CONVERGENCE OF A KINETIC EQUATION TO A FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. A linear Boltzmann equation is interpreted as the forward equation for the probability density of a Markov process $(K(t), Y(t))$ on $(\mathbb{T} \times \mathbb{R})$, where $\mathbb{T}$ is the one-dimensional torus. $K(t)$ is a autonomous reversible jump process, with waiting times between two jumps with finite expectation value but infinite variance. $Y(t)$ is an additive functional of K, defined as $\int_0^t v(K(s))ds$, where $|v| \sim 1$ for small k . We prove that the rescaled process $N^{-2/3}Y(Nt)$ converge in distribution to a symmetric Lévy process, stable with index $\alpha = 3/2$.

1. INTRODUCTION.

The understanding of thermal conductance in both classical and quantum mechanical systems is one of the fundamental problems of non-equilibrium statistical mechanics. A particular aspect that has attracted much interest is the observation that autonomous translation invariant systems in dimensions one and two exhibit anomalously large conductivity. The canonical example here is a chain of anharmonic oscillators introduced by Fermi-Pasta-Ulam (FPU)[13], for which numerical evidence shows a super-diffusive spreading of energy (see [19] for a general review). However, the rigorous analysis of energy transport mechanism presents serious mathematical difficulties and few results are obtained starting from microscopic dynamics.

The canonical approach to this problem, starting with the pioneering work of Peierls [23] for the case of weak non-linearities, is to derive a Boltzmann-type equation that will describe the energy transport in a kinetic limit. Recently, this approach was carried out rigorously for weakly anharmonic FPU chains [24, 1, 21]. A linear Boltzmann equation was derived in [20] for the harmonic chain with random masses. The same linear Boltzmann equation appears also as limit of a random Schrödinger equation (see for example [11], [10], [25], [2]).

In [3] a kinetic limit was performed for a system of harmonic oscillators perturbed by a conservative stochastic noise and the following linear Boltzmann equation is deduced for the the energy density distribution of the normal modes, or phonons, characterized by a wave-number $k \in [-1/2, 1/2]$:

$$\partial_t W(t,u,k) + v(k)\partial_u W(t,u,k) = \int_{\mathbb{T}} dk'\ R(k,k')[W(t,u,k') - W(t,u,k)]. \tag{1.1}$$

The exact form of the scattering kernel R and the velocity $v(k)$ will be given below. The crucial features are, however, that the kernel R behaves like k^2 for small k,

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$\forall k'$, and like k'^2 for small k', $\forall k$, while $|v(k)| \to 1$, as $|k| \downarrow 0$. This conforms to the intuitive picture that phonons with wave number k travel with a velocity $v(k)$ and are scattered with a rate $R(k, k')$. It is well known that super-diffusive spreading of energy is connected with the fact that the mean free path of phonons with small wave number k has a macroscopic length (ballistic transport), which follows essentially from the smallness of the rate with which these phonons are scattered, together with the fact that they travel with finite velocity. In fact, in [3] it is proved that this system exhibits anomalous conductance.

To analyse the Boltzmann equation (1.1) is to exploit the fact that it can be reinterpreted as the forward equation for the probability density of a Markov process $(K(t), Y(t))$ on $(\mathbb{T} \times \mathbb{R})$. Here $K(t)$ is a reversible jump process with generator R and $Y(t)$ is an additive functional of K, given as $Y(t) = \int_0^t ds\ v(K(s))$. In a phononic picture, the process $Y(t)$ describes the trajectory of a phonon. For system with diffusive energy spreading, one expects that the law of the rescaled process $Y(Nt)/\sqrt{N}$ converges to the solution of heat equation. On the other hand, we expect that this is not true for systems exhibiting ballistic transport.

To understand heuristically what is to be expected, it is convenient to introduce a discrete time Markov chain, X_i, that records the sequence of values assumed by the continuous time chain $K(t)$ and the holding times $\tau(X_i)$, i.e. the time the chain $K(t)$ remains in the state visited in step number i. Then the process $Y(t)$, at the time of the n-th jump of $K(t)$, can be written as $S_n = \sum_{i=1}^n \tau(X_i)v(X_i)$. We will see later that in our case, due to the fact that the transition kernel behaves as k'^2 for small k' with respect to the second argument, the stationary distribution of the chain X_i is of the form $\pi(dk) = \phi(k)dk$ where $\phi(k) \sim k^2$, for $k << 1$. On the other hand, the distribution of the holding time $\tau(k)$ is of the form $\mathbb{P}(\tau(k) > s) \sim e^{-sk^2}$. Hence, in the stationary distribution, we have that

$$\mathbb{P}(\tau(X_i) > s) \sim \int \pi(dk) e^{-sk^2} \sim s^{-3/2}. \tag{1.2}$$

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Thus, since v is antisymmetric and $|v(0)| = 1$, we expect $\tau(X_i)v(X_i)$ to be in the domain of attraction of a stable law of index $\alpha = 3/2$. Then, if the $\tau(X_i)v(X_i)$ were independent random variables, $n^{-1/\alpha}S_n$ would converge to the corresponding stable law and $n^{-1/\alpha}S_{[ns]}$ to a stable Lévy process.

The corresponding problems of the convergence of dependent random variables with heavy tail distributions to a stable process has been studied extensively in the literature. A general, and very efficient, approach is explained e.g. in Durrett and Resnick [9]: first one uses methods from the theory of extremes of dependent random variables to study the convergence of the point process of scaled summands to a Poisson point process. Then one writes the sum as an integral with respect to the point process and uses a moment method to show that this integral converges to the corresponding integral with respect to the Poisson point process, which is a Lévy process.

The conditions needed to establish such a result are mainly required to assure the convergence to the Poisson process. Durrett and Resnick [9] express these in terms of the asymptotic behaviour of the conditional distribution. Davis [6] considers stationary processes and the corresponding well known mixing conditions (see [18]), while in [16] and [8] a non stationary generalisation is done, requiring some mixing conditions. Further results are to be found in e.g. [7, 14, 15].

In this paper, we will follow the general strategy outlined above and use a criterion for Poisson convergence that can be found in in [5] and which has the advantage of being rather easily verified for an ergodic Markov chain. It requires asymptotic factorisation conditions for probabilities which hold on average. This will allow us to prove convergence first of the sum $n^{-1/\alpha}S_{[nt]}$ to an $3/2$-stable Lévy process in the J_1-Skorokhod topology. Since $S_n = Y(T_n)$, where $T_n = \sum_{i=1}^n \tau(X_i)$, and since $Y(t)$ is the piecewise linear interpolation of this function at the random sequence to times T_n, once we will have shown that $T_{[nt]}/n$ converges to t, we will obtain that the rescaled process $n^{-1/\alpha}Y(nt)$ is the function given by converges to the same limit process in the Skorokhod M_1-topology, which is the appropriate topology for the convergence of continuous process to a process with càdlàg paths.

Let us note that Jara et al [17] prove similar convergence results for additive functionals of Markov processes using Martingale and coupling methods and apply their results to the same model we consider here. Their methods do, however, only yield convergence of finite dimensional marginals. The methods we use here seem more straightforward and direct, and give stronger results.

Finally, using convergence of $n^{-1/\alpha}Y(nt)$ to an α-stable Lévy process, $\alpha \in (1, 2)$, one can prove that the rescaled solution of the linear Boltzmann equation (1.1) converges to the solution of the following fractional diffusion equation

$$\partial_t \bar{W}(t,u) = -(-\partial_u^2)^{\alpha/2}\bar{W}(t,u).$$

We refer to [17] for the proof. Convergence of the rescaled solution of a linear Boltzmann equation to the solution of a fractional diffusion equation was independently proved by Mellet et al [22], with purely analytical techniques.

2. The model

We consider the process $(K(t), Y(t))$ described by equation (1.1). Denoting by $\mathbb{T}$ the one-dimensional torus, we choose $v : \mathbb{T} \to \mathbb{R}$ and $R : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ as in [3], namely

$$v(k) = \frac{\sin(2\pi k)}{|\sin(\pi k)|}, \tag{2.1}$$

$$R(k,k') = \frac{4}{3}\left[2\sin^2(2\pi k)\sin^2(\pi k') + 2\sin^2(2\pi k')\sin^2(\pi k) - \sin^2(2\pi k)\sin^2(2\pi k')\right]. \tag{2.2}$$

Observe that the rate kernel R is symmetric, not negative and it is equal to zero only if $k = 0$ or $k' = 0$. We remark that despite the special case we consider, results depend essentially on the behaviour of v and $\mathbb{R}$ for small k, i.e. for $k \ll 1$ $|v(k)| \sim 1$ and $R(k,k') \sim k^{-2}$, $\forall k' \in \mathbb{T}/\{0\}$, and $R(k,k') \sim k'^{-2}$ for $k \ll 1$, $\forall k \in \mathbb{T}/\{0\}$. The jump process $K(t)$ is determined by the generator

$$Lf(k) = \phi(k)\int_{\mathbb{T}} dk'\ R(k,k')[f(k') - f(k)], \quad \forall f \in C(\mathbb{T}) \tag{2.3}$$

where $\phi : \mathbb{T} \to \mathbb{R}_+$ is given by

$$\phi(k) = \int_{\mathbb{T}} dk'\ R(k,k') = \frac{4}{3}\sin^2(\pi k)(1 + 2\cos^2(\pi k)). \tag{2.4}$$

The process $Y(t)$ is an additive functional of $K(t)$, defined as $Y(t) = \int_0^t ds\ v(K(s))$.

Disregarding the time, the stochastic sequence $\{X_i\}_{i\geq 0}$ of states visited by $K(t)$ is a Markov chain with value in $\mathbb{T}$, with a probability kernel P concentrated on $\mathbb{T}$ given by $P(k, dk') = \phi(k)^{-1} R(k, k')dk'$. We denote with P^m, $m \geq 2$, the m-th convolution integral of P. Since the probability kernel P is regular and strictly positive and defined on a compact set, it is ergodic, i.e. therer exists a strictly positive probability distribution π such that $\forall k \in \mathbb{T}$, $P^n(k, \cdot) \to \pi(\cdot)$, weakly, as $n \uparrow \infty$. The stationary measure, π, is given by $\pi(dk) = \phi(k)dk$.

