta)}\left[e^{\beta\sqrt{N}E_\eta} + \sum_{\sigma\in T^c} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma)\right]. \quad (3.8)$$

The factor in front of the square brackets was estimated in Proposition 2.1, iv). Plugging in this estimate yields

$$\mathbb{E}(\tau_{T\setminus\eta}^{\eta}) = \frac{1}{1-\frac{1}{M}}\left[e^{\beta\sqrt{N}E_\eta} + \sum_{\sigma\in T^c} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma)\right](1+O(1/N)) \quad (3.9)$$

and we are left to study the term

$$I \equiv \sum_{\sigma\in T^c} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma). \quad (3.10)$$

To do so, we proceed as follows: for $\varepsilon > 0$ a constant, let $\mathcal{B}_{\sqrt{\varepsilon N}}(\eta)$ and $W_\varepsilon(T)$ be defined as in (2.1) and (2.2) and set

$$V_\varepsilon(T) \equiv \bigcup_{\eta\in T}\left(T^c \cap \mathcal{B}_{\sqrt{\varepsilon N}}(\eta)\right). \quad (3.11)$$

Observing that

$$T^C = V_\varepsilon(T) \cup W_\varepsilon(T) \quad (3.12)$$

I may be decomposed as

$$I = I_1 + I_2 \quad (3.13)$$

with

$$\begin{aligned} I_1 &\equiv \sum_{\sigma\in V_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma),\\ I_2 &\equiv \sum_{\sigma\in W_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma). \end{aligned} \quad (3.14)$$

Now obviously,

$$0 \le I_1 \le \sum_{\sigma\in V_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma} \quad (3.15)$$

while, by Proposition 2.1, i), for all $\omega \in \mathcal{E}$ and large enough N, I_2 obeys the bound

$$\left| I_2 - \frac{1}{M}\sum_{\sigma\in W_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma}\right| \le \frac{d}{NM}(1+c\delta(N)). \quad (3.16)$$

Therefore, setting

$$Z(V_\varepsilon(T)) \equiv \sum_{\sigma\in V_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma},$$

$$Z(T^c) \equiv \sum_{\sigma \in T^c} e^{\beta\sqrt{N}E_\sigma}, \tag{3.17}$$

and combining (3.15) and (3.16) together with (3.13), we arrive at

$$\left| I - \frac{1}{M} Z(T^c) \left[1 + (M-1)\frac{Z(V_\varepsilon(T))}{Z(T^c)} \right] \right| \le \frac{d}{NM}(1 + c\delta(N)), \tag{3.18}$$

and it remains to study the behavior of the random variables $Z(V_\varepsilon(T))$ and $Z(T^c)$. As this depends on the cardinality of $V_\varepsilon(T)$, we will first establish that:

Lemma 3.1. *Assume that* $0 < \varepsilon < 1/2$ *and set*

$$J(x) = (1-x)\ln \tfrac{1}{1-x} + x \ln \tfrac{1}{x}, \quad 0 < x < 1. \tag{3.19}$$

Then, for all $\omega \in \mathcal{E}$ *and large enough* N*, there exist constants,* $0 < c^- \le c^+ < \infty$*, such that*

$$c^- M N^{-1/2} e^{NJ(\varepsilon/4)} - M \le |V_\varepsilon(T)| \le c^+ M N^{1/2} e^{NJ(\varepsilon/4)}. \tag{3.20}$$

Proof. Under the assumptions of Lemma 2.12,

$$|V_\varepsilon(T)| = \sum_{\eta \in T} |T^c \cap \mathcal{B}_{\sqrt{\varepsilon N}}(\eta)| = \sum_{\eta \in T} |\mathcal{B}_{\sqrt{\varepsilon N}}(\eta) \setminus \eta| = \sum_{\eta \in T} |\mathcal{B}_{\sqrt{\varepsilon N}}(\eta)| - M. \tag{3.21}$$

