METASTABILITY AND AGEING IN STOCHASTIC DYNAMICS

ANTON BOVIER
Weierstrass-Institut für Angewandte Analysis und Stochastik
Mohrenstrasse 39, D-10117 Berlin, Germany
bovier@wias-berlin.de

Abstract. In these notes I review recent results on metastability and ageing in stochastic dynamics. The first part reviews a somewhat novel approach to the computation of key quantities such as mean exit times in metastable systems and small eigenvalues of the generator of metastable Markov chain developed over the last years with M. Eckhoff, V. Gayrard and M. Klein. This approach is based on extensive use of potential theoretic ideas and allows, at least in the case of reversible dynamics, to get very accurate results with comparatively little effort. This methods have also been used in recent joint work with G. Ben Arous and V. Gayrard on the dynamics of the random energy model. The second part of these lectures is devoted to a review of this work.

Contents

1. Introduction

In these lectures I will review a somewhat new approach to the old issue of metastability and the rather more recent issue of "ageing" in the framework of stochastic dynamics, or more precisely Markov processes. This approach addresses, in the context of metastability, the issue of precise asymptotics, that, although frequently considered in the physical literature for decades, had been lacking a fully rigorous mathematical treatment. Apart from potential intrinsic interest, this question is strongly motivated by the second aspect, ageing, where, as will become clear, this will become indispensable if anything is to be understood at all.

Metastability as well as ageing (at least in the context we will be interested in) can be seen as special aspects of the far wider issue of understanding the dynamic behaviour of a complicated system on particular time scales in terms of a simplified dynamics with a reduced state space. Such a coarse grained description will always have to be chosen in accordance with the problem at hand. The phenomena we are trying to capture here concern the existence of two (or more) *time scales* at which the systems considered behave in distinct ways: on the short time scales, the system performs some local motion that appears to be reasonably recurrent, suggesting that equilibrium is reached, while at the larger time scale, the systems undergoes dramatic changes that seem to alter its character entirely. Our purpose is thus to characterize the long time dynamics in terms of a process that describes only the sequence of these global changes and ignores all of the local short time motion. Our task is then to devise in a given setting such a coarse grained description and to derive the properties of the corresponding process and from the underlying microscopic dynamics.

The setting sketched above is common in a wide variety of natural phenomena. From the most classical setting of dynamical phase transitions in solid state physics, conformational states of macromolecules, macroclimatological states, or more speculatively, macro-economic states, there is hardly a branch of science where one is not confronted with the same type of general situation. While in the classical setting of metastability the number of relevant "macro-states" is finite or even small, systems with a large, resp. infinite, number of such states tend to "age", meaning in the broadest sense that their long time behaviour is to be described in terms of a more complicated process than just a Markov chain with finite state space. Ageing is observed any many important materials like glasses, glassy polymers, bio-molecules and plastic, to name just a few.

The study of metastability in the context of probabilistic models dates back at least to the work of Eyring, Polanyi, Wigner, and Kramers [28, 29, 62, 45] in the context of chemical reactions. An early textbook reference is [35]. Kramers was the first to use a model of a particle in a drift field subject to forcing by Brownian motion, i.e. stochastic differential equations.

Stochastic differential equations form indeed one major context in which metastability was studied extensively. In the physics literature, the main tools were based on perturbation theory in the spirit of the quantum mechanical WKB method (a selection of references is [42, 43, 50, 51, 59]). Mathematical rigorous work was based mostly on the large deviation methods developed by Wentzell and Freidlin [31] who developed a systematic approach to the problem in the mid 1970's.

Over the last few year, in collaboration with M. Eckhoff, V. Gayrard, and M. Klein [8, 9, 10, 11] we have developed a somewhat novel approach to the metastability problem that is to a large extend based on potential theoretic ideas and makes extensive use of capacities and associated variational formulas. It has the virtue of being widely applicable while yielding rather precise results that improve on the exponential asymptotics obtained with large deviation methods. This approach has also allowed us, in collaboration with G. Ben Arous and V. Gayrard, to get some results on systems that show ageing, in particular in the Random Energy Model [1, 2]. In these lectures I want to review the key aspects of these developments.

2. A General Setting for Metastability and some Key Issues

The most general setting we will consider can be described as follows. We consider a Markov process X_t on a measure space $\Gamma \supset X_t$ with discrete or continuous time t. In continuous time the process is characterized by its *generator*, L, and in discrete time by the transition matrix, P. We will usually assume that the process is uniquely ergodic with invariant measure $\mathbb{Q}$. We will denote the law of this process by $\mathbb{P}$. Moreover, we will denote by $\mathbb{P}_x$ the law of the process conditioned to verify $X(0) = x$. We will denote by τ_D, $D \subset \Gamma$, the first entrance time of $X(t)$ in D, i.e.

$$\tau_D \equiv \inf\{t > 0, X(t) \in D\} \tag{2.1}$$

We would like to say that the process is exhibiting metastability, if Γ can be partitioned into subsets S_i such that the process starting anywhere in S_i explores any S_i on a time-scale T_{fast} that is much shorter than the typical times it takes T_{slow} to leave S_i. Of course such a statement is imprecise and makes strictly speaking no sense. In particular the requirement that the process explores all of S_i must be qualified. There are a number of possible attempts to formalize this notion (see e.g. [37]) and the choice we will give below is certainly only one among a number of possibilities; however, it is useful inasmuch it implies a number of key properties of metastable systems while being sufficiently flexible to be applicable in a broad range of situations. We stress, however, that we do not necessarily consider this as the ultimate choice.

To make precise statements, we will also need a small parameter. Thus instead of a single Markov process, we will from now on always think of a family of processes X_t^ϵ (where in principle all key objects like $\Gamma, \mathbb{P}, \mathbb{Q}$ are allowed to depend on ϵ.