We define two functions of the Markov chain $\{X_i\}_{i\geq 0}$: the clock process, $T_n \in \mathbb{R}^+$, and the position process, $S_n \in \mathbb{R}$, by

$$T_n = \sum_{i=0}^{n-1} e_i [\phi(X_i)]^{-1}, \tag{2.5}$$

and

$$S_n = \sum_{i=0}^{n-1} e_i [\phi(X_i)]^{-1} v(X_i). \tag{2.6}$$

Here $\{e_i\}_{i\geq 0}$ are i.i.d. exponential random variables with parameter 1. The clock process, T_n, is the time of the n-the jump of the process $K(t)$. It is a sum of positive random variables with finite expectation, as one can easily check using the explicit form of the probability density (see Eq. (5.2) below). The position process, S_n, is the value of the position of $Y(t)$ at time T_n, i.e. $S_n = Y(T_n)$. It is a sum of real variables with zero mean and infinite variance. More precisely, for any $i \in \mathbb{N}$, for large λ

$$\mathbb{P}[|e_i \phi(X_i)^{-1} v(X_i)| > \lambda] \sim \lambda^{-3/2}, \tag{2.7}$$

Let T^{-1} denote the right-continuous inverse of T_n, i.e. let

$$T^{-1}(t) \equiv \inf\{n : T_n \geq t\}. \tag{2.8}$$

We can represent the original processes, $(K(t), Y(t))$, as follows:

$$\begin{aligned} K(t) =& X_{[T^{-1}(t)-1]} \\ Y(t) =& S_{[T^{-1}(t)-1]} + v(X_{[T^{-1}(t)-1]})(t - T_{[T^{-1}(t)-1]}). \end{aligned} \tag{2.9}$$

In particular, $Y(t)$ is the function defined by linear interpolation between its values S_n at the random points T_n (we take $S_0 = 0$).

3. Main results.

We assume that the initial distribution, μ, of the process X satisfies the condition

$$\int_T d\mu(k) k^{-2} < \infty, \tag{3.1}$$

which guarantees in particular that $\mathbb{E}_\mu\left[e_0 \phi(X_0)^{-1}\right] < \infty$.

We define the rescaled processes

$$T_N(\theta) = \frac{1}{N} T_{\lfloor N\theta \rfloor}, \quad S_N(\theta) = \frac{1}{N^{2/3}} S_{\lfloor N\theta \rfloor}, \quad T_N^{-1}(\theta) = \frac{1}{N} T^{-1}(N\theta), \tag{3.2}$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part of $\cdot$. Since T_n is a sum of positive variables with finite expectation, we expect that both $T_N(\theta)$ and $T_N^{-1}(\theta)$ converge in probability (and thus in distribution) to θ, in the topology of uniform convergence on compact intervals. This will be proven in Proposition 8.2.

On the other hand, S_n is a sum of centred random variables whose tail behaviour is given in (2.7). Thus we expect that the rescaled process S_N converges to a stable process with index $3/2$. This is the content of the following theorem.

Theorem 3.1. *Let S_N be the process defined in (3.2). Then for any $0 < \mathcal{T} < \infty$, the process $\{S_N(\theta)\}_{0\le\theta\le\mathcal{T}}$ converges to $\{V(\theta)\}_{0\le\theta\le\mathcal{T}}$, where V is a symmetric Lévy process stable with index $3/2$. Convergence is in distribution on the Skorokhod space of càdlàg functions equipped with the $J_1-topology$.*

Combining this theorem with Proposition 8.2, we will prove that $S_N \circ T_N^{-1}$ converges in distribution to V. This will imply our main theorem.

Theorem 3.2. *Let S_N, T_N^{-1} be the processes defined in (3.2). For every $0 < \mathcal{T} < \infty$, the process $\{S_N(T_N^{-1}(\theta))\}_{0\le\theta\le\mathcal{T}}$, converges to $\{V(\theta)\}_{0\le\theta\le\mathcal{T}}$, where V is a symmetric Lévy process stable with index $3/2$. Convergence is in distribution on the Skorokhod space of càdlàg functions equipped with the $J_1-topology$.*

Moreover, for every $N\ge 1$ the process $Y_N(t) = \frac{1}{N^{2/3}}\int_0^{Nt} ds\ v(K_s)$ is the function defined by linear interpolation between its values $S_N(\theta)$ at points $T_N(\theta)$, with $\theta\in[0, T_N^{-1}(\mathcal{T})]$. In particular, Y_N converges to V in distribution in the M_1-Skorokhod topology.

4. Sketch of the proof of Theorem 3.1

In this section we present the key steps of the proof of Theorem 1. The technical details will be given in Sections 5, 6, and 7. As we mentioned in the introduction, we follow the strategy of considering the sequence of the point processes associated to S_N. At the first step we define this sequence of point processes and we show that it converge to a Poisson point process. Then we prove that the limit process for S_N exists and it is a Lévy process stable with index $3/2$. Finally, we prove the tightness for the sequence S_N.

4.1. Point processes.

Define the real valued random variables

$$\psi_n \equiv \phi(X_n)^{-1} v(X_n), \quad n\in\mathbb{N}_0. \tag{4.1}$$

For some fixed $c>0$, we decompose $S_N(\theta)$ into two parts, $S_N(\theta) = S_N^>(\theta) + S_N^<(\theta)$, where

$$\begin{aligned} S_N^>(\theta) &= N^{-2/3}\sum_{n=0}^{[N\theta]-1} e_n\psi_n \mathbb{1}_{\{e_n|\psi_n|>cN^{2/3}\}} \\ S_N^<(\theta) &= N^{-2/3}\sum_{n=0}^{[N\theta]-1} e_n\psi_n \mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}. \end{aligned} \tag{4.2}$$

We will see later that $S_N^<(\theta)$ vanishes as $N\to\infty$ and then $c\to 0$. More precisely:

Lemma 4.1. *Let $S_N^<(\theta)$, $\theta\in[0,\mathcal{T}]$ be the process defined in (4.2). Then for every $\theta\in[0,\mathcal{T}]$*

$$\mathbb{E}\Big[\big|S_N^<(\theta)\big|^2\Big] \le C_0\theta\sqrt{c} + \mathcal{O}(N^{-4/3}), \tag{4.3}$$

where C_0 is a positive constant.

We will prove this lemma in the next section.
On the other hand, $S_N^>$ will be connected to two Poisson processes.
We split the sum $S_N^>(\theta)$ into two parts:

$$\begin{aligned} S_N^>(\theta) &= \frac{1}{N^{2/3}} \sum_{n=0}^{[N\theta]-1} e_n\psi_n \, \mathbb{1}_{\{e_n\psi_n > cN^{2/3}\}} \\ &+ \frac{1}{N^{2/3}} \sum_{n=0}^{[N\theta]-1} e_n\psi_n \, \mathbb{1}_{\{e_n\psi_n < -cN^{2/3}\}}. \end{aligned} \tag{4.4}$$

Defining the random variables

$$X_{i,N}^+ = \frac{e_i\psi_i}{N^{2/3}} \, \mathbb{1}_{\{\psi_i \geq 0\}}, \qquad X_{i,N}^- = \frac{e_i\psi_i}{N^{2/3}} \, \mathbb{1}_{\{\psi_i \leq 0\}}, \tag{4.5}$$

def:X+X-

with values in $\mathbb{R}_+$, $\mathbb{R}_-$, respectively, and the associated point processes $\mathcal{R}_N^+$ and $\mathcal{R}_N^-$,

$$\mathcal{R}_N^+ = \sum_{i\in\mathbb{N}_0} \delta_{i/N, X_{i,N}^+}, \quad \mathcal{R}_N^- = \sum_{i\in\mathbb{N}_0} \delta_{i/N, X_{i,N}^-}. \tag{4.6}$$

def:RN

we can rewrite $S_N^>(\theta)$ as

$$S_N^>(\theta) = \int_0^\theta \int_c^\infty x \, \mathcal{R}_N^+(ds, dx) + \int_0^\theta \int_{-\infty}^{-c} x \, \mathcal{R}_N^-(ds, dx) \tag{4.7}$$

split.2

The following Proposition states that the two point processes $\mathcal{R}_N^+$ and $\mathcal{R}_N^+$ converge to Poisson point processes.

prop:PPP

Proposition 4.2. *Let $\mathcal{R}_N^+$, $\mathcal{R}_N^-$ be the point processes defined in (4.6). Then the point process $\mathcal{R}_N^\pm$ converges in distribution to Poisson point processes $\mathcal{R}^\pm$ on $\mathbb{R}_+ \times \mathbb{R}_+$, resp. $\mathbb{R}_+ \times \mathbb{R}_-$, with intensity measures $dt \times d\nu^\pm(x) \equiv \frac{3}{2}a|x|^{-5/2}dx$, with $a > 0$ some explicite constant.*

We will prove this proposition in Section 6.

4.2. **Limit process for S_N.** We define a process V on $\mathbb{R}_+$ with values in $\mathbb{R}$ by

$$V(\theta) = \int_0^\theta \int_0^\infty \mathcal{R}^+(ds, dx)x + \int_0^\theta \int_{-\infty}^0 \mathcal{R}^-(ds, dx)x, \tag{4.8}$$

def:V

and

$$V_c^>(\theta) = \int_0^\theta \int_c^\infty \mathcal{R}^+(ds, dx)x + \int_0^\theta \int_{-\infty}^{-c} \mathcal{R}^-(ds, dx)x, \tag{4.9}$$

def:Vc

where $\mathcal{R}_+$, $\mathcal{R}_-$ are the Poisson point processes defined in Proposition 4.2.

levy-ito

Proposition 4.3. *Let be V, the process defined in (4.8). V is well defined and it is symmetric a Lévy process stable of index $3/2$, with Laplace functional $\Psi(\lambda) = C_0|\lambda|^{3/2}$, where $C_0 = a\int dy|y|^{-5/2}(1-\cos y)$.*

Moreover, $V_c \to V$ as $c \downarrow 0$, in the topology of uniform convergence on compact intervals.

Proof. We split $V(\theta)$ into two parts $V(\theta) = V_c^>(\theta) + V_c^<(\theta)$. Clearly, the two processes, if they exist, are independent. Since, for any compact interval, I, the total intensity of $I \times (c, \infty)$ equals

$$\int_I ds \int_c^\infty d\nu^+(x) = |I| a c^{-3/2} < \infty \tag{4.10}$$

the process $V_c^>(\theta)$ is a finite sum, almost surely.