Now, for all $\eta \in T$,

$$\binom{N}{\varepsilon N/4} \le |\mathcal{B}_{\sqrt{\varepsilon N}}(\eta)| \le \frac{\varepsilon N}{4}\binom{N}{\varepsilon N/4}, \tag{3.22}$$

where we used that $\binom{N}{k}$ is an increasing function of k for $0 \le k \le \varepsilon N/4$. By Stirling's formula, for large enough N, there exist constants, $0 < a^- \le a^+ < \infty$ such that

$$\frac{a^-}{\sqrt{\pi\varepsilon(1-\varepsilon/4)}} e^{NJ(\varepsilon/4)} \le \binom{N}{\varepsilon N/4} \le \frac{a^+}{\sqrt{\pi\varepsilon(1-\varepsilon/4)}} e^{NJ(\varepsilon/4)}. \tag{3.23}$$

Inserting (3.23) in (3.22) and using that, by assumption, $0 < \varepsilon < 1/2$ we obtain

$$c^- N^{-1/2} e^{NJ(\varepsilon/4)} \le |\mathcal{B}_{\sqrt{\varepsilon N}}(\eta)| \le c^+ N^{1/2} e^{NJ(\varepsilon/4)} \tag{3.24}$$

for some constants, $0 < c^- \le c^+ < \infty$. Inserted in (3.21), (3.24) yields (3.20), concluding the proof of Lemma 3.1. □

We are now ready to prove the following two lemmata.

Lemma 3.2. *Let* $Z(V_\varepsilon(T))$ *be as in (3.17). Under the assumptions and with the notation of Lemma 3.1), the following holds: there exists a constant* $0 < c < \infty$ *such that, for all* $0 < \varepsilon < 1/2$ *, and large enough* N*,*

$$P\left(Z(V_\varepsilon(T)) \ge |V_\varepsilon(T)| e^{2\beta\sqrt{N \ln |V_\varepsilon(T)|}} \;\middle|\; T(\omega) = T \right) \le \frac{c}{\sqrt{J(\varepsilon/4)}} e^{-NJ(\varepsilon/4)}. \tag{3.25}$$

Lemma 3.3. *Let $Z(T^c)$ and $\mathcal{W}$ be as in (3.1) and (3.3). Then, for all N large enough,*

$$P\left(Z(T^c) \le e^{\beta N\sqrt{\ln 2}} \,\middle|\, T(\omega) = T\right) \le e^{-e^{N(\ln 2)/4}} \tag{3.26}$$

and

$$Z(T^c) = M\mathcal{W}(1 + O(1/N)). \tag{3.27}$$

Proof of Lemma 3.2. For $\delta > 0$, set $a = e^{\beta\sqrt{2(\delta+1)N\ln|V_\varepsilon(T)|}}$,

$$\begin{aligned} &P\left(Z(V_\varepsilon(T)) \ge a|V_\varepsilon(T)| \mid T(\omega) = T\right) \\ &\le P\left(|V_\varepsilon(T)| \max_{\sigma\in V_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma} \ge a|V_\varepsilon(T)| \,\middle|\, T(\omega) = T\right) \\ &\le |V_\varepsilon(T)| P\left(e^{\beta\sqrt{N}E_\sigma} \ge a \,\middle|\, T(\omega) = T\right) \\ &= |V_\varepsilon(T)| P\left(e^{\beta\sqrt{N}E_\sigma} \ge a \,\middle|\, E_\sigma < u_N(E)\right), \end{aligned} \tag{3.28}$$

where the second inequality holds true for all $\sigma \in V_\varepsilon(T)$ (thereby implying the last equality). In explicit form, the probability appearing in the last line of (3.28) reads

$$P\left(e^{\beta\sqrt{N}E_\sigma} \ge a \,\middle|\, E_\sigma < u_N(E)\right) = \frac{P\left(\sqrt{2(\delta+1)\ln|V_\varepsilon(T)|} \le E_\sigma < u_N(E)\right)}{P\left(E_\sigma < u_N(E)\right)}. \tag{3.29}$$