Provisional Definition 2.1. A family of Markov processes is called metastable, if there exists a collection of disjoint sets $B_i \subset \Gamma$ (possibly depending on ϵ), such that

$$\frac{\sup_{x \notin \cup_i B_i} \mathbb{E}_x \tau_{\cup_i B_i}}{\inf_{x \in \cup_i B_i} \mathbb{E}_x \tau_{\cup_{k \neq i} B_k}} \to 0, \quad \text{as} \quad \epsilon \downarrow 0 \tag{2.2}$$

Remark. Note that both numerator and denominator represent time scales that depend on the choice of the sets B_i. In fact, metastability may hold with respect to different decompositions on different time scales in the same system. This is often important in applications.

For many purposes it is useful to chose the B_i as small as possible. E.g. it will be very useful to have the property that

$$\frac{\sup_{x \in B_i} \mathbb{E}_x \tau_{B_k} - \inf_{x \in B_i} \mathbb{E}_x \tau_{B_k}}{\inf_{x \in B_i} \mathbb{E}_x \tau_{B_k}} \to 0, \quad \text{as} \quad \epsilon \downarrow 0 \tag{2.3}$$

Note that this requirement should be balanced against the main requirement (2.2).

Definition 2.1 defines metastability in terms of physical properties of a system we would like to consider as metastable. The problem is that it is not immediately verifiable, since it involves derived quantities, i.e. mean first hitting times, that are nor immediately computable. Our first problem will thus be to re-express the mean hitting times in terms of more manageable quantities.

The second problem will be to derive further properties of metastable systems. Since the definition implies frequent returns to the small starting set B_i before transit to another set B_j, this suggests an exponential law for the transit times. This also suggests that we may expect to describe the process of successive visits to distinct B_i asymptotically as a Markov process.

The most fundamental result we want to achieve, however, is a characterization of the spectrum of the generator, resp. the transition matrix of a metastable process. Our purpose here is to derive spectral information from the Definition above (an not the converse, which is much simpler).

3. Markov Processes and Potential Theory

The intimate relation between Markov processes and potential theory is well-known since the work of of Kakutani [39] and is the subject of numerous textbooks (see in particular the fundamental monograph by Doob [24], and for the discrete case [58]). This connection has found numerous and widespread applications both in probability theory and in analysis (see e.g. [25, 57] and references wherein). Our approach to metastability relies heavily on this connection. In the following pages we give a brief survey of some key facts (all of which are classical) that we will need later. While it would be possible to treat the continuous and the discrete case in unified way, this introduces some notational absurdities (in the discrete case) that we would rather avoid. Since all formulas in the case of diffusion processes can be found in [10], we will present here the formulas in the discrete case, i.e. when Γ is a discrete set.

Thus let Γ be a discrete set, $\mathbb{Q}$ be a positive measure on Γ, P a (irreducible) stochastic matrix on Γ. We will denote by L the generator of the process in the case of continuous time, and set $L = \mathbb{1} - P$ in the case of discrete time. We assume that L is symmetric on the space $L^2(\Gamma, \mathbb{Q})$.

Green's function. Let $\Omega \subset \Gamma$. Consider for $\lambda \in \mathbb{C}$ and g a real valued function on Ω the Dirichlet problem

$$\begin{aligned} (L-\lambda)f(x) &= g(x), \quad x \in \Omega \\ f(x) &= 0, \quad x \in \Omega^c \end{aligned} \tag{3.1}$$

The associated Dirichlet Green's function $G^\lambda_\Omega(x,y)$ is the kernel of the inverse of the operator $(L-\lambda)^\Omega$, i.e. for any $g \in C_0(\Omega)$,

$$f(x) = \sum_{y\in\Omega} G^\lambda_\Omega(x,y)g(y) \tag{3.2}$$

Note that the Green's function is symmetric with respect to the measure $\mathbb{Q}$, i.e.

$$G^\lambda_\Omega(x,y) = \mathbb{Q}(y)G^\lambda_\Omega(y,x)\mathbb{Q}(x)^{-1} \tag{3.3}$$

Recall that the spectrum of L (more precisely the Dirichlet spectrum of the restriction of L to Ω, which we will sometimes denote by L^Ω), is the complement of the set of values λ for which G^λ_Ω defines a bounded operator.

Poisson Kernel. Consider for $\lambda \in \mathbb{C}$ the boundary value problem

$$\begin{aligned} (L-\lambda)f(x) &= 0, \quad x \in \Omega \\ f(x) &= \phi(x), \quad x \in \Omega^c \end{aligned} \tag{3.4}$$

We denote by H^λ_Ω the associated solution operator.

Equilibrium Potential and Equilibrium Measure. Let $A, D \subset \Gamma$. Then the equilibrium potential (of the capacitor (A,D)), $h^\lambda_{A,D}$, is defined as the solution of the Dirichlet problem

$$\begin{aligned} (L-\lambda)h^\lambda_{A,D}(x) &= 0, \quad x \in (A\cup D)^c \\ h^\lambda_{A,D}(x) &= 1, \quad x \in A \\ h^\lambda_{A,D}(x) &= 0, \quad x \in D \end{aligned} \tag{3.5}$$

Note that (3.5) has a unique solution provided λ is not in the spectrum of $L^{(A\cup B)^c}$.

The equilibrium measure, $e^\lambda_{A,D}$, is defined as the unique measure on A, such that

$$h^\lambda_{A,D}(x) = \sum_{y\in A} G^\lambda_{D^c}(x,y)e^\lambda_{A,D}(y) \tag{3.6}$$

(3.6) may also be written as

$$e^\lambda_{A,D}(y) = -(L-\lambda)h^\lambda_{A,D}(y) \tag{3.7}$$

In fact, (3.7) defines a measure supported on both A and D that we will henceforth call the equilibrium measure. Note that this measure (in the case $\lambda = 0$) is zero on the interior of $A\cup D$, i.e. on the part of $A \cup D$ which is not connected by a non-zero element of the transition matrix P.