Moreover, by direct computation and using the fact that $\mathbb{E}[V_c^<(\theta)] = 0$, we get

$$\mathbb{E}\left[|V_c^<(\theta)|^2\right] = \int_0^\theta ds \int_0^c d\nu^+(x) x^2 + \int_0^\theta ds \int_{-c}^0 d\nu^-(x) x^2 = 6\theta a c^{1/2}, \tag{4.11}$$

hence also $V^<(\theta)$ is almost surely finite, and tends to zero in probability, as $c \downarrow 0$.

Then $V(\theta)$ is almost surely finite, and since it is the sum of two independent processes which are right-continuous and have independents increments, it satisfies the hypothesis of a Lévy process, and it is full characterized by the one-dimensional distribution. This is uniquely determined by its characteristic exponent $\Psi : \mathbb{R} \to \mathbb{C}$ defined as

$$e^{-t\Psi(\lambda)} = \mathbb{E}\big[\exp\{i\lambda V(\theta)\}\big] \tag{4.12}$$

which by direct computation is given by

$$\Psi(\lambda) = \int_{-\infty}^{\infty} d\nu(x)\big(1 - e^{i\lambda x}\big), \tag{4.13}$$

with the Lévy measure ν defined on $\mathbb{R} \setminus \{0\}$ given by $d\nu(x) = a|x|^{-5/2} dx$. For every $\lambda \in \mathbb{R}$

$$\Psi(\lambda) = |\lambda|^{3/2} a \int_{\mathbb{R}} dy\, |y|^{-5/2}(1 - \cos y) = C_0 |\lambda|^{3/2}, \tag{4.14}$$

which is the characteristic exponent of a symmetric Lévy process stable with index 3/2. □

Proposition 4.3 together with Lemma 4.1 and Proposition 4.2 implies convergence of the finite dimensional distributions of S_N to V. Since V has stationary and independent increments, it is in fact enough to prove the convergence of the one-dimensional distributions.

Corollary 4.4. *For any $\theta \in \mathbb{R}_+$,*

$$S_N(\theta) \to V(\theta), \tag{4.15}$$

in distribution.

Proof. From the representation (4.6) and Proposition 4.2, and since the intensity of $[0, \theta] \times (c, \infty)$ is finite, it follows readily that

$$S_N^>(\theta) \to V_c(\theta), \quad \text{as } N \uparrow \infty, \tag{4.16}$$

in distribution. Moreover, $V_c(\theta) \to V(\theta)$, as $c \downarrow 0$, and

$$\lim_{c \downarrow 0} \limsup_{N \uparrow \infty} |S_N^<(\theta)| = 0, \tag{4.17}$$

in probability, by Lemma 4.1. This implies the assertion of the corollary. □

To conclude the proof of Theorem 1, we need to complement this corollary with the proof of tightness of the sequence S_N. This will be postponed to Section 6.

5. Moment estimations.

In this section we collect and prove some useful moment estimates.

We start with some preliminary results on the transition probability density for the Markov chain $\{X_n\}$. They are given by

$$\begin{aligned} p(k,k') &= \phi(k)^{-1}R(k,k') \\ &= 8\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}\sin^4(\pi k') + 8\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\sin^2(\pi k')\cos^2(\pi k'). \end{aligned} \tag{5.1}$$

We denote by $p^m(k,k')$ the m-step transition densities, i.e.

$$p^2(k,k') = p(k,\cdot)\circ p(\cdot,k') \equiv \int_{\mathbb{T}} dk_1\ p(k,k_1)p(k_1,k')$$

and in the same way, for every $m \geq 1$

$$p^{m+1}(k,k') = p^m(k,\cdot)\circ p(\cdot,k') = p(k,\cdot)\circ p^m(\cdot,k').$$

Observe that $P^m(k,dk') = p^m(k,k')dk'$, where P^m is the m-th convolution integral of the probability kernel P.

In the next proposition we give an explicit formula for p^m.

Proposition 5.1. *For every $m \geq 1$, $p^m(k,k')$ has the following form*

$$\begin{aligned} p^m(k,k') =& 8\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}\left[a_m\sin^4(\pi k') + b_m\sin^2(\pi k')\cos^2(\pi k')\right] \\ &+8\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\left[c_m\sin^4(\pi k') + d_m\sin^2(\pi k')\cos^2(\pi k')\right] \end{aligned} \tag{5.2}$$

form:prob

where

$$\begin{cases} a_1 = d_1 = 1, \quad b_1 = c_1 = 0, & m = 1 \\ 0 < a_m, b_m, c_m, d_m < 1, & m \geq 2 \end{cases} \tag{5.3}$$

coeff

Proof. By direct computation,

$$\begin{aligned} p^2(k,k') =& 8\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}\left[a_2\sin^4(\pi k') + b_2\sin^2(\pi k')\cos^2(\pi k')\right] \\ &+8\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\left[c_2\sin^4(\pi k') + d_2\sin^2(\pi k')\cos^2(\pi k')\right] \end{aligned} \tag{5.4}$$

with a_2, b_2, c_2, d_2 positive, $a_2 = d_2$. In the same way for $m \geq 2$ we find expression form:prob (5.2), where the coefficients a_m, b_m, c_m, d_m are given by the following recursive formula

$$\begin{pmatrix} a_{m+1} & b_{m+1} \\ c_{m+1} & d_{m+1} \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}, \quad m \geq 2 \tag{5.5}$$

In particular a_m, b_m, c_m, d_m are positive for every $m \geq 2$. Moreover, by condition $\int_{\mathbb{T}} dk'\ p^m(k,k') = 1$, $\forall m \geq 1$, we get the relations

$$3a_m + b_m = 3, \qquad 3c_m + d_m = 1$$

which says that the coefficients are uniformly bounded, i.e. $a_m, b_m, c_m, d_m < 1$ for every $m \geq 1$. □

We now give the proof of Lemma proof:S- 4.1.

Proof. (of Lemma 4.1). Let us write

$$\begin{aligned}\mathbb{E}\Big[\big|S_N^{<}(\theta)\big|^2\Big] &= N^{-4/3}\sum_{n=0}^{\lfloor N\theta\rfloor-1}\mathbb{E}\Big[\big|e_n\psi_n\big|^2\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\Big]\\ &+ 2N^{-4/3}\sum_{n=1}^{\lfloor N\theta\rfloor-1}\sum_{0\le m<n}\mathbb{E}\Big[e_n\psi_n e_m\psi_m\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\mathbb{1}_{\{e_m|\psi_m|\le cN^{2/3}\}}\Big].\end{aligned}\tag{5.6}$$

Let us focus on the first sum on the right hand side. For every $n\ge 1$ we have

$$\begin{aligned}&\mathbb{E}\Big[\big|e_n\psi_n\big|^2\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\Big]\\ &= \int_0^\infty dz\; e^{-z}z^2\int_{\mathbb{T}}d\mu(k_0)\int_{\mathbb{T}}dk\; p^n(k_0,k)|\psi(k)|^2\mathbb{1}_{\{z|\psi(k)|\le cN^{2/3}\}},\end{aligned}\tag{5.7}$$

where

$$|\psi(k)| = |\phi(k)^{-1}v(k)| = \frac{3|\cos(\pi k)|}{2\sin^2(\pi k)(1+2\cos^2(\pi k))}.\tag{5.8}$$

Using the explicit form of p^n given in (5.2), an elementary computation reveals that

$$\mathbb{E}\Big[\big|e_n\psi_n\big|^2\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\Big]\le C_0\sqrt{c}N^{1/3},\tag{5.9}$$

with $C_0<\infty$ a positive constant. Then

$$N^{-4/3}\sum_{n=0}^{\lfloor N\theta\rfloor-1}\mathbb{E}\Big[\big|e_n\psi_n\big|^2\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\Big]\le C_0\theta\sqrt{c}+\mathcal{O}(N^{-4/3}).\tag{5.10}$$

The second sum on the r.h.s. of Eq. (5.6) is in fact equal to zero. Namely, for all $n>m\ge 1$ we have

$$\begin{aligned}&\mathbb{E}\Big[e_n\psi_n e_m\psi_m\mathbb{1}_{\{e_n|\psi_n|\le cN^{2/3}\}}\mathbb{1}_{\{e_m|\psi_m|\le cN^{2/3}\}}\Big]\\ &= \int_0^\infty dz\; ze^{-z}\int_0^\infty du\; ue^{-u}\int_{\mathbb{T}}d\mu(k_0)\int_{\mathbb{T}}dk\; p^m(k_0,k)\psi(k)\mathbb{1}_{\{|\psi(k)|\le cN^{2/3}z^{-1}\}}\\ &\times\int_{\mathbb{T}}dk\; p^{(n-m)}(k,k')\psi(k')\mathbb{1}_{\{|\psi(k')|\le cN^{2/3}u^{-1}\}} = 0,\end{aligned}\tag{5.11}$$

since $\psi(k)=-\psi(-k)$ and $p^\ell(k,k')=p^\ell(k,-k')$ $\forall k,k'\in\mathbb{T}$, $\forall\ell\ge 1$. This concludes the proof of Lemma 4.1. □

The following related lemma will be needed in Section 8.

Lemma 5.2. *There exists $C<\infty$, such that for $n\ge 1$ and for all N large enough,*

$$\mathbb{E}\left[e_n|\psi_n|\mathbb{1}_{\{e_i|\psi_n|>N^{2/3}\}}\right]\le CN^{-1/3}.\tag{5.12}$$

Proof. For every $n\ge 1$ we have

$$\mathbb{E}\left[e_n|\psi_n|\mathbb{1}_{\{e_n|\psi_n|>N^{2/3}\}}\right] = \int_0^\infty dz\; ze^{-z}\int_{\mathbb{T}}d\mu(k_0)\int_{\mathbb{T}}dk\; p^n(k_0,k)\big|\psi(k)\big|\mathbb{1}_{\{z|\psi(k)|>N^{2/3}\}},\tag{5.13}$$

and using the explicit formula (5.2) for p^n one easily finds that this expression is bounded by $C_0N^{-1/3}+\mathcal{O}(N^{-1})$, with $C_0>0$. □

ESS **Lemma 5.3.** *Let $\tilde S_N^{<}(\theta)$, $\theta\in[0,\mathcal{T}]$ be the process defined in* def:sti *(7.12). Then for every $0\le r\le s\le t\le\mathcal{T}$*

$$\mathbb{E}\Big[|\tilde S_N^{<}(s)-\tilde S_N^{<}(r)|^2|\tilde S_N^{<}(t)-\tilde S_N^{<}(s)|^2\Big]\le A_0(t-r)^2, \tag{5.14}$$

with A_0 positive constant.