By a standard upper tail estimate for Gaussian random variables,

$$\begin{aligned} &P\left(\sqrt{2(\delta+1)\ln|V_\varepsilon(T)|} \le E_\sigma < u_N(E)\right) \\ &\le P\left(E_\sigma \ge \sqrt{2(\delta+1)\ln|V_\varepsilon(T)|}\right) \\ &\le \frac{1}{|V_\varepsilon(T)|^{\delta+1}\sqrt{4\pi(\delta+1)\ln|V_\varepsilon(T)|}} \end{aligned} \tag{3.30}$$

while

$$P\left(E_\sigma < u_N(E)\right) = 1 - 2^{-N}e^{-E}. \tag{3.31}$$

Inserting (3.30) and (3.31) in (3.29) and combining with (3.28) yields

$$\begin{aligned} &P\left(Z(V_\varepsilon(T)) \ge a|V_\varepsilon(T)| \mid T(\omega) = T\right) \\ &\le \frac{1}{|V_\varepsilon(T)|^{\delta}(1 - 2^{-N}e^{-E})\sqrt{4\pi(\delta+1)\ln|V_\varepsilon(T)|}}. \end{aligned} \tag{3.32}$$

Choosing $\delta = 1$, (3.32) together with the lower bound on $|V_\varepsilon(T)|$ of Lemma 3.1 gives (3.25). This proves the lemma. $\square$

Proof of Lemma 3.3. We first prove (3.27). Recall from Theorem 1.4 that $\mathbb{E}_T(\,.\,)$ and $\mathbb{V}_T(\,.\,)$ denote the expectation and the variance with respect to the conditional distribution $P(\,.\mid T(\omega) = T)$ and set

$$X_\beta^\sigma = e^{\beta\sqrt{N}E_\sigma}\mathbb{I}_{\{E_\sigma < u_N(E)\}}. \tag{3.33}$$

Observing that

$$\begin{aligned}
&\mathbb{E}_T(Z(T^c)) = |T^c|\mathbb{E}_T(X^\sigma_\beta),\\
&\mathbb{V}^2_T(Z(T^c)) = \mathbb{E}_T(Z(T^c)^2) - \mathbb{E}^2_T(Z(T^c))^2 = |T^c|[\mathbb{E}_T(X^\sigma_{2\beta}) - \mathbb{E}^2_T(X^\sigma_\beta)],
\end{aligned} \tag{3.34}$$

the computation of the mean and variance of $Z(T^c)$ reduces to that of $\mathbb{E}_T(X^\sigma_\beta)$. Now

$$\mathbb{E}_T(X^\sigma_\beta) = \frac{\frac{1}{\sqrt{2\pi}}\int_0^{u_N(E)} e^{-\frac{x^2}{2}+\beta\sqrt{N}x}dx}{P\left(E_\sigma < u_N(E)\right)}. \tag{3.35}$$

Decomposing the integral above as

$$\frac{1}{\sqrt{2\pi}}\int_0^{u_N(E)} e^{-\frac{x^2}{2}+\beta\sqrt{N}x}dx = r_{1,N} - r_{0,N} \tag{3.36}$$

with

$$\begin{aligned}
r_{1,N} &= \frac{e^{\beta^2 N/2}}{\sqrt{2\pi}}\int_{-\infty}^{u_N(E)-\beta\sqrt{N}} e^{-\frac{y^2}{2}}dy\\
r_{0,N} &= \frac{e^{\beta^2 N/2}}{\sqrt{2\pi}}\int_{-\infty}^{-\beta\sqrt{N}} e^{-\frac{y^2}{2}}dy
\end{aligned} \tag{3.37}$$

we have, by standard tail estimates for the Gaussian,

$$\frac{\beta\sqrt{N}}{\sqrt{2\pi}(1+\beta^2 N)} \le r_{0,N} \le \frac{1}{\sqrt{2\pi\beta^2 N}} \tag{3.38}$$

while, for $\beta > \sqrt{2\ln 2}$,

$$r_{1,N} = \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{2^N(\alpha-1)}(1+O(1/N)). \tag{3.39}$$

Inserting these bounds (3.36) and making use of (3.31), we get

$$\mathbb{E}_T(X^\sigma_\beta) = \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{2^N(\alpha-1)}(1+O(1/N)). \tag{3.40}$$