Note that (3.6) implies a representation formula of the Poisson-kernel H^λ_Ω, namely

$$(H^\lambda_\Omega \phi)(x) = \sum_{z\in\Omega^c} \phi(z)G^\lambda_{\Omega\cup z}(x,z)e^\lambda_{z,\Omega^c\setminus z}(z) \tag{3.8}$$

which is the discrete analog to the usual Poisson-Green's formula.

Capacity. Given a capacitor, (A,D), and $\lambda \in \mathbb{R}$, the λ-capacity of the capacitor is defined as

$$\mathrm{cap}^\lambda_A(D) \equiv \sum_{y\in A} \mathbb{Q}(y)e^\lambda_{A,D}(y) \tag{3.9}$$

Using (3.7) one derives after some algebra that in discrete time

$$\begin{aligned} \mathrm{cap}^\lambda_A(D) &= \tfrac{1}{2}\sum_{x,y} \mathbb{Q}(x)\left[p(x,y)\left\|h^\lambda_{A,D}(x)-h^\lambda_{A,D}(y)\right\|^2 - \lambda\left(h^\lambda_{A,D}(x)\right)^2\right] \\ &\equiv \Phi^\lambda\left(h^\lambda_{A,D}\right) \end{aligned} \tag{3.10}$$

where $p(x,y)$ are the transition probabilities (in discrete time) respectively transition rates (in discrete time). Φ^λ_Ω is called the Dirichlet form (or energy) for the operator $L-\lambda$ on Ω.

A fundamental consequence of (3.10) is the variational representation of the capacity when λ is real and non-positive, namely

$$\mathrm{cap}_A^\lambda(D) = \inf_{h\in\mathcal{H}_{A,D}} \Phi^\lambda(h) \tag{3.11}$$

where $\mathcal{H}_{A,D}$ denotes the set of function

$$\mathcal{H}_{A,D} \equiv \{h : \Gamma \to [0,1] : h(x) = 0, x \in D, h(x) = 1, x \in A\} \tag{3.12}$$

Probabilistic Interpretation: Equilibrium Potential.
If $\lambda = 0$, the equilibrium potential has a natural probabilistic interpretation in terms of hitting probabilities of this process, namely,

$$h_{A,D}(x) \equiv h^0_{A,D}(x) = \mathbb{P}_x[\tau_A < \tau_D] \tag{3.13}$$

The equilibrium measure has a nice interpretation in the discrete time case as well:

$$e_{A,D}(y) = \mathbb{P}_y[\tau_D < \tau_A] \tag{3.14}$$

if $y \in A$. In particular, if $A = \{y\}$,

$$\mathbb{P}_y[\tau_D < \tau_A] = e_{y,D}(y) = \frac{\mathrm{cap}_y(D)}{\mathbb{Q}(y)} \tag{3.15}$$

If $\lambda \neq 0$, the equilibrium potential still has a probabilistic interpretation in terms of the Laplace transform of the hitting time τ_A of the process starting in x and killed in D. Namely, we have for general λ, that

$$h^\lambda_{A,D}(x) = \mathbb{E}_x e^{\lambda\tau_A} 1\!\!1_{\tau_A<\tau_D} \tag{3.16}$$

for $x \in (A\cup D)^c$, whenever the right-hand side exists.
Note that (3.16) implies that

$$\frac{d}{d\lambda} h^{\lambda=0}_{A,D}(x) = \mathbb{E}_x \tau_A 1\!\!1_{\tau_A<\tau_D} \tag{3.17}$$

Differentiating the defining equation of $h^\lambda_{A,D}$ then implies that the function

$$w_{A,D}(x) = \begin{cases} \mathbb{E}_x\tau_A 1\!\!1_{\tau_A<\tau_D}, & x \in (A\cup D)^c \\ 0, & x \in A\cup D \end{cases} \tag{3.18}$$

solves the inhomogeneous Dirichlet problem (to simplify notation, we set from now on $h_{A,D} \equiv h^0_{A;D}$, etc.)

$$\begin{aligned} Lw_{A,D}(x) &= h_{A,D}(x), \quad x \in (A\cup D)^c \\ w_{A,D}(x) &= 0, \quad x \in A\cup D \end{aligned} \tag{3.19}$$

Therefore, the mean hitting time in A of the process killed in D can be represented in terms of the Green's function as

$$\mathbb{E}_x \tau_A \mathbb{1}_{\tau_A<\tau_D} = \sum_{y\in(A\cup D)^c} G_{(A\cup D)^c}(x,y) h_{A,D}(y) \tag{3.20}$$

Note that in the particular case when $D=\emptyset$, we get the familiar Dirichlet problem

$$\begin{aligned} Lw_A(x) &= 1, \quad x\in A^c \\ w_A(x) &= 0, \quad x\in A \end{aligned} \tag{3.21}$$

and the representation

$$\mathbb{E}_x \tau_A = \sum_{y\in A^c} G_{A^c}(x,y) \tag{3.22}$$

The full beauty of all this comes out when combining (3.6) with (3.20), resp. (3.22). Then, using Fubini's theorem,

$$\begin{aligned} \mathbb{Q}(z)\mathbb{E}_z \tau_A e_{z,A}(z) &= \sum_{y\in A^c} \mathbb{Q}(y) G_{A^c}(y,z) e_{z,A}(z) \\ &= \sum_{y\in A^c} \mathbb{Q}(y) h_{z,A}(y) \end{aligned} \tag{3.23}$$

and

$$\mathbb{Q}(z)\mathbb{E}_z \tau_A \mathbb{1}_{\tau_A<\tau_D} e_{z,A\cup D}(z) = \sum_{y\in(A\cup D)^c} \mathbb{Q}(y) h_{z,A\cup D}(y) h_{A,D}(y) \tag{3.24}$$

(3.24) yield directly formulae for mean hitting times in terms of capacities and equilibrium potentials.