Proof. For every $0\le r<s<t\le\mathcal{T}$ we have:

$$\begin{aligned}&|\tilde S_N^{<}(s)-\tilde S_N^{<}(r)|^2|\tilde S_N^{<}(t)-\tilde S_N^{<}(s)|^2\\&=\frac{1}{N^{8/3}}\sum_{i=\lfloor Nr\rfloor}^{\lfloor Ns\rfloor-1}\sum_{j=\lfloor Nr\rfloor}^{\lfloor Ns\rfloor-1}e_i\psi_i\mathbb{1}_{\{e_i|\psi_i|\le N^{2/3}\}}e_j\psi_j\mathbb{1}_{\{e_j|\psi_j|\le N^{2/3}\}}\\&\times\sum_{h=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor-1}\sum_{l=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor-1}e_h\psi_h\mathbb{1}_{\{e_h|\psi_h|\le N^{2/3}\}}e_l\psi_l\mathbb{1}_{\{e_l|\psi_l|\le N^{2/3}\}}.\end{aligned}\tag{5.15}$$

By simm (5.11), the only terms with expectation value different from zero are

$$\frac{1}{N^{8/3}}\sum_{i=\lfloor Nr\rfloor}^{\lfloor Ns\rfloor-1}\left|e_i\psi_i\right|^2\mathbb{1}_{\{e_i|\psi_i|\le N^{2/3}\}}\sum_{h=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor-1}\left|e_h\psi_h\right|^2\mathbb{1}_{\{e_h|\psi_h|\le N^{2/3}\}}. \tag{5.16}$$

For every $h>i$ we have

$$\begin{aligned}&\mathbb{E}\Big[\left|e_i\psi_i\right|^2\mathbb{1}_{\{e_i|\psi_i|\le N^{2/3}\}}\left|e_h\psi_h\right|^2\mathbb{1}_{\{e_h|\psi_h|\le N^{2/3}\}}\Big]\\&=\int_0^\infty dz\ z^2e^{-z}\int_0^\infty du\ u^2e^{-u}\int_{\mathbb{T}}d\mu(k_0)\\&\times\int_{\mathbb{T}}dk\ p^i(k_0,k)|\psi(k)|^2\mathbb{1}_{\{z|\psi(k)|\le N^{2/3}\}}\int_{\mathbb{T}}dk'\ p^{(h-i)}(k,k')|\psi(k')|^2\mathbb{1}_{\{u|\psi(k')|\le N^{2/3}\}}.\end{aligned}\tag{5.17}$$

Using form:prob (5.2), we find that for all $k\in\mathbb{T}$:

$$\begin{aligned}&\int_{\mathbb{T}}dk'\ p^{(h-i)}(k,k')|\psi(k')|^2\mathbb{1}_{\{u|\psi(k')|\le N^{2/3}\}}\\&\le\frac{1}{\sqrt{u}}N^{1/3}\left(B\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}+D\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\right)+\mathcal{O}(1),\end{aligned}\tag{5.18}$$

where B,D are finite constants. Then $\forall k_0\in\mathbb{T}$

$$\begin{aligned}&\int_{\mathbb{T}}dk\ p^i(k_0,k)|\psi(k)|^2\mathbb{1}_{\{z|\psi(k)|\le N^{2/3}\}}\int_{\mathbb{T}}dk'\ p^{(h-i)}(k,k')|\psi(k')|^2\mathbb{1}_{\{u|\psi(k')|\le N^{2/3}\}}\\&\le\frac{1}{\sqrt{z}}\frac{1}{\sqrt{u}}N^{2/3}\left(B_1\frac{\cos^2(\pi k_0)}{1+2\cos^2(\pi k_0)}+D_1\frac{\sin^2(\pi k_0)}{1+2\cos^2(\pi k_0)}\right)+\mathcal{O}(N^{2/3}),\end{aligned}\tag{5.19}$$

with $B_1,D_1<\infty$. Finally we get

$$\begin{aligned}&\mathbb{E}\Big[|\tilde S_N^{<}(s)-\tilde S_N^{<}(r)|^2|\tilde S_N^{<}(t)-\tilde S_N^{<}(s)|^2\Big]\\&\le\frac{1}{N^{8/3}}\sum_{i=\lfloor Nr\rfloor}^{\lfloor Ns\rfloor-1}\sum_{h=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor-1}A_0\ N^{2/3}\\&\le\frac{1}{N^{8/3}}N^2(s-r)(t-s)A_0\ N^{2/3}\le A_0(t-r)^2,\end{aligned}\tag{5.20}$$

which is the assertion of the lemma. ☐

6. Proof of Proposition 4.2

This section is devoted to verify conditions of the Theorem 2.1 in [5], which guarantees the convergence of the point processes defined in (4.6) to Poisson point processes. We recall the statement of Theorem 2.1 of [5].

Theorem 6.1. *[Theorem 2.1 in [5].] Denote by $\sum_{\alpha_N(\ell)}$ the sum over all the ordered sequences of different indices $(i_1,..,i_\ell)$ with values in $\{1,..,N\}$. Let be $\{Z^+_{i,N}\}$ (resp. $\{Z^-_{i,N}\}$), $N\geq 1$, $1\leq i\leq N$, an array of random variables with values in $\mathbb{R}_+$ (resp. in $\mathbb{R}_-$). Assume that for every $\ell>0$ and all sets of positive constants $(\tau_1,..,\tau_\ell)$*

$$\sum_{\alpha_N(\ell)}\mathbb{P}\Big[|Z^\pm_{i_1,N}|>\tau_1,...,|Z^\pm_{i_\ell,N}|>\tau_\ell)\Big]\to a^\ell\frac{1}{\tau_1^{3/2}}...\frac{1}{\tau_\ell^{3/2}},\quad \text{as } N\to\infty, \tag{6.1}$$

for some $a>0$. Then the point process $\sum_{i=0}^{N-1}\delta_{i/N,Z^\pm_{i,N}}$ converges in distribution to a Poisson point process $\mathcal{R}^\pm$ on $[0,T]\times\mathbb{R}_\pm$ with intensity measure $dt\times d\nu^\pm(x)$, where $d\nu^\pm(x)=\frac{3}{2}a|x|^{-5/2}dx$.

Our goal is to verify these conditions for the random variables $\{X^+_{i,N}\}$, $\{X^-_{i,N}\}$ defined in (4.5). In order to do it, we need the following Lemma.

Lemma 6.2. *For every $\ell>0$, for every sequence of different indices $i_1,..,i_\ell\in\{1,..,N\}$ and all set of positive constants $(\tau_1,..,\tau_\ell)$ the following statements hold:*

(i) *if $(i_{j+1}-i_j)\geq 2$ for every $j=1,..,\ell-1$, then*

$$\lim_{N\to\infty} N^\ell\tau_1^{3/2}...\tau_\ell^{3/2}\,\mathbb{P}\Big[e_1\big|\psi_{i_1}\big|>N^{2/3}\tau_1,...,e_\ell\big|\psi_{i_\ell}\big|>N^{2/3}\tau_\ell\Big]=\tilde{C}_\ell, \tag{6.2}$$

with $\tilde{C}_\ell>0$.

(ii) *If $(i_{j+1}-i_j)=1$ for some $j=1,..,\ell-1$, then*

$$\lim_{N\to\infty} N^\ell\tau_1^{3/2}...\tau_\ell^{3/2}\,\mathbb{P}\Big[e_1\big|\psi_{i_1}\big|>N^{2/3}\tau_1,...,e_\ell\big|\psi_{i_\ell}\big|>N^{2/3}\tau_\ell\Big]=0. \tag{6.3}$$

Proof. Let us consider

$$\int_0^\infty dz\,e^{-z}\int_{\mathbb{T}}dk_1\,p^m(k,k')\mathbb{1}_{\{|\psi(k')|>N^{2/3}\tau z^{-1}\}}, \tag{6.4}$$

with $m\geq 1$. Using (5.2) and the explicit expression for ψ (5.8), one easily finds that for $m\geq 2$ this quantity is bounded by

$$\left[\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}+\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\right]\left(C_1\frac{1}{N}\frac{1}{\tau^{3/2}}+\mathcal{O}(N^{-5/3})\right), \tag{6.5}$$

with $C_1>0$. On the other hand, for $m=1$, we get

$$\begin{aligned}&\int_0^\infty dz\,e^{-z}\int_{\mathbb{T}}dk'\,p(k,k')\mathbb{1}_{\{|\psi(k')|>N^{2/3}\tau z^{-1}\}}\\&\leq\left[\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}+\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\right]C_0\frac{1}{N^{5/3}}\frac{1}{\tau^{5/2}},\end{aligned}\tag{6.6}$$

with $C_0>0$.

Now let's consider

$$\begin{aligned}&\int_0^\infty du\ e^{-u}\int_{\mathbb{T}} dk_1\ p^n(k,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 u^{-1}\}}\int_0^\infty dz\ e^{-z}\\ &\times\int_{\mathbb{T}} dk'\ p^m(k_1,k')\mathbb{1}_{\{|\psi(k')|>N^{2/3}\tau_2 z^{-1}\}}.\end{aligned}\tag{6.7}$$

By (6.5), (6.6) we compute that for $n, m \geq 2$

$$\begin{aligned}&\int_0^\infty du\ e^{-u}\int_{\mathbb{T}} dk_1\ p^n(k,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 u^{-1}\}}\int_0^\infty dz\ e^{-z}\\ &\quad\times\int_{\mathbb{T}} dk'\ p^m(k_1,k')\mathbb{1}_{\{|\psi(k')|>N^{2/3}\tau_2 z^{-1}\}}\\ &\leq\left[\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}+\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\right]\left(C_2\frac{1}{N^2}\frac{1}{\tau_1^{3/2}}\frac{1}{\tau_2^{3/2}}+\mathcal{O}(N^{-8/3})\right),\end{aligned}\tag{6.8}$$

with $C_2 > 0$, while if $m = 1$ and/or $n = 1$, then (6.7) is of order $N^{-8/3}$. By repeating this procedure, one finds that for every $n_1, .., n_\ell \geq 2$ the following inequality holds:

$$\begin{aligned}&\int_0^\infty dz_1\ e^{-z_1}\int_{\mathbb{T}} dk_1\ p^{n_1}(k,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 z_1^{-1}\}}\int_0^\infty dz_2\ e^{-z_2}\\ &\quad\times\int_{\mathbb{T}} dk_2\ p^{n_2}(k_1,k_2)\mathbb{1}_{\{|\psi(k_2)|>N^{2/3}\tau_2 z_2^{-1}\}}\int_0^\infty dz_3 e^{-z_3}\int_{\mathbb{T}}\dots\int_0^\infty dz_\ell e^{-z_\ell}\\ &\quad\times\int_{\mathbb{T}} dk_\ell\ p^{n_\ell}(k_{\ell-1},k_\ell)\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}}\\ &\leq\frac{1}{N^\ell}C_\ell\frac{1}{\tau_1^{3/2}}\cdots\frac{1}{\tau_\ell^{3/2}}\left[\frac{\cos^2(\pi k)}{1+2\cos^2(\pi k)}+\frac{\sin^2(\pi k)}{1+2\cos^2(\pi k)}\right]+o\left(\frac{1}{N^\ell}\right).\end{aligned}\tag{6.9}$$

where C_ℓ is a positive constant for $n_1, .., n_\ell \geq 2$. If $n_i = 1$ for some i, then $C_\ell = 0$.