Remembering that $|T^c| = 2^N - M$, it follows from (3.34) and (3.40) that

$$\mathbb{E}_T(Z(T^c)) = \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{(\alpha-1)}(1+O(1/N)) \tag{3.41}$$

and

$$\mathbb{V}^2_T(Z(T^c)) = \frac{e^{(2\alpha-1)E+2\beta\sqrt{N}u_N(0)}}{(2\alpha-1)}(1+O(1/N)). \tag{3.42}$$

Hence, recalling from (3.2) that $\mathcal{V} = [Z(T^c) - \mathbb{E}_T(Z(T^c))]/[\mathbb{V}_T(Z(T^c))]$, we have

$$Z(T^c) = \mathbb{E}_T(Z(T^c))\left[1 + \mathcal{V}\frac{\mathbb{V}_T(Z(T^c))}{\mathbb{E}_T(Z(T^c))}\right]$$

$$
\begin{aligned}
&= \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{(\alpha-1)}\left[1+\mathcal{V}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right](1+O(1/N))\\
&= M\mathcal{W}(1+O(1/N)),
\end{aligned}
\tag{3.43}
$$

where the last line follows from the definition of $\mathcal{W}$. Thus (3.27) is proven. To prove (3.26) we use the rather abrupt bound

$$
\begin{aligned}
P\left(Z(T^c)\leq e^{\beta N\sqrt{\ln 2}}\,\middle|\, T(\omega)=T\right) &\leq P\left(\forall_{\sigma\in T^c} E_\sigma < \sqrt{N\ln 2}\,\middle|\, T(\omega)=T\right)\\
&\leq \prod_{\sigma\in T^c} P\left(E_\sigma < \sqrt{N\ln 2}\,\middle|\, E_\sigma < u_N(E)\right).
\end{aligned}
\tag{3.44}
$$

Since $\sqrt{N\ln 2} < u_N(E)$ for fixed E and large enough N, we have

$$
P\left(E_\sigma < \sqrt{2N\ln 2}/2\,\middle|\, E_\sigma < u_N(E)\right) = \frac{\frac{2}{\sqrt{2\pi}}\int_0^{\sqrt{N\ln 2}} e^{-\frac{x^2}{2}}dx}{P\left(E_\sigma < u_N(E)\right)},
\tag{3.45}
$$

and it follows from (3.31) together with the bound

$$
\frac{2}{\sqrt{2\pi}}\int_0^{\sqrt{N\ln 2}} e^{-\frac{x^2}{2}}dx = 1-\frac{2}{\sqrt{2\pi}}\int_{\sqrt{N\ln 2}}^{\infty} e^{-\frac{x^2}{2}}dx \geq 1-\frac{1}{4\sqrt{N}}e^{-N(\ln 2)/2}
\tag{3.46}
$$

that

$$
\begin{aligned}
P\left(\forall_{\sigma\in T^c} E_\sigma < \sqrt{N\ln 2}\,\middle|\, T(\omega)=T\right) &\leq \left(\frac{1-\frac{1}{4\sqrt{N}}e^{-N(\ln 2)/2}}{1-2^{-N}e^{-E}}\right)^{|T^c|}\\
&\leq \left(1-\frac{1}{8\sqrt{N}}e^{-N(\ln 2)/2}\right)^{2^N(1-2^{-N}M)}\\
&\leq \exp\left(-e^{N(\ln 2)/2}(1-2^{-N}M)/8\sqrt{N}\right)\\
&\leq \exp\left(-e^{N(\ln 2)/4}\right).
\end{aligned}
\tag{3.47}
$$

This proves (3.26) and concludes the proof of Lemma 3.3. □

We are now ready to complete the proof of assertion i) of Theorem 1.4. For $\beta \geq \sqrt{2\ln 2}$ and fixed $0 < \varepsilon < 1/2$ (to be chosen appropriately later), let $\widetilde{\mathcal{E}}_N \equiv \widetilde{\mathcal{E}}_N(\beta,\varepsilon)$ be defined as

$$
\widetilde{\mathcal{E}}_N \equiv \mathcal{E}\cap\left\{\omega\in\Omega \mid Z(V_\varepsilon(T)) \leq |V_\varepsilon(T)|e^{2\beta\sqrt{N\ln|V_\varepsilon(T)|}},\, Z(T^c)\geq e^{\beta N\sqrt{\ln 2}}\right\},
\tag{3.48}
$$

where $\mathcal{E}$ is taken from Proposition 2.1. Now set

$$
\widetilde{\mathcal{E}} \equiv \bigcup_{N_0}\bigcap_{N>N_0}\widetilde{\mathcal{E}}_N.
\tag{3.49}
$$