Indeed (3.6) yield a formula for the Green function

$$G^\lambda_{D^x}(x,y) = \frac{h^\lambda_{y,D}(x)}{e^\lambda_{y,D}(y)} = \frac{\mathbb{Q}(y) h^\lambda_{y,D}(x)}{\mathrm{cap}^\lambda_y(D)} = \frac{\mathbb{Q}(y) h^\lambda_{x,D}(y)}{\mathrm{cap}^\lambda_x(D)} \tag{3.25}$$

which will play a key rôle in the sequel.

Remark. Equations (3.23)-(3.25) rely explicitly on the discrete structure on the state space, or more precisely that for any $x\in\Gamma$, $\mathbb{Q}(x)>0$. In the case of continuous state space, such formulas do not hold in the strict sense, or are not useful, but suitable "integral versions", involving integrals over suitably chosen small neighborhoods of e.g. the points z in (3.23) are still valid, and can be used to more or less the same effect as the exact relations in the discrete case. This entails, however, some extra technical difficulties. In these notes we will therefore restrict our attention to the discrete case, where the principle ideas can be explained without being obscured by technicalities.

4. Metastability in Terms of Capacities and Mean Hitting Times

We now use the observations made in Section 2 to derive the desired alternative characterization of metastability in terms of potential theoretic quantities, namely capacities.

Definition 4.1 Assume that Γ is a discrete set. Then a family of Markov processes X_t^ϵ is metastable with respect to the set of points $\mathcal{M} \subset \Gamma$, if

$$\frac{\sup_{x \notin \mathcal{M}} \mathbb{Q}(x)/\mathrm{cap}_x(\mathcal{M})}{\inf_{x \in \mathcal{M}} \mathbb{Q}(x)/\mathrm{cap}_x(\mathcal{M}\backslash x)} \leq \rho(\epsilon) \tag{4.1}$$

for some $\rho(\epsilon)$ that tends to zero as $\epsilon \downarrow 0$.

We will see that this definition is essentially equivalent to the provisional definition given in Section 2, at least in situations where the latter is reasonable. Definition 4.1 has far reaching consequences. As a first step we will show that it implies the mean values of transition times from minima to other minima can be computed very precisely.

Let $x \in \mathcal{M}$, $x \notin J \subset \mathcal{M}$. We want to compute $\mathbb{E}_x \tau_J$. This computation will be based on the formula (3.22), which yields that

$$\mathbb{E}_x \tau_J = \frac{\mathbb{Q}(x)}{\mathrm{cap}_x(J)} \sum_{y \in J^c} \frac{\mathbb{Q}(y)}{\mathbb{Q}(x)} h_{x,J\backslash x}(y) \tag{4.2}$$

Thus we have only to control expressions like

$$\sum_{y \in \mathcal{M}^c} \frac{\mathbb{Q}(y)}{\mathbb{Q}(x)} h_{x,\mathcal{M}\backslash x}(y) \tag{4.3}$$

The analysis of such sums requires some knowledge of the equilibrium potential. One of the cornerstones of our approach is the observation that the equilibrium potential, too, can be estimated in terms of capacities, and that in many cases, these estimates yield very sharp results. The reason underlying this fact is that it turns out that in metastable systems, we tend to have a dichotomy of the type: *Either* the equilibrium potential $h_{x,J}(y)$ is close to zero or to one, *or* the invariant measure $\mathbb{Q}(y)$ is very small.

Renewal Estimates. The estimation of the equilibrium through capacities is based on a renewal argument, that in the case of discrete state space is very simple.

Lemma 4.1 Let $A, D \subset \Gamma$ be disjoint sets, and let $x \notin A \cup D$. Then

$$h_{A,D}(x) \leq \frac{\operatorname{cap}_x(A)}{\operatorname{cap}_x(D)} \tag{4.4}$$

Proof. In the discrete case, this result is extremely easy to prove. Just use the simple observation that going from x to A the process either does or does not re-visit x, or does so a first time, one gets

$$h_{A,D}(x) = \mathbb{P}_x[\tau_A < \tau_D] = \mathbb{P}_x[\tau_A < \tau_{D\cup x}] + \mathbb{P}_x[\tau_x < \tau_{A\cup D}]\mathbb{P}_x[\tau_A < \tau_D] \tag{4.5}$$

Hence

$$h_{A,D}(x) \leq \frac{\mathbb{P}_x[\tau_A < \tau_{D\cup x}]}{1 - \mathbb{P}_x[\tau_x < \tau_{A\cup D}]} = \frac{\mathbb{P}_x[\tau_A < \tau_{D\cup x}]}{\mathbb{P}_x[\tau_{A\cup D} < \tau_x]} \leq \frac{\mathbb{P}_x[\tau_A < \tau_x]}{\mathbb{P}_x[\tau_D < \tau_x]} = \frac{\operatorname{cap}_x(A)}{\operatorname{cap}_x(D)} \tag{4.6}$$

Since even large probabilities do not exceed 1, the lemma is proven. □

Remark. Note that the power of Lemma 4.1 is more than doubled by judicious use of the elementary fact that $h_{A,D}(x) = 1 - h_{D,A}(x)$.

Remark. The bound (4.5) can easily be improved to $h_{A,D}(x) \leq \min\left(\frac{\operatorname{cap}_x(A)}{\operatorname{cap}_x(D\cup A)}, 1\right)$, but this is seldom very useful.

Ultrametricity. An important fact that allows to obtain general results under our Definition of metastability is the fact that it implies approximate ultrametricity of capacities. This has been noted in [9].