We can choose, without loss of generality, $i_1 < i_2 < .. < i_\ell$, then we have

$$\begin{aligned}&N^\ell\ \mathbb{P}\Big[e_1|\psi_{i_1}|>N^{2/3}\tau_1,\dots,e_l|\psi_{i_\ell}|>N^{2/3}\tau_\ell\Big]\\ &=N^\ell\int_{\mathbb{T}} d\mu(k_0)\int_0^\infty dz_1\ e^{-z_1}\int_{\mathbb{T}} dk_1\ p^{i_1}(k_0,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 z_1^{-1}\}}\\ &\quad\times\int_0^\infty dz_2\ e^{-z_2}\int_{\mathbb{T}} dk_2\ p^{i_2-i_1}(k_1,k_2)\mathbb{1}_{\{|\psi(k_2)|>N^{2/3}\tau_2 z_2^{-1}\}}\int_0^\infty dz_3 e^{-z_3}\int_{\mathbb{T}}\dots\\ &\quad\times\int_0^\infty dz_\ell e^{-z_\ell}\int_{\mathbb{T}} dk_\ell\ p^{i_\ell-i_{\ell-1}}(k_{\ell-1},k_\ell)\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}}.\end{aligned}\tag{6.10}$$

The proof of the Lemma is just an application of (6.9). □

Recalling the definition of $X^+_{i,N}$, $X^-_{i,N}$ given in (4.5), by the symmetry of probability density we have

$$\begin{aligned}&\mathbb{P}\Big[\big|X^+_{i_1,N}\big|>\tau_1,\dots,\big|X^+_{i_\ell,N}\big|>\tau_\ell\Big]=\mathbb{P}\Big[\big|X^-_{i_1,N}\big|>\tau_1,\dots,\big|X^-_{i_\ell,N}\big|>\tau_\ell\Big]\\ &=\mathbb{P}\Big[e_1\big|\psi_{i_1}\big|>N^{2/3}\tau_1,\dots,e_l\big|\psi_{i_\ell}\big|>N^{2/3}\tau_\ell\Big].\end{aligned}\tag{6.11}$$

Let us denote by $\sum_{\beta_N(\ell)}$ the sum over all not ordered sequences of different indices $[i_1, .., i_\ell]$ with values in $\{1, .., N\}$. Then $\sum_{\alpha_N(\ell)} = \ell! \sum_{\beta_N(\ell)}$.

We choose $i_1 < i_2 < ... < i_\ell$ and by denoting with $m_1 = i_1$, $m_j = i_j - i_{j-1}$, $\forall j = 2, .., \ell$, we get

$$\begin{aligned} &\sum_{\alpha_N(\ell)} \mathbb{P}\Big[e_1\big|\psi_{i_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{i_\ell}\big| > N^{2/3}\tau_\ell\Big] \\ &= \ell! \sum_{\substack{m_1,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big]. \end{aligned} \tag{6.12}$$

Fixed $M \ll N$, we split the sum on m_1 in two parts:

$$\begin{aligned} &\sum_{\substack{m_1,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \\ &= \sum_{m_1=1}^{M} \sum_{\substack{m_2,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \\ &\quad + \sum_{m_1=M+1}^{N-\ell+1} \sum_{\substack{m_2,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \end{aligned} \tag{6.13}$$

Using Lemma 6.2, we find that the first sum on the r. h. s. of (6.13) is bounded by $\tilde{C}_\ell \tau_1^{-3/2}...\tau_\ell^{-3/2}(M/N)$, with $\tilde{C}_\ell > 0$. Let us consider the second sum on the r.h.s. of (6.13). We repeat this procedure, and we split the sum on m_2 in two parts. We get

$$\begin{aligned} &\sum_{m_1=M+1}^{N-\ell+1} \sum_{\substack{m_2,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \\ &= \sum_{m_1=M+1}^{N-\ell+1} \sum_{m_2=M+1}^{N-\ell+1} \sum_{\substack{m_3,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \\ &\quad + \mathcal{O}(M/N). \end{aligned} \tag{6.14}$$

By repeating this procedure for all the sums, finally we get

$$\begin{aligned} &\sum_{\substack{m_1,..,m_\ell=1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \\ &= \sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\leq N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] + \mathcal{O}(M/N). \end{aligned} \tag{6.15}$$

Now we show that in the last expression we can replace the probability $\mathbb{P}$ with the invariant measure π. We have

$$
\begin{aligned}
&\sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\le N}}^{N} \mathbb{P}\big[e_1\big|\psi_{m_1}\big|>N^{2/3}\tau_1,...,e_l\big|\psi_{(m_1+...+m_\ell)}\big|>N^{2/3}\tau_\ell\big]\\
&=\sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\le N}}^{N}\int_{\mathbb{T}} d\mu(k_0)\int_0^\infty dz_1\int_{\mathbb{T}} dk_1\; p^{m_1}(k_0,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 z_1^{-1}\}}\int_{\mathbb{T}}\cdots\\
&\quad\times\int_0^\infty dz_\ell\int_{\mathbb{T}} dk_\ell\;\phi(k_\ell)\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}}\\
&\quad+\sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\le N}}^{N}\int_{\mathbb{T}} d\mu(k_0)\int_0^\infty dz_1\int_{\mathbb{T}} dk_1\; p^{m_1}(k_0,k_1)\mathbb{1}_{\{|\psi(k_1)|>N^{2/3}\tau_1 z_1^{-1}\}}\int_{\mathbb{T}}\cdots\\
&\quad\times\int_0^\infty dz_\ell\int_{\mathbb{T}} dk_\ell\;[p^{m_\ell}(k_{\ell-1},k_\ell)-\phi(k_\ell)]\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}}.
\end{aligned}
\tag{6.16}
$$

Let us consider the second sum on the r.h.s. of (6.16). We have

$$
\begin{aligned}
&\left|\int_0^\infty dz_\ell\int_{\mathbb{T}} dk_\ell\;\big[p^{m_\ell}(k_{\ell-1},k_\ell)-\phi(k_\ell)\big]\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}}\right| \qquad (6.17)\\
&\le\sup_{k,k'\in\mathbb{T}}\Big\{\phi(k')^{-1}\big|p^{m_\ell}(k,k')-\phi(k')\big|\Big\}\int_0^\infty dz_\ell\int_{\mathbb{T}} dk_\ell\;\phi(k_\ell)\mathbb{1}_{\{|\psi(k_\ell)|>N^{2/3}\tau_\ell z_\ell^{-1}\}},
\end{aligned}
$$

which is finite since $p^m(k,k')\phi(k')^{-1}\in\mathbb{C}^\infty(\mathbb{T}\times\mathbb{T})$ for all $m\ge 1$. Moreover, since by direct computation

$$
\int_0^\infty dz\;\int_{\mathbb{T}} dk\;\phi(k)\mathbb{1}_{\{|\psi(k)|>N^{2/3}\tau z^{-1}\}}\le C_0\frac{1}{N}\frac{1}{\tau^{3/2}},
$$

with $C_0>0$, using Lemma 6.2, we find that the second sum in (6.16) for large N is bounded from above by

$$
\begin{aligned}
&\frac{(N-\ell-M)^\ell}{N^\ell}C_\ell\frac{1}{\tau_1^{3/2}}\cdots\frac{1}{\tau_\ell^{3/2}}\sup_{m\ge M+1}\sup_{k,k'\in\mathbb{T}}\Big\{\phi(k')^{-1}\big|p^m(k,k')-\phi(k')\big|\Big\}\\
&\le\tilde{C}_\ell\frac{1}{\tau_1^{3/2}}\cdots\frac{1}{\tau_\ell^{3/2}}\sup_{m\ge M+1}\sup_{k,k'\in\mathbb{T}}\Big\{\phi(k')^{-1}\big|p^m(k,k')-\phi(k')\big|\Big\}.
\end{aligned}
\tag{6.18}
$$

Now we consider the first sum of (6.16). By repeating ℓ-times this procedure, we can replace the transition probability densities $p_i^m(k,k')$, $i=1,..,\ell-1$,with the invariant density $\phi(k')$. This gives an error $\mathcal{E}_N(\ell,M,\tau_1,..,\tau_\ell)$, which satisfies the inequality

$$
\big|\mathcal{E}_N(\ell,M,\tau_1,..,\tau_\ell)\big|\le C_\ell\frac{1}{\tau_1^{3/2}}\cdots\frac{1}{\tau_\ell^{3/2}}\sup_{m\ge M+1}\sup_{k,k'\in\mathbb{T}}\Big\{\phi(k')^{-1}\big|p^m(k,k')-\phi(k')\big|\Big\}.
\tag{6.19}
$$

By ergodicity, for every $\ell\ge 1$ and all sets of constants $\tau_i>0$, $i=1,..,\ell$

$$
\lim_{M\to\infty}\big|\mathcal{E}_N(\ell,M,\tau_1,..,\tau_\ell)\big|=0,\quad \forall N\in\mathbb{N}.
\tag{6.20}
$$

Then for large N

$$\begin{aligned}
&\sum_{\substack{m_1,..,m_\ell=1\\ m_1+..+m_\ell\le N}}^{N} \mathbb{P}\Big[e_1\big|\psi_{m_1}\big| > N^{2/3}\tau_1, ..., e_l\big|\psi_{(m_1+...+m_\ell)}\big| > N^{2/3}\tau_\ell\Big] \qquad (6.21)\\
&= \sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\le N}}^{N} \Pi_{i=1}^{\ell}\left(\int_0^\infty dz\ \int_{\mathbb{T}} dk\ \phi(k)\mathbb{1}_{\{|\psi(k)|>N^{2/3}\tau_i z^{-1}\}}\right)\\
&\quad +\mathcal{E}_N(\ell, M, \tau_1, .., \tau_\ell) + \mathcal{O}(M/N)\\
&= \sum_{\substack{m_1,..,m_\ell=M+1\\ m_1+..+m_\ell\le N}}^{N} \left[\frac{a^\ell}{N^\ell}\frac{1}{\tau_1^{3/2}}\cdots\frac{1}{\tau_\ell^{3/2}} + \mathcal{O}(N^{-5/3})\right] + \mathcal{E}_N(\ell, M, \tau_1, .., \tau_\ell) + \mathcal{O}(M/N).
\end{aligned}$$

We take the limit $N, M \to \infty$ with $M/N \to 0$. The proof of the proposition follows by (6.20) and by relation

$$\lim_{N\to\infty} \sum_{\substack{m_1,..,m_\ell=1\\ m_1+..+m_\ell\le N}}^{N} N^{-\ell} = 1/\ell!$$

7. Tightness.

In this section we conclude the proof of Theorem 1 by showing tightness of the sequence S_N. This is relatively easy due to the strong convergence properties stemming from the weak convergence of the point processes $\mathcal{R}_N$.