Obviously, by (3.25), (3.26) and Lemma 2.11,

$$
P\left(\widetilde{\mathcal{E}}\right) = 1.
\tag{3.50}
$$

Assume from now on that $\omega \in \widetilde{\mathcal{E}}_N$ (and that N is large enough). By Lemma 3.1 and Lemma 3.2,

$$Z(V_\varepsilon(T)) \leq e^{c\beta N\sqrt{\varepsilon}} \tag{3.51}$$

for some numerical constant $0 < c < \infty$. Thus, by 3.26 of Lemma 3.3,

$$0 < \frac{Z(V_\varepsilon(T))}{Z(T^c)} < e^{-c(\varepsilon)\beta N}, \tag{3.52}$$

where $c(\varepsilon) = \ln 2 - c\sqrt{\varepsilon}$. Let ε_0 be defined through $c(\varepsilon_0) = 0$. Then, choosing $0 < \varepsilon < (1/2 \wedge \varepsilon_0)$,

$$\frac{1}{M}Z(T^c)\left[1 + (M-1)\frac{Z(V_\varepsilon(T))}{Z(T^c)}\right] = \frac{1}{M}Z(T^c)\left[1 + O\left(e^{-c(\varepsilon)\beta N}\right)\right] \tag{3.53}$$

which, combined with (3.27) of Lemma 3.3 yields,

$$\frac{1}{M}Z(T^c)\left[1 + (M-1)\frac{Z(V_\varepsilon(T))}{Z(T^c)}\right] = \mathcal{W}(1 + O(1/N)). \tag{3.54}$$

This inserted in turn in (3.18) gives,

$$I = \mathcal{W}(1 + O(1/N)). \tag{3.55}$$

Combining (3.55) with (3.10) and (3.9) concludes the proof of assertion i) of Theorem 1.4. $\square$

Proof of assertion ii) of Theorem 1.4. It follows from (3.7) that, for all $\sigma \notin T$,

$$\mathbb{E}(\tau_T^\sigma) = \frac{1}{e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_T^\sigma < \tau_\sigma^\sigma)}\left[e^{\beta\sqrt{N}E_\sigma} + \sum_{\sigma' \in T^c\setminus\sigma} e^{\beta\sqrt{N}E_{\sigma'}}\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_T^{\sigma'})\right]. \tag{3.56}$$

Assuming again that the assumptions of Proposition 2.1 are satisfied, it follows from Proposition 2.1, v), that

$$\left[e^{\beta\sqrt{N}E_\sigma} + I'\right] \leq \mathbb{E}(\tau_T^\sigma) \leq \frac{1}{1-\frac{1}{M}}\left[e^{\beta\sqrt{N}E_\sigma} + I'\right](1 + O(1/N)), \tag{3.57}$$

where

$$I' \equiv \sum_{\sigma' \in T^c\setminus\sigma} e^{\beta\sqrt{N}E_{\sigma'}}\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_T^{\sigma'}). \tag{3.58}$$

Then, decomposing I' as

$$I' = I_1' + I_2' \tag{3.59}$$

with (for $\varepsilon > 0$ a constant and with $V_\varepsilon(T)$ and $W_\varepsilon(T)$ defined as in (3.11) and (3.12))

$$I_1' \equiv \sum_{\sigma' \in V_\varepsilon(T\cup\sigma)} e^{\beta\sqrt{N}E_{\sigma'}}\mathbb{P}(\tau_\sigma^{\sigma'} < \tau_T^{\sigma'}),$$

$$I'_2 \equiv \sum_{\sigma' \in W_\varepsilon(T\cup\sigma)} e^{\beta\sqrt{N}E_{\sigma'}} \mathbb{P}(\tau_\sigma^{\sigma'} < \tau_T^{\sigma'}). \tag{3.60}$$