Lemma 4.2 Assume that $x, y \in \Gamma$, $D \subset \Gamma$. Then, if for $0 < \delta < \frac{1}{2}$, $\operatorname{cap}_y(D) \leq \delta \operatorname{cap}_y(x)$, then

$$\frac{1-2\delta}{1-\delta} \leq \frac{\operatorname{cap}_x(D)}{\operatorname{cap}_y(D)} \leq \frac{1}{1-\delta} \tag{4.7}$$

Proof. The proof of this lemma given in [9] is probabilistic and uses splitting and renewal ideas. It should be possible to prove this result with purely potential theoretic arguments, but I have not worked this out. □

Lemma 4.2 has the following immediate corollary, which is the version of the ultrametric triangle inequality we are looking for:

Corollary 4.1 *Let $x, y, z \in \mathcal{M}$. Then*

$$\operatorname{cap}_x(y) \geq \frac{1}{3} \min\left(\operatorname{cap}_x(z), \operatorname{cap}_y(z)\right) \tag{4.8}$$

In the sequel it will be useful to have the notion of a "valley" or "attractor" of a point in $\mathcal{M}$. We set for $x \in \mathcal{M}$,

$$A(x) \equiv \left\{ z \in \Gamma \,|\, \mathbb{P}_z[\tau_x = \tau_{\mathcal{M}}] = \sup_{y \in \mathcal{M}} \mathbb{P}_z[\tau_y = \tau_{\mathcal{M}}] \right\} \tag{4.9}$$

Note that valleys may overlap, but from Lemma 4.2 it follows easily that the intersection has a vanishing invariant mass. The notion of a valley in the case of a diffusion process coincides with the intuitive notion.

The following simple corollary will be most useful:

Corollary 4.2 *Let $m \in \mathcal{M}$, $y \in A(m)$, and $J \subset \mathcal{M}\backslash m$. Then either*

$$\frac{1}{2} \leq \frac{\mathrm{cap}_m(J)}{\mathrm{cap}_y(J)} \leq \frac{3}{2} \tag{4.10}$$

or

$$\frac{\mathbb{Q}(y)}{\mathbb{Q}(m)} \leq 3|\mathcal{M}| \frac{\mathbb{Q}(y)}{\mathrm{cap}_y(\mathcal{M})} \frac{\mathrm{cap}_m(J)}{\mathbb{Q}(m)} \tag{4.11}$$

Proof. Lemma 4.2 implies that *if* $\mathrm{cap}_m(y) \geq 3\mathrm{cap}_m(J)$, then (4.10) holds. Otherwise,

$$\frac{\mathbb{Q}(y)}{\mathbb{Q}(m)} \leq 3 \frac{\mathbb{Q}(y)}{\mathrm{cap}_y(m)} \frac{\mathrm{cap}_m(J)}{\mathbb{Q}(m)} \tag{4.12}$$

Since $y \in A(m)$,

$$\mathrm{cap}_y(\mathcal{M}) \leq \sum_{z \in \mathcal{M}} \mathrm{cap}_y(z) \leq |\mathcal{M}| \sup_{z \in \mathcal{M}} \mathrm{cap}_y(z) = |\mathcal{M}| \mathrm{cap}_y(m) \tag{4.13}$$

which yields (4.11). $\square$

We want to use this corollary in order to estimate the summands in the sum (4.2). We will set $\inf_y \mathbb{Q}(y)^{-1}\mathrm{cap}_y(\mathcal{M}) = a_\epsilon$.

Lemma 4.3 Let $x \in \mathcal{M}$ and $J \subset \mathcal{M}$ with $x \notin J$. Then:
(i) If $x = m$, either

$$h_{x,J}(y) \geq 1 - \frac{3}{2}|\mathcal{M}| \frac{\mathrm{cap}_m(J)}{\mathrm{cap}_y(\mathcal{M})} \tag{4.14}$$

or

$$\frac{\mathbb{Q}(y)}{\mathbb{Q}(x)} \leq 3|\mathcal{M}| a_\epsilon^{-1} \frac{\mathrm{cap}_m(J)}{\mathbb{Q}(m)} \tag{4.15}$$

(ii) If $m \in J$, then

$$\mathbb{Q}(y) h_{x,J}(y) \leq \frac{3}{2}|\mathcal{M}| a_\epsilon^{-1} \mathrm{cap}_m(x) \tag{4.16}$$

(iii) If $m \notin J \cup x$, then either

$$h_{x,J}(y) \leq 3\frac{\text{cap}_m(x)}{\text{cap}_m(J)} \tag{4.17}$$

and

$$h_{x,J}(y) \geq 1 - 3\frac{\text{cap}_m(J)}{\text{cap}_m(x)} \tag{4.18}$$

or

$$\mathbb{Q}(y) \leq 3|\mathcal{M}|a_\epsilon^{-1} \max\left(\text{cap}_m(J), \text{cap}_m(x)\right) \tag{4.19}$$

Proof. We make use of the fact that by Lemma 4.1,

$$0 \leq h_{x,J}(y) \leq \frac{\text{cap}_y(x)}{\text{cap}_y(J)} \tag{4.20}$$

and

$$1 \geq h_{x,J}(y) \geq 1 - \frac{\text{cap}_y(J)}{\text{cap}_y(x)} \tag{4.21}$$

In case (i), we anticipate that only (4.21) will be useful. To get the first dichotomy, we use Corollary 4.2 to replace the numerator by $\text{cap}_m(J)$. To get the second assertion, note simply that

$$\frac{\text{cap}_m(J)}{\text{cap}_y(m)} \leq \frac{\mathbb{Q}(y)\text{cap}_m(J)}{\text{cap}_y(x)\mathbb{Q}(m)} \frac{\mathbb{Q}(m)}{\mathbb{Q}(y)} \tag{4.22}$$

and rewrite this inequality for $\frac{\mathbb{Q}(y)}{\mathbb{Q}(m)}$.