As criterion for tightness in the J_1 topology we use slight variant of Theorem 13.2 (with condition 13.5 replaced by 13.8) from [4].

We define the modulo of continuity on D

$$w_f(\delta) = \sup\{(|f(t) - f(t_1)| \wedge |f(t_2) - f(t)|) : t_1 \le t \le t_2, t_2 - t_1 \le \delta\}. \qquad (7.1)$$

The sequence $\{P_n\}$ of probability measures on $(D, \mathcal{D})$ is tight in the J_1-topology if and only if

(i) For each positive ϵ there exist τ such that

$$P_n[f : \|f\|_\infty > \tau] \le \epsilon, \qquad n \ge 1. \qquad (7.2)$$

(ii) For each $\epsilon > 0$ and $\eta > 0$, there exist $\delta > 0$, and a integer n_0 such that $\forall n \ge n_0$

$$P_n[f : w_f(\delta) \ge \eta] \le \epsilon, \qquad n \ge n_0, \qquad (7.3)$$

and

$$P_n[f : w_f(\delta) \ge \eta] \le \epsilon, \qquad (7.4)$$

$$P_n[f : |f(\delta) - f(0)| \ge \eta] \le \epsilon, \qquad (7.5)$$

$$P_n[f : |f(\mathcal{T}) - f(\mathcal{T} - \delta)| \ge \eta] \le \epsilon. \qquad (7.6)$$

In order to verify these conditions, we start with some definitions and preliminary results. For every $0 \le r \le s \le t \le \mathcal{T}$, let us denote by

$$m_{rst} = \min\Big(\big|S_N(s) - S_N(r)\big|, \big|S_N(t) - S_N(s)\big|\Big) \qquad (7.7)$$

and

$$L_{\mathcal{T}} = \sup_{0\le r\le s\le t\le \mathcal{T}} m_{rst} \tag{7.8}$$

The following inequalities holds (see [4], (10.4), (10.6)):

$$\sup_{\theta\in[0,\mathcal{T}]} |S_N(\theta)| \le L_{\mathcal{T}} + |S_N(\mathcal{T})|, \tag{7.9}$$

$$\sup_{\theta\in[0,\mathcal{T}]} |S_N(\theta)| \le 3L_{\mathcal{T}} + \max_{0\le i\le [N\mathcal{T}]-1} N^{-2/3} e_i|\psi_i|. \tag{7.10}$$

Fix $N \ge 1$. We have

$$\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} S_N(\theta) > \tau\right] \le \mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \tilde{S}_N^{<}(\theta) > \tau/2\right] + \mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \tilde{S}_N^{>}(\theta) > \tau/2\right], \tag{7.11}$$

where

$$\begin{aligned} \tilde{S}_N^{<}(\theta) &= \frac{1}{N^{2/3}} \sum_{i=0}^{[N\theta]-1} e_i\psi_i \mathbb{1}_{\{e_i|\psi_i|<N^{2/3}\}} \\ \tilde{S}_N^{>}(\theta) &= \frac{1}{N^{2/3}} \sum_{i=0}^{[N\theta]-1} e_i\psi_i \mathbb{1}_{\{e_i|\psi_i|>N^{2/3}\}}. \end{aligned} \tag{7.12}$$

Let us consider inequality (7.9) with $\tilde{S}_N$ replaced by $\tilde{S}_N^{<}$. We have

$$\mathbb{P}\left[\sup_{\theta\in[0,T]} \tilde{S}_N^{<}(\theta) > \tau/2\right] \le \mathbb{P}\Big[L_T^{<} > \tau/4\Big] + \mathbb{P}\Big[\big|\tilde{S}_N^{<}(T)\big| > \tau/4\Big]. \tag{7.13}$$

By direct computation, for all $0 \le r \le s \le t \le \mathcal{T}$ we have

$$\begin{aligned} &\mathbb{P}\Big[m_{rst}^{<} \ge \tau/4\Big] \le \mathbb{P}\Big[\big|\tilde{S}_N^{<}(s) - \tilde{S}_N^{<}(r)\big|\big|\tilde{S}_N^{<}(t) - \tilde{S}_N^{<}(s)\big| \ge \tau^2/16\Big] \\ &\le \left(\frac{16}{\tau^2}\right)^2 \mathbb{E}\Big[\big|\tilde{S}_N^{<}(s) - \tilde{S}_N^{<}(r)\big|^2\big|\tilde{S}_N^{<}(t) - \tilde{S}_N^{<}(s)\big|^2\Big] \le \frac{A}{\tau^4}(t-r)^2, \end{aligned} \tag{7.14}$$

with $A > 0$. The last inequality is proved in the next section (Lemma 5.3). Then we can use Theorem 10.3 of [4] to get

$$\mathbb{P}\Big[L_T^{<} \ge \tau/4\Big] \le \frac{B}{\tau^4}\mathcal{T}^2, \tag{7.15}$$

where B is a positive constant. Moreover, by inequality (4.3) with $c = 1$, $\theta = \mathcal{T}$ we get

$$\mathbb{P}\Big[\big|\tilde{S}_N^{<}(\mathcal{T})\big| \ge \tau/4\Big] \le \frac{16}{\tau^2}\mathbb{E}\Big[\big|\tilde{S}_N^{<}(\mathcal{T})\big|^2\Big] \le \frac{D}{\tau^2}\mathcal{T}, \tag{7.16}$$

with D positive constant. Finally by (7.13), (7.15) and (7.16), we get that for every $N \ge 1$

$$\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \tilde{S}_N^{<}(\theta) \ge \tau/2\right] \le \frac{B}{\tau^4}\mathcal{T}^2 + \frac{D}{\tau^2}\mathcal{T}. \tag{7.17}$$

In order to estimate $\mathbb{P}\big[\sup_{\theta\in[0,\mathcal{T}]} \tilde{S}_N^{>}(\theta) > \tau/2\big]$, we observe that

$$\sup_{\theta\in[0,\mathcal{T}]} \tilde{S}_N^{>}(\theta) \le \frac{1}{N^{2/3}} \sum_{i=0}^{[N\mathcal{T}]-1} e_i|\psi_i| \mathbb{1}_{\{e_i|\psi_i|>N^{2/3}\}}.$$

Then by a straightforward computation (see Lemma 5.2)

$$\begin{aligned}\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\tilde{S}_N^{>}(\theta)>\tau/2\right] &\leq \frac{2}{\tau}\frac{1}{N^{2/3}}\sum_{i=0}^{[N\mathcal{T}]-1}\mathbb{E}\Big[e_i|\psi_i|\mathbb{1}_{\{e_i|\psi_i|>N^{2/3}\}}\Big]\\ &\leq \frac{2}{\tau}\frac{1}{N^{2/3}}N\mathcal{T}\frac{K_0}{N^{1/3}}\\ &= K\mathcal{T}\frac{1}{\tau},\end{aligned}\tag{7.18}$$

with $K_0, K>0$. This inequality with (7.11) and (7.17) prove (7.2).

Now we prove (7.4). Again, for each $\eta>0$

$$\mathbb{P}\Big[w_{\tilde{S}_N}(\delta)\geq\eta\Big]\leq\mathbb{P}\Big[w_{\tilde{S}_N^{<}}(\delta)\geq\eta/2\Big]+\mathbb{P}\Big[w_{\tilde{S}_N^{>}}(\delta)\geq\eta/2\Big].\tag{7.19}$$

To estimate $\mathbb{P}\big[w_{\tilde{S}_N^{<}}(\delta)\geq\eta/2\big]$, we divide $[0,\mathcal{T}]$ in intervals of length $\delta>0$. If $w_{\tilde{S}_N^{<}}(\delta)\geq\eta/2$, then exists k, $1\leq k<\mathcal{T}/\delta$, such that

$$\begin{aligned}&\sup\Big\{\min\Big(\big|\tilde{S}_N^{<}(s)-\tilde{S}_N^{<}(r)\big|,\big|\tilde{S}_N^{<}(t)-\tilde{S}_N^{<}(s)\big|\Big),\ (k-1)\delta\leq r\leq s\leq t\leq(k+1)\delta\Big\}\\ &\geq\eta/2.\end{aligned}$$

Observe that

$$\min\Big(\big|\tilde{S}_N^{<}(s)-\tilde{S}_N^{<}(r)\big|,\big|\tilde{S}_N^{<}(t)-\tilde{S}_N^{<}(s)\big|\Big)=m_{rst}^{<},$$

thus by (7.14)

$$\mathbb{P}\Big[m_{rst}^{<}\geq\eta/2\Big]\leq A\frac{1}{\eta^4}(t-r)^2,\qquad (k-1)\delta\leq r\leq t\leq(k+1)\delta.\tag{7.20}$$

Then

$$\mathbb{P}\Big[\sup\big\{m_{rst}^{<}:(k-1)\delta\leq r\leq s\leq t\leq(k+1)\delta\big\}\geq\eta/2\Big]\leq\frac{C_0}{\eta^4}\delta^2,\tag{7.21}$$

with $C_0>0$ (see Theorem 10.3 in [4]). Using this result we get

$$\begin{aligned}&\mathbb{P}\Big[w_{\tilde{S}_N^{<}}(\delta)\geq\eta/2\Big]\\ &\leq\sum_{k=1}^{[\mathcal{T}/\delta]}\mathbb{P}\Big[\sup\big\{m_{rst}^{<}:(k-1)\delta\leq r\leq s\leq t\leq(k+1)\delta\big\}\geq\eta/2\Big]\\ &\leq\frac{C_1}{\eta^4}\mathcal{T}\delta\end{aligned}\tag{7.22}$$

with $C_1>0$.