But clearly,

$$0 \le I'_1 \le \sum_{\sigma' \in V_\varepsilon(T\cup\sigma)} e^{\beta\sqrt{N}E_{\sigma'}}, \tag{3.61}$$

while by Lemma 2.1, vi), I'_2 obeys the bounds

$$\begin{aligned} I'_2 &\ge \frac{1}{M+1} \sum_{\sigma' \in W_\varepsilon(T\cup\sigma)} e^{\beta\sqrt{N}E_{\sigma'}} \left(1 + \frac{d}{N}(1 - c\delta(N))\right) \\ &\le I'_2 \le \frac{1}{M} \sum_{\sigma' \in W_\varepsilon(T\cup\sigma)} e^{\beta\sqrt{N}E_{\sigma'}} \left(1 + \frac{d}{N}(1 + c\delta(N))\right). \end{aligned} \tag{3.62}$$

Therefore, recalling the definition of $Z(V_\varepsilon(T))$ and $Z(T^c)$ from (3.17),

$$\begin{aligned} I' &\ge \frac{Z((T\cup\sigma)^c)}{M+1}\left[1 - (M-1)\frac{Z(V_\varepsilon(T\cup\sigma))}{Z((T\cup\sigma)^c)}\right]\left(1 + \frac{d}{NM}(1 + c\delta(N))\right) \\ I' &\le \frac{Z((T\cup\sigma)^c)}{M}\left[1 + (M-1)\frac{Z(V_\varepsilon(T\cup\sigma))}{Z((T\cup\sigma)^c)}\right]\left(1 + \frac{d}{NM}(1 + c\delta(N))\right). \end{aligned} \tag{3.63}$$

In other words, comparing (3.63) with (3.18), and noting that the difference between $Z(T^c)$ and $Z((T \cup \sigma)^c$ is even in the worst case not larger than $e^E(\alpha - 1)Z(T^c)$, I' obeys virtually the same upper bound as does I in the proof of assertion i). From here on the proof of assertion ii) follows step by step that of the first assertion.

Proof of assertion iii) of Theorem 1.4. As is the proof of the first two assertions we will assume that the assumptions of Proposition 2.1 are satisfied. By (3.6) we have, for all $\eta, \bar\eta \in T$, $\eta \ne \bar\eta$,

$$\begin{aligned} \mathbb{E}(\tau_{\bar\eta}^\eta \mid \tau_{\bar\eta}^\eta \le \tau_{T\setminus\eta}^\eta) &= \frac{1}{e^{\beta\sqrt{N}E_\eta}\mathbb{P}(\tau_{T\setminus\eta}^\eta < \tau_\eta^\eta)} \\ &\quad \times \left[e^{\beta\sqrt{N}E_\eta} + \sum_{\sigma\in T^c} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma)\frac{\mathbb{P}(\tau_{\bar\eta}^\sigma \le \tau_{T\setminus\eta}^\sigma)}{\mathbb{P}(\tau_{\bar\eta}^\eta \le \tau_{T\setminus\eta}^\eta)}\right]. \end{aligned} \tag{3.64}$$

Recalling the definition of I from (3.10) and comparing Eqs. (3.64) and (3.8), we see that their right hand sides are identical up to the extra factor $\mathbb{P}(\tau_{\bar\eta}^\sigma \le \tau_{T\setminus\eta}^\sigma)/\mathbb{P}(\tau_{\bar\eta}^\eta \le \tau_{T\setminus\eta}^\eta)$ that multiplies each of the terms of the sum in I. Mimicking the proof of assertion i), set

$$I' \equiv I'_1 + I'_2, \tag{3.65}$$

where, for $\varepsilon > 0$ a constant and with $V_\varepsilon(T)$ and $W_\varepsilon(T)$ defined as in (3.11) and (3.12),