In case (ii), we use (4.20) and apply Corollary 4.2 to $\text{cap}_y(x)$.

In case (iii), we admit both possibilities and apply the corollary to both the numerators and the denominators. $\square$

Remark. Case (iii) in the preceding lemma is special in as much as it will not always give sharp estimates, namely whenever $\text{cap}_m(J) \sim \text{cap}_m(y)$. If this situation occurs, and the corresponding terms contribute to leading order, we cannot get sharp estimates with the tools we are exploiting here, and better estimates on the equilibrium potential will be needed.

Mean Times. Let us now apply this lemma to the computation of the sum (4.3) (we ignore the fact that the sets $A(m)$ may not be disjoint, as the overlaps give no significant contribution).

$$\begin{aligned} \sum_{y \in \mathcal{M}^c} \frac{\mathbb{Q}(y)}{\mathbb{Q}(x)} h_{x,J}(y) &= \sum_{m \in \mathcal{M}} \frac{\mathbb{Q}(m)}{\mathbb{Q}(x)} \sum_{y \in A(m) \setminus m} \frac{\mathbb{Q}(y)}{\mathbb{Q}(m)} h_{x,J}(y) \\ &\equiv \sum_{m \in \mathcal{M}} \frac{\mathbb{Q}(m)}{\mathbb{Q}(x)} L(m) \end{aligned} \tag{4.23}$$

(we ignore the fact that the sets $A(m)$ may not be disjoint, as the overlaps give no significant contribution).

We now estimate the terms $L(m)$ with the help of Lemma 4.3.

Lemma 4.4 With the notation introduced above and the assumptions of Lemma 4.3, we have that

(i) If $m = x$

$$L(x) \leq \frac{\mathbb{Q}(A(x))}{Q(x)} \tag{4.24}$$

and

$$L(x) \geq \frac{\mathbb{Q}(A(x))}{Q(x)} \left(1 - 6C|\mathcal{M}|a_\epsilon^{-1} \frac{\mathrm{cap}_m(J)}{\mathbb{Q}(A(m))}\right) \tag{4.25}$$

(ii) If $m \in J$, then

$$L(m) \leq Ca_\epsilon^{-1}|\mathcal{M}| \frac{\mathrm{cap}_m(x)}{\mathbb{Q}(m)} \left|\{y \in A(m) : \mathbb{Q}(y) \geq a_\epsilon^{-1}|\mathcal{M}|\mathrm{cap}_m(x)\}\right| \tag{4.26}$$

for some constant C independent of ϵ.

(iii) If $m \notin J \cup x$, then

$$L(m) \leq \frac{\mathbb{Q}(A(m))}{\mathbb{Q}(m)} \tag{4.27}$$

Moreover,

(iii.1) if $\mathrm{cap}_m(J) \leq \frac{1}{3}\mathrm{cap}_m(x)$, then

$$L(m) \geq \frac{\mathbb{Q}(A(m))}{\mathbb{Q}(m)} \left(1 - 3\frac{\mathrm{cap}_m(J)}{\mathrm{cap}_m(x)}\right) \left(1 - C|\mathcal{M}|a_\epsilon^{-1} \frac{\mathrm{cap}_m(x)}{\mathbb{Q}(A(m))}\right) \tag{4.28}$$

and

(iii.2) if $\mathrm{cap}_m(J) \geq \frac{1}{3}\mathrm{cap}_m(x)$, then

$$L(m) \leq \frac{\mathbb{Q}(A(m))}{\mathbb{Q}(m)} - C|\mathcal{M}|a_\epsilon^{-1} \frac{\mathrm{cap}_m(x)}{\mathbb{Q}(m)} \tag{4.29}$$

Proof. The proof of this lemma is rather straightforward and will be left as an exercise. Just note that to get (4.27) involves an optimal choice of δ in the application of Lemma 4.3. □

Remark. The statement of Lemma 4.4 looks a little complicated due to the rather explicit error terms. Ignoring all small factors, its statement boils down to:

(i) There is always the term

$$L(x) \approx \frac{\mathbb{Q}(A(x))}{\mathbb{Q}(x)}$$

(ii) If $m \in J$, roughly,

$$L(m) \leq \frac{\operatorname{cap}_m(x)}{\mathbb{Q}(m)}$$

(iii) If $m \notin J \cup x$, it is always true that

$$L(m) \leq \frac{\mathbb{Q}(A(m))}{\mathbb{Q}(m)}$$

i.e. roughly of order one. This bound is achieved, if

(iii.1) $\operatorname{cap}_m(J) \ll \operatorname{cap}_m(x)$, whereas in the opposite case
(iii.2) $\operatorname{cap}_m(J) >> \operatorname{cap}_m(x)$,

$$L(m) \leq \max\left(\frac{\operatorname{cap}_m(x)}{\mathbb{Q}(m)}, \frac{\operatorname{cap}_m(x)}{\operatorname{cap}_m(J)}\right)$$

Of course these arguments use that quantities like $\frac{\mathbb{Q}(m)}{\mathbb{Q}(A(m))}$ are not too small, i.e. that the most massive points in a metastable set have a reasonably large mass (compared to say, $\rho(\epsilon)$). If this condition is violated, the idea to represent metastable sets by single points is clearly mislead. We will discuss later what has to be done in such cases.

Taking into account that the $L(m)$ appear with the prefactor $\mathbb{Q}(m)/\mathbb{Q}(x)$ in the expression (4.23), we see that contributions from case (ii) are always sub-dominant; in particular, when $J = \mathcal{M}\backslash x$, the term $m = x$ gives always the main contribution. The terms from case (iii) have a chance to contribute only if $\mathbb{Q}(m) \geq \mathbb{Q}(x)$. If that is the case, and we are in sub-case (iii.1), they indeed contribute, and potentially dominate the sum, whereas in sub-case (iii.2) they never contribute, just as in case (ii).