Now let us consider the quantity

$$m_{rst}^{>}=\min\Big(\big|\tilde{S}_N^{>}(s)-\tilde{S}_N^{>}(r)\big|,\big|\tilde{S}_N^{>}(t)-\tilde{S}_N^{>}(s)\big|\Big)$$

for $0 \le r \le s \le t \le \mathcal{T}$, with $t - r < \delta$. If $m^{>}_{rst} > 0$, then $e_i|\psi_i| > N^{2/3}$, $e_j|\psi_j| > N^{2/3}$ for some $[rN] \le i \le [sN] - 1$, $[sN] \le j \le [tN] - 1$. We get

$$\begin{aligned}
&\mathbb{P}\Big[w_{\tilde{S}^{>}_N}(\delta) \ge \eta/2\Big] \\
&\le \mathbb{P}\Big[\exists\, i, j < N\mathcal{T}, |i-j| \le 2\delta N : e_i|\psi_i| > N^{2/3},\ e_j|\psi_j| > N^{2/3}\Big] \\
&\le \sum_{i=0}^{[N\mathcal{T}]-1} \sum_{j=i+1}^{i+[2N\delta]} \mathbb{P}\Big[e_i|\psi_i| > N^{2/3},\ e_j|\psi_j| > N^{2/3}\Big] \\
&\le C\mathcal{T}\delta
\end{aligned} \tag{7.23}$$

with $C > 0$, where the last inequality follows by Lemma 6.2. Equation (7.23) together with (7.22) prove (7.4).

The proofs of equations (7.5),(7.6) are easier. We give only the proof of (7.6), since the other is similar. We have

$$\begin{aligned}
&\mathbb{P}\Big[\big|S_N(\mathcal{T}) - S_N(\mathcal{T}-\delta)\big| \ge \eta\Big] \\
&\le \mathbb{P}\Big[\big|\tilde{S}^{<}_N(\mathcal{T}) - \tilde{S}^{<}_N(\mathcal{T}-\delta)\big| \ge \eta/2\Big] + \mathbb{P}\Big[\big|\tilde{S}^{>}_N(\mathcal{T}) - \tilde{S}^{>}_N(\mathcal{T}-\delta)\big| \ge \eta/2\Big],
\end{aligned} \tag{7.24}$$

where, using a second moment estimate (essentially the same proof of Lemma 4.1), we get

$$\begin{aligned}
&\mathbb{P}\Big[\big|\tilde{S}^{<}_N(\mathcal{T}) - \tilde{S}^{<}_N(\mathcal{T}-\delta)\big| \ge \eta/2\Big] \\
&= \mathbb{P}\left[\left|\frac{1}{N^{2/3}} \sum_{n=\lfloor N(\mathcal{T}-\delta)\rfloor}^{\lfloor N\mathcal{T}\rfloor - 1} e_n\psi_n \mathbb{1}_{\{|e_n\psi_n| \le N^{2/3}\}}\right| \ge \eta/2\right] \\
&\le \frac{4}{\eta^2}\frac{1}{N^{4/3}}\mathbb{E}\left[\left|\sum_{n=\lfloor N(\mathcal{T}-\delta)\rfloor}^{\lfloor N\mathcal{T}\rfloor - 1} e_n\psi_n \mathbb{1}_{\{|e_n\psi_n| N^{2/3}\}}\right|^2\right] \\
&\le \frac{C}{\eta^2}\delta,
\end{aligned} \tag{7.25}$$

with $C > 0$. Moreover,

$$\begin{aligned}
&\mathbb{P}\Big[\big|\tilde{S}^{>}_N(\mathcal{T}) - \tilde{S}^{>}_N(\mathcal{T}-\delta)\big| \ge \eta/2\Big] \\
&\le \mathbb{P}\left[\frac{1}{N^{2/3}} \sum_{n=\lfloor N(\mathcal{T}-\delta)\rfloor}^{\lfloor N\mathcal{T}\rfloor - 1} |e_n\psi_n|\ \mathbb{1}_{\{|e_n\psi_n| > N^{2/3}\}} \ge \eta/2\right] \\
&\le \frac{2}{\eta}\frac{1}{N^{2/3}} \sum_{n=\lfloor N(\mathcal{T}-\delta)\rfloor}^{\lfloor N\mathcal{T}\rfloor - 1} \mathbb{E}\Big[|e_n\psi_n|\ \mathbb{1}_{\{|e_n\psi_n| > N^{2/3}\}}\Big] \\
&\le \frac{K}{\eta}\delta,
\end{aligned} \tag{7.26}$$

and this, together with (7.25), proves (7.5).

8. Proof Theorem 3.2

Let us consider the two rescaled processes T_N, T_N^{-1} defined in (3.2). We want to prove the convergence in probability of both processes to the function θ. We start with the following Lemma.

Lemma 8.1. *Let T_N be the process defined in (3.2). Then* $\forall \varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\Big[\big|T_N(\theta) - \theta\big| > \varepsilon\Big] = 0, \qquad \forall\theta \in [0, \mathcal{T}]. \tag{8.1}$$

Proof. Le us denote by $\mathbb{E}_\pi$ the expectation value with respect to the invariant measure π. Since $\mathbb{E}_\pi\left[e_i\phi(X_i)^{-1}\right] = 1$, $\forall i = 0, 1, ...,$ then

$$\mathbb{P}\Big[\big|T_N(\theta) - \theta\big| > \varepsilon\Big] \le \mathbb{P}\left[\frac{1}{N}\Big|\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\Big(e_i\phi(X_i)^{-1} - \mathbb{E}_\pi\left[e_i\phi(X_i)^{-1}\right]\Big)\Big| > \varepsilon - \frac{1}{N}\right].$$

We denote by $\varepsilon' = \varepsilon - N^{-1}$, which is positive for N large enough. Let us introduce the following notations:

$$\tau_i^< \equiv e_i\phi(X_i)^{-1}\mathbb{1}_{\{e_i\phi(X_i)^{-1}\le N^{2/3}\}}, \quad \tau_i^> \equiv e_i\phi(X_i)^{-1}\mathbb{1}_{\{e_i\phi(X_i)^{-1}> N^{2/3}\}}, \tag{8.2}$$

$\forall i = 0, 1, ...$ We have

$$\begin{aligned}
&\mathbb{P}\left[\frac{1}{N}\Big|\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\Big(e_i\phi(X_i)^{-1} - \mathbb{E}_\pi\left[e_i\phi(X_i)^{-1}\right]\Big)\Big| > \varepsilon'\right] \\
&\le \mathbb{P}\left[\frac{1}{N}\Big|\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\left(\tau_i^< - 1\right)\Big| > \frac{\varepsilon'}{2}\right] + \mathbb{P}\left[\frac{1}{N}\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\tau_i^> > \frac{\varepsilon'}{2}\right].
\end{aligned} \tag{8.3}$$

By a first moment estimation (see Lemma 5.2), we get

$$\mathbb{E}\Big[N^{-1}\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\tau_i^>\Big] \le A_0\mathcal{T}\frac{1}{N^{1/3}},$$

with $A_0 > 0$, thus we can neglect the second term on the r.h.s. of (8.3). For the first term we have

$$\begin{aligned}
&\mathbb{P}\left[\frac{1}{N}\Big|\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\left(\tau_i^< - 1\right)\Big| > \frac{\varepsilon'}{2}\right] \\
&\le \frac{4}{\varepsilon^2}\frac{1}{N^2}\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\mathbb{E}\Big[\left(\tau_i^< - 1\right)^2\Big] + \frac{4}{\varepsilon^2}\frac{1}{N^2}\sum_{i,j=0,\, i\ne j}^{\lfloor N\theta\rfloor - 1}\mathbb{E}\Big[\tau_i^<\tau_j^< - \mathbb{E}_\pi\left[\tau_i^<\right]\mathbb{E}_\pi\left[\tau_j^<\right]\Big] \\
&\quad + \mathcal{O}(N^{-1/3}),
\end{aligned} \tag{8.4}$$

where,using a second moment estimate (see for example (5.10)), we get

$$\frac{4}{\varepsilon^2}\frac{1}{N^2}\sum_{i=0}^{\lfloor N\theta\rfloor - 1}\mathbb{E}\Big[\left(\tau_i^< - 1\right)^2\Big] \le \frac{1}{\varepsilon^2}B_0\mathcal{T}\frac{1}{N^{2/3}} + \mathcal{O}(N^{-1}). \tag{8.5}$$

Let us consider the second sum in (8.4). Fixed $1 < M < N$, we split it in three parts:

$$
\begin{aligned}
&\sum_{i,j=0,\, i\neq j}^{\lfloor N\theta\rfloor -1} \mathbb{E}\Big[\tau_i^< \tau_j^< - \mathbb{E}_\pi\big[\tau_i^<\big]\mathbb{E}_\pi\big[\tau_j^<\big]\Big] \\
&= \sum_{\substack{i,j=0\\ 1\leq |j-i|\leq M}}^{\lfloor N\theta\rfloor -1} \mathbb{E}\Big[\tau_i^< \tau_j^< - \mathbb{E}_\pi\big[\tau_i^<\big]\mathbb{E}_\pi\big[\tau_j^<\big]\Big] + \sum_{i=0}^{M-1}\sum_{\substack{j=0\\ |j-i|>M}}^{\lfloor N\theta\rfloor -1} \mathbb{E}\Big[\tau_i^< \tau_j^< - \mathbb{E}_\pi\big[\tau_i^<\big]\mathbb{E}_\pi\big[\tau_j^<\big]\Big] \\
&\quad + \sum_{i=M}^{\lfloor N\theta\rfloor -1}\sum_{\substack{j=0\\ |j-i|>M}}^{\lfloor N\theta\rfloor -1} \mathbb{E}\Big[\tau_i^< \tau_j^< - \mathbb{E}_\pi\big[\tau_i^<\big]\mathbb{E}_\pi\big[\tau_j^<\big]\Big].
\end{aligned}
\tag{8.6}
$$