$$\begin{aligned} I'_1 &\equiv \sum_{\sigma\in V_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma)\frac{\mathbb{P}(\tau_{\bar\eta}^\sigma \le \tau_{T\setminus\eta}^\sigma)}{\mathbb{P}(\tau_{\bar\eta}^\eta \le \tau_{T\setminus\eta}^\eta)}, \\ I'_2 &\equiv \sum_{\sigma\in W_\varepsilon(T)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ(\tau_\eta^\sigma < \tau_{T\setminus\eta}^\sigma)\frac{\mathbb{P}(\tau_{\bar\eta}^\sigma \le \tau_{T\setminus\eta}^\sigma)}{\mathbb{P}(\tau_{\bar\eta}^\eta \le \tau_{T\setminus\eta}^\eta)}. \end{aligned} \tag{3.66}$$

It follows from Proposition 2.1, iii) that, for $\sigma \in V_\varepsilon(T)$,

$$0 \leq \frac{\mathbb{P}(\tau^\sigma_{\bar\eta} \leq \tau^\sigma_{T\setminus\eta})}{\mathbb{P}(\tau^\eta_{\bar\eta} \leq \tau^\eta_{T\setminus\eta})} \leq (M-1)(1+O(1/N)), \tag{3.67}$$

where we trivially used that $0 \leq \mathbb{P}(\tau^\sigma_{\bar\eta} \leq \tau^\sigma_{T\setminus\eta}) \leq 1$ in the numerator. In the case where $\sigma \in W_\varepsilon(T)$, observing that $\mathbb{P}(\tau^\sigma_{\bar\eta} \leq \tau^\sigma_{T\setminus\eta}) = \mathbb{P}(\tau^\sigma_{\bar\eta} < \tau^\sigma_{(T\setminus\eta)\setminus\bar\eta})$, and making use of assertion i) of Proposition 2.1 with T replaced by $T\setminus\eta$ (hence, since $M=|T|$, with M replaced by $M-1$) we get,

$$\left|\frac{\mathbb{P}(\tau^\sigma_{\bar\eta} \leq \tau^\sigma_{T\setminus\eta})}{\mathbb{P}(\tau^\eta_{\bar\eta} \leq \tau^\eta_{T\setminus\eta})} - 1\right| \leq O(1/N). \tag{3.68}$$

Thus, with $I = I_1 + I_2$, I_1 and I_2 being defined as in (3.14), we get

$$\begin{aligned} |I - I'| &\leq |I_1' - I_1| + |I_2' - I_2| \\ &\leq M(1+O(1/N))I_1 + O(1/N)I_2 \\ &= [M(I_1/I_2)(1+O(1/N)) + O(1/N)]\, I_2 \\ &\leq [M(I_1/I_2)(1+O(1/N)) + O(1/N)]\, I, \end{aligned} \tag{3.69}$$

where for $Z(V_\varepsilon(T))$ and $Z(T^c)$ defined as in (3.17) we have, in view of (3.15) and (3.16),

$$\begin{aligned} \frac{I_1}{I_2} &\leq \frac{MZ(V_\varepsilon(T))}{Z(T^c) - Z(V_\varepsilon(T)) - \frac{d}{N}(1+c\delta(N))} \\ &\leq \frac{M[Z(V_\varepsilon(T))/Z(V_\varepsilon(T))]}{1 - [Z(V_\varepsilon(T))/Z(V_\varepsilon(T))]\left(1+\frac{d}{N}(1+c\delta(N))\right)}. \end{aligned} \tag{3.70}$$

With $\widetilde{\mathcal{E}}_N$ defined as in (3.48), choosing ε as in the line following (3.52), and inserting the bound (3.52) in (3.69) we get, for large enough N,

$$\frac{I_1}{I_2} \leq 2e^{-c(\varepsilon)\beta N}, \quad \text{on } \widetilde{\mathcal{E}}_N. \tag{3.71}$$