From Lemma 4.3 we can now derive precise formulae for the mean arrival times in a variety of special cases. In particular,

Theorem 4.1 *Let $x \in \mathcal{M}$ and $J \subset \mathcal{M}\backslash x$ be such a that for all $m \notin J \cup x$ either $\mathbb{Q}(m) \ll \mathbb{Q}(x)$ or $\operatorname{cap}_m(J) m \operatorname{cap}_m(x)$, then*

$$\mathbb{E}_x \tau_J = \frac{\mathbb{Q}(A(x))}{\operatorname{cap}_x(J)}\left(1 + o(1)\right) \tag{4.30}$$

Proof. The proof of this result is straightforward from (4.2), (4.23) and Lemma 4.3. □

Remark. In much the same way one can compute conditional mean times such as $\mathbb{E}_x[\tau_J | \tau_J \leq \tau_I]$. Formulae are given in [8, 9] and we will not go into these issues any further here.

Finally we want to compute the mean time to reach $\mathcal{M}$ starting from a general point.

Lemma 4.5 Let $z \notin \mathcal{M}$. Then

$$\mathbb{E}_z \tau_{\mathcal{M}} \leq a_\epsilon^{-2} \left(|\{y : \mathbb{Q}(y) \geq \mathbb{Q}(z)|\} + C \right) \tag{4.31}$$

Proof. Using Lemma 4.1, we get that

$$\begin{aligned}
\mathbb{E}_z \tau_{\mathcal{M}} &\leq \frac{\mathbb{Q}(z)}{\operatorname{cap}_z(\mathcal{M})} \sum_{y \in \mathcal{M}^c} \frac{\mathbb{Q}(y)}{\mathbb{Q}(z)} \max\left(1, \frac{\operatorname{cap}_y(z)}{\operatorname{cap}_y(\mathcal{M})}\right) \\
&= \frac{\mathbb{Q}(z)}{\operatorname{cap}_z(\mathcal{M})} \sum_{y \in \mathcal{M}^c} \frac{\mathbb{Q}(y)}{\mathbb{Q}(z)} \max\left(1, \frac{\mathbb{P}_y[\tau_z < \tau_y]}{\mathbb{P}_y[\tau_{\mathcal{M}} < \tau_y]}\right) \\
&\leq \sup_{y \in \mathcal{M}^c} \left(\frac{\mathbb{Q}(y)}{\operatorname{cap}_y(\mathcal{M})}\right)^2 \sum_{y \in \mathcal{M}^c} \max\left(\frac{\mathbb{Q}(y)}{\mathbb{Q}(z)}, \mathbb{P}_z[\tau_y < \tau_z]\right) \qquad (4.32) \\
&\leq \sup_{y \in \mathcal{M}^c} \left(\frac{\mathbb{Q}(y)}{\operatorname{cap}_y(\mathcal{M})}\right)^2 \left(\sum_{y : \mathbb{Q}(y) \leq \mathbb{Q}(z)} \frac{\mathbb{Q}(y)}{\mathbb{Q}(z)} + \sum_{y : \mathbb{Q}(y) > \mathbb{Q}(z)} 1 \right) \\
&\leq \sup_{y \in \mathcal{M}^c} \left(\frac{\mathbb{Q}(y)}{\operatorname{cap}_y(\mathcal{M})}\right)^2 \left(C + |\{y : \mathbb{Q}(y) > \mathbb{Q}(z)\}| \right)
\end{aligned}$$

which proves the lemma. □

Remark. If Γ is finite (resp. not growing to fast with ϵ), the above estimate combined with Theorem 4.1 shows that the two definitions of metastability we have given in terms of mean times rep. capacities are equivalent. On the other hand, in the case of infinite state space Γ, we cannot expect the supremum over $\mathbb{E}_z \tau_{\mathcal{M}}$ to be finite, which shows that our first definition was somewhat naive. We will later see that this definition can rectified in the context of spectral estimates.

5. Metastability and Spectral Theory

We now turn to the characterisation of metastability through spectral data. The connection between metastable behaviour and the existence of small eigenvalues of the generator of the Markov process has been realised for a very long time. Some key references are [17, 18, 19, 31, 33, 36, 43, 46, 49, 56, 60, 61]

We will show that Definition 4.1 implies that the spectrum of $1 - P$ decomposes into a cluster of $|\mathcal{M}|$ very small real eigenvalues that are separated by a gap from the rest of the spectrum.

A Priori Estimates. The first step of our analysis consists in showing that the matrix $(1 - P)^{\mathcal{M}}$ that has Dirichlet conditions in all the points of $\mathcal{M}$ has a minimal eigenvalue that is not smaller than $O(a_\epsilon)$.

Lemma 5.1 Let λ^0 denote the infimum of the spectrum of $L_\epsilon^{\mathcal{M}}$. Then

$$\lambda^0 \geq \frac{1}{\sup\limits_{z\in\Omega} \mathbb{E}_z \tau_{\mathcal{M}}} \tag{5.1}$$

Proof. This result is a classical result of Donsker and Varadhan [26] (in the diffusion setting; see [9] for a simple proof in the discrete case). □

In the case when Γ is a finite set, (5.1) together with the estimate of Lemma 4.5 will yield a sufficiently good estimate. If $|\Gamma| = \infty$, the supremum on the right may be infinite and the estimate becomes useless. However, it is easy to modify the proof of Lemma 5.1 to yield an improvement.