For every $i, m \geq 1$, we have

$$
\begin{aligned}
\mathbb{E}\Big[\tau_i^< \tau_{i+m}^<\Big] = \int_0^\infty dz\; z e^{-z} \int_0^\infty du\; u e^{-u} \int_{\mathbb{T}} d\mu(k_0) \int_{\mathbb{T}} dk\; p^i(k_0,k)\phi(k)^{-1}\mathbb{1}_{\{z\phi(k)^{-1}<N^{2/3}\}} \\
\times \int_{\mathbb{T}} dk'\; p^m(k,k')\phi(k')^{-1}\mathbb{1}_{\{z\phi(k')^{-1}<N^{2/3}\}}.
\end{aligned}
$$

Using (5.2) one finds that $\mathbb{E}\big[\tau_i^< \tau_{i+m}^<\big] \sim \mathcal{O}(1)$, thus the first and the second sum in (8.6) are $\mathcal{O}(M/N)$. Moreover since $\pi(dk) = \phi(k)dk$, one get

$$
\begin{aligned}
&\mathbb{E}\Big[\tau_i^< \tau_{i+m}^<\Big] \\
&= \mathbb{E}_\pi\Big[\tau_i^< \tau_{i+m}^<\Big] + \int_0^\infty dz\; e^{-z} z \int_{\mathbb{T}} d\mu(k_0) \int_{\mathbb{T}} dk\; p^i(k_0,k)\phi(k)^{-1}\mathbb{1}_{\{z\phi(k)^{-1}<N^{2/3}\}} \\
&\quad \times \int_0^\infty du\; e^{-u} u \int_{\mathbb{T}} dk'\; \big(p^m(k,k') - \phi(k')\big)\phi(k')^{-1}\mathbb{1}_{\{u\phi(k')^{-1}<N^{2/3}\}} \\
&\quad + \int_0^\infty dz\; e^{-z} z \int_{\mathbb{T}} d\mu(k_0) \int_{\mathbb{T}} dk\; \big(p^i(k_0,k) - \phi(k)\big)\phi(k)^{-1}\mathbb{1}_{\{z\phi(k)^{-1}<N^{2/3}\}}\mathbb{E}_\pi\Big[\tau_{i+m}^<\Big],
\end{aligned}
$$

and then $\forall i, m \geq 1$

$$
\Big|\mathbb{E}\Big[\tau_i^< \tau_{i+m}^<\Big] - \mathbb{E}_\pi\Big[\tau_i^<\Big]\mathbb{E}_\pi\Big[\tau_{i+m}^<\Big]\Big| \leq C \int_{\mathbb{T}} dk \max_{i>M} \sup_{k_0\in\mathbb{T}} \Big\{\phi(k)^{-1}\big|p^i(k_0,k) - \phi(k)\big|\Big\}.
$$

Thus the third sum in (8.6) is bounded by

$$
\varepsilon^{-2}\mathcal{T}^2 C \int_{\mathbb{T}} dk \max_{i>M} \sup_{k_0\in\mathbb{T}} \Big\{\phi(k)^{-1}\big|p^i(k_0,k) - \phi(k)\big|\Big\},
$$

which, by ergodicity, goes to zero for $M \to \infty$. We prove the Lemma choosing $N, M \to \infty$, with $M/N \to 0$. □

Now we prove that the processes $T_N(\theta)$, $T_N^{-1}(\theta)$ converge in probability (and thus in distribution) to the function θ.

Proposition 8.2. *Let T_N, T_N^{-1} be the processes defined in (3.2). Then* $\forall \epsilon > 0$, $\forall \delta > 0$ $\exists N_0$ *such that* $\forall N \geq N_0$

$$
\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]} \Big\{\big|T_N(\theta) - \theta\big|\Big\} > \epsilon\right] < \delta, \tag{8.7}
$$

and

$$\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{\left|T_N^{-1}(\theta)-\theta\right|\right\}>\epsilon\right]<\delta. \tag{8.8}$$

Proof. We give the proof only for (8.8), the proof of (8.7) is similar.

Fixed $\eta>0$, we divide $[0,\mathcal{T}]$ in $\lfloor \mathcal{T}/\eta\rfloor+1$ intervals of length less or equal than η. We observe that

$$\begin{aligned}\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{\left|T_N^{-1}(\theta)-\theta\right|\right\}>\epsilon\right] &\le \mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{T_N^{-1}(\theta)-\theta\right\}>\epsilon\right]\\ &\quad+\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{\theta-T_N^{-1}(\theta)\right\}>\epsilon\right].\end{aligned} \tag{8.9}$$

We have

$$\begin{aligned}\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{T_N^{-1}(\theta)-\theta\right\}>\epsilon\right] &\le \sum_{k=0}^{\lfloor \mathcal{T}/\eta\rfloor}\mathbb{P}\left[\sup_{\theta\in[k\eta,(k+1)\eta]}\left\{T_N^{-1}(\theta)-\theta\right\}>\epsilon\right]\\ \le \sum_{k=0}^{\lfloor \mathcal{T}/\eta\rfloor}\mathbb{P}\Big[\Big\{T_N^{-1}[(k+1)\eta]-(k+1)\eta\Big\}>\epsilon-\eta\Big] \qquad (8.10)\\ = \sum_{k=1}^{\lfloor \mathcal{T}/\eta\rfloor+1}\mathbb{P}\Big[T_N^{-1}(k\eta)-k\eta>\epsilon-\eta\Big]\end{aligned}$$

where we used the fact that T_N^{-1} is non-decreasing. For every $\epsilon>0$, we choose η such that $\epsilon'\equiv\epsilon-\eta$ is positive, and we rewrite the relation $T_N^{-1}(k\eta)-k\eta>\epsilon'$ as

$$T^{-1}(Nk\eta)>Nk\eta+N\epsilon', \tag{8.11}$$

where $T^{-1}(t)$ is defined in (2.8). By definition the following relation holds:

$$T_{[T^{-1}(Nk\eta)-1]}<Nk\eta\le T_{[T^{-1}(Nk\eta)]}, \tag{8.12}$$

then inequality (8.11) implies

$$Nk\eta>T_{\lfloor Nk\eta+N\epsilon'-1\rfloor}, \tag{8.13}$$

and thus

$$\begin{aligned}&\sum_{k=1}^{\lfloor \mathcal{T}/\eta\rfloor+1}\mathbb{P}\Big[T_N^{-1}(k\eta)-k\eta>\epsilon'\Big]\\ &\le \sum_{k=1}^{\lfloor \mathcal{T}/\eta\rfloor+1}\mathbb{P}\Big[(k\eta+\epsilon'-N^{-1})-T_N(k\eta+\epsilon'-N^{-1})>\epsilon'-N^{-1}\Big].\end{aligned} \tag{8.14}$$

In the same way one can easily prove that

$$\mathbb{P}\left[\sup_{\theta\in[0,\mathcal{T}]}\left\{\theta-T_N^{-1}(\theta)\right\}>\epsilon\right]\le\sum_{k=0}^{\lfloor \mathcal{T}/\eta\rfloor}\mathbb{P}\Big[T_N(k\eta-\epsilon')-(k\eta-\epsilon')>\epsilon'\Big]. \tag{8.15}$$

By Lemma 8.1, $\forall\epsilon'>0$, $\forall\delta'>0$, $\exists N_0$ such that $\mathbb{P}\Big[T_N(\theta)-\theta>\epsilon'\Big]<\delta'$, $\forall N\ge N_0$, for all $\theta\in[0,\mathcal{T}]$, and this conclude the proof of (8.8). □

In order to prove that $S_N \circ T_N^{-1}$ converges in distribution to the Lévy process V, we have just to show that the distance between the two process S_N, $S_N \circ T_N^{-1}$ goes to zero in probability (see Theorem 3.1 in [4]). By denoting with $\rho(\cdot,\cdot)$ the distance in the Skorokhod J_1 topology,

$$\lim_{N\to\infty} \mathbb{P}\Big[\rho(S_N, S_N \circ T_N^{-1}) > \epsilon\Big] \le \mathbb{P}\left[\sup_{t\in[0,\mathcal{T}]} \big|S_N(\lambda_N(t)) - S_N \circ T_N^{-1}(t)\big| > \epsilon\right], \quad (8.16)$$

where $\lambda_N : [0,\mathcal{T}] \to [0,\mathcal{T}]$ is a sequence of increasing homeomorphisms such that $\|\lambda_N - I\|_\infty \to 0$. We have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\in[0,\mathcal{T}]} \big|S_N(\lambda_N(t)) - S_N \circ T_N^{-1}(t)\big| > \epsilon\right] \\
&\le \mathbb{P}\left[\left\{\sup_{t\in[0,\mathcal{T}]} \big|S_N(\lambda_N(t)) - S_N \circ T_N^{-1}(t)\big| > \epsilon\right\} \cap \left\{\sup_{t\in[0,\mathcal{T}]} \big|T_N^{-1}(t) - t\big| \le \sigma\right\}\right] \\
&\quad + \mathbb{P}\left[\sup_{t\in[0,\mathcal{T}]} \big|T_N^{-1}(t) - t\big| > \sigma\right],
\end{aligned}$$

for every $\sigma > 0$. The second term on the r.h.s. goes to zero for $N \to \infty$ (see Proposition 8.2), while the first term is bounded by

$$\mathbb{P}\left[\sup\left\{\big|S_N(\lambda_N(t)) - S_N(s)\big|, |t-s| \le \sigma,\ t,s \in [0,\mathcal{T}]\right\} > \epsilon\right],$$

which goes to zero for the tightness (see (7.4)) and the fact that $\|\lambda_N - I\|_\infty \to 0$.

It remains to prove the convergence of the process Y_N to the stable process V. The basic idea is that the step-function sequence $\{S_N(\theta)\}_{0\le\theta\le\mathcal{T}}$ and the continuous-time sequence, gived by the linear interpolation of $\{S_N(\theta)\}$, are *asymptotically equivalent*, i.e. if either converges in distribution as $\mathbb{N} \to \infty$, then so should the other, and they should have the same limit. This is proved, for example, in [26], Section 6.2. Then one can easily extend this result to $\{S_N(T_N^{-1})\}_{0\le\theta\le\mathcal{T}}$ and $\{Y_N\}_{0\le\theta\le\mathcal{T}}$, since the waiting time between two jumps has finite average.

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