From this and (3.55), (3.69) yields

$$|I' - I| \leq \mathcal{W}O(1/N). \tag{3.72}$$

Finally, combining (3.64) and (3.8), and using the previous bound,

$$\left|\mathbb{E}(\tau^\eta_{\bar\eta} \mid \tau^\eta_{\bar\eta} \leq \tau^\eta_{T\setminus\eta}) - \mathbb{E}(\tau^\eta_{T\setminus\eta})\right| = e^{\beta\sqrt{N}E_\eta}\mathbb{P}(\tau^\eta_{T\setminus\eta} < \tau^\eta_{\bar\eta})|I' - I| \leq \frac{1}{1-\frac{1}{M}}\mathcal{W}O(1/N), \tag{3.73}$$

where the pre-factor of $|I' - I|$ was estimated by means of Proposition 2.1, iv). This completes the proof of the last assertion of Theorem 1.4. □

Proof of Theorem 1.5. The proof of this theorem is very similar to that of assertion (i) of Theorem 1.4. The only difference is that this time, the partial partition function $Z(V_\epsilon(E))$ is negligible compared to $Z_{\beta,N}$. Finally, for $\beta < \sqrt{2\ln 2}$, $Z_{\beta,N} = e^{N\beta^2/2}(1+O(N^{-1/4}))$ with probability tending to one faster than any polynomial, as follows from easy estimates (see [BKL or Bo]), and this proves the theorem. □

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A. Appendix

We state and prove a simple lemma that is used in Sect. 2.

Lemma 4.1. *Let $\Delta_k \subset \Gamma_N$, $1 \leq k \leq K$, be a collection of subgraphs of Γ_N and let $\widetilde{\mathbb{R}}^{\circ}_{\Delta_k}$ denote the law of the Markov chain with transition rates*

$$\widetilde{r}^{\circ}_{\Delta_k}(x', x'') = \begin{cases} r^{\circ}_N(x', x''), & \text{if} \quad x' \neq x'', \text{and } (x', x'') \in E(\Delta_k) \\ 0, & \text{otherwise} \end{cases}. \tag{4.1}$$

Assume that

$$E(\Delta_k) \cap E(\Delta_{k'}) = \emptyset, \quad \forall k, k' \in \{1, \ldots, K\}, k \neq k' \tag{4.2}$$

and that

$$y, x \in \bigcap_{k=1}^{K} V(\Delta_k). \tag{4.3}$$

Then

$$\mathbb{R}^{\circ}\left(\tau^y_x < \tau^y_y\right) \geq \sum_{k=1}^{K} \widetilde{\mathbb{R}}^{\circ}_{\Delta_k}\left(\tau^y_x < \tau^y_y\right). \tag{4.4}$$

Proof. This lemma is a straightforward generalisation of Lemma 2.1 of [BEGK1]. Let $\mathcal{H}^y_x$ denote the space of functions

$$\mathcal{H}^y_x \equiv \{h : \Gamma_N \to [0, 1] : h(y) = 0, h(x) = 1\} \tag{4.5}$$

and define the Dirichlet forms

$$\begin{aligned} \Phi_N(h) &\equiv \sum_{x', x'' \in \Gamma_N} \mathbb{Q}_N(x') r^{\circ}_N(x', x'')[h(x') - h(x'')]^2, \\ \Phi_{\Delta_k}(h) &\equiv \sum_{x', x'' \in \Delta_k} \widetilde{\mathbb{Q}}^{\circ}_{\Delta_k}(x') r^{\circ}_{\Delta_k}(x', x'')[h(x') - h(x'')]^2, \end{aligned} \tag{4.6}$$

where $\widetilde{\mathbb{Q}}^{\circ}_{\Delta_k}(y) = \mathbb{Q}_N(y)/\mathbb{Q}_N(\Delta_k)$. Then $\Phi_N(h) \geq \sum_{k=1}^{K} \mathbb{Q}_N(\Delta_k)\Phi_{\Delta_k}(h)$, implying that

$$\inf_{h \in \mathcal{H}^y_x} \Phi_N(h) \geq \inf_{h \in \mathcal{H}^y_x} \sum_{k=1}^{K} \mathbb{Q}_N(\Delta_k)\Phi_{\Delta_k}(h) \geq \sum_{k=1}^{K} \mathbb{Q}_N(\Delta_k) \inf_{h \in \mathcal{H}^y_x} \Phi_{\Delta_k}(h) \tag{4.7}$$

from which the lemma follows by an application of Theorem 2.2 of [BEGK1]. □

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