Lemma 5.2 Let λ^0 denote the infimum of the spectrum of $(1-P)^{\mathcal{M}}$ and denote by ϕ the corresponding eigenfunction . Let $D \subset \Gamma$ be any compact set, Then

$$\lambda^0 \geq \frac{1}{\sup\limits_{z\in D} \mathbb{E}_z \tau_{\mathcal{M}_\epsilon}} \left(1 - \sum_{y\in D^c} \mathbb{Q}(y)|\phi(y)|^2\right) \tag{5.2}$$

Moreover, for any $\delta > 0$, there exists D finite such that

$$\lambda^0 \geq \frac{1}{\sup\limits_{z\in D} \mathbb{E}_z \tau_{\mathcal{M}_\epsilon}}(1-\delta) \tag{5.3}$$

Proof. Let $w(x)$ denote the solution of the Dirichlet problem

$$\begin{aligned} (1-P)w(x) &= 1, \quad x \in \Gamma\backslash\mathcal{M} \\ w(x) &= 1, \quad x \in \mathcal{M} \end{aligned} \tag{5.4}$$

Recall that $w(x) = \mathbb{E}_x \tau_{\mathcal{M}_\epsilon}$. Using that for any $C > 0$, $ab \leq \frac{1}{2}(Ca^2 + b^2/C)$ with $ab = \phi(x)\phi(y)$ and $C = w(y)/w(x)$, one shows readily that

$$\begin{aligned} \sum_{x\in\Gamma\backslash\mathcal{M}} \mathbb{Q}(x)\phi(x)(1-P)\phi(x) &\geq \sum_{x\in\Gamma\backslash\mathcal{M}} \mathbb{Q}(x)\tfrac{\phi(x)}{w(x)}(1-P)w(x)\phi(x) \\ &= \sum_{x\in\Gamma\backslash\mathcal{M}} \mathbb{Q}(x)\tfrac{\phi(x)}{w(x)}\phi(x) \\ &\geq \frac{1}{\sup\limits_{x\in D} w(x)} \sum_{x\in D\cap\Gamma\backslash\mathcal{M}} \phi^2(x) \end{aligned} \tag{5.5}$$

Choosing ϕ as the normalized eigenfunction with maximal eigenvalue yields (5.2).

We now claim that for any $\gamma > 0$,

$$\sum_{y\in\Gamma} \mathbb{Q}(y)^{-\gamma}|\phi(y)|^2 < C_\gamma < \infty \tag{5.6}$$

This clearly implies (5.7). The estimate (5.6) follows from a standard Combes-Thomas estimate for the ground-state eigenfunction, ϕ. It is convenient to introduce $v(y) \equiv \mathbb{Q}(y)^{1/2}\phi(y)$, which is the corresponding ground state eigenfunction of the operator

$$H_\epsilon \equiv \mathbb{Q}(y)^{1/2}(1-P)\mathbb{Q}(y)^{-1/2} \tag{5.7}$$

which is a symmetric operator on $\ell^2(\Gamma)$. By a standard computation,

$$t(\alpha)[u] \equiv \sum_{y\in\Gamma} u^*(y)\mathbb{Q}(y)^{-i\alpha}H_\epsilon\mathbb{Q}(y)^{i\alpha}u(y) \tag{5.8}$$

defines a closed sectoral form (in the sense of Kato [38]), which is analytic in the strip $|\Im\alpha| < 1/2$. An adaptation of the Combes-Thomas estimate (see e.g. [53]) then implies that v satisfies

$$\sum_{y\in\mathbb{Q}} \mathbb{Q}(y)^{(1-\gamma)}|v(y)|^2 < C_\gamma < \infty \tag{5.9}$$

which is equivalent to (5.6). This completes the proof of the lemma. $\square$

If we combine this result with the estimate from Lemma 4.5, we obtain the following proposition.

Proposition 5.1 *Let λ^0 denote the principal eigenvalue of the operator $(1-P)^{\mathcal{M}}$. Then there exists a constant $C > 0$, independent of ϵ, such that for all ϵ small enough,*

$$\lambda^0 \geq Ca_\epsilon^{-2} \tag{5.10}$$

Remark. Proposition 5.1 links the fast time scale to the smallest eigenvalue of the Dirichlet operator, as should be expected. Note that the relation is not very precise. We will soon derive a much more precise relation between times and eigenvalues for the cluster of small eigenvalues.

Characterization of Small Eigenvalues. We will now obtain a representation formula for all eigenvalues that are smaller than λ^0. It is clear that there will be precisely $|\mathbb{M}|$ such eigenvalues. This representation was first exploited in [9], but already in 1973 Wentzell put forward very similar ideas (in the case of general Markov processes). As will become clear (hopefully more so then on the original paper [9], this is extremely simple in the context of discrete processes (see [11] for the more difficult continuous case.

The basic idea is to use the fact that the solution of the Dirichlet problem

$$\begin{aligned} (L-\lambda)f(x) &= 0, \quad x \notin \mathcal{M} \\ f(x) &= \phi_x, \quad x \in \mathcal{M} \end{aligned} \tag{5.11}$$

already solves the eigenvalue equation $L\phi(x) = \lambda\phi(x)$ everywhere except possibly on $\mathcal{M}$. The question if whether an appropriate choice of boundary conditions and the right choice of the value of λ will actually lead to a solution. This is indeed the case.

Lemma 5.3 Assume that $\lambda < \lambda^0$ is an eigenvalue of L and $\phi(x)$ is the corresponding eigenfunction. Then the unique solution of (5.11) with $\phi_x = \phi(x)$, $x \in \mathcal{M}$, satisfies $f(y) = \phi(y)$, for all $y \in \Gamma$.

Proof. Since $\phi(x) = f(x)$ on $\mathcal{M}$, we have that $(f-\phi)(x)$ solves the Dirichlet problem

$$\begin{aligned} (L-\lambda)^{\mathcal{M}}(f-\phi)(x) &= 0, \quad x \notin \mathcal{M} \\ (f-\phi)(x) &= 0, \quad x \in \mathcal{M} \end{aligned} \tag{5.12}$$

But since λ is not in the spectrum of $L_\epsilon^{\mathcal{S}_k}$, (5.12) has a unique solution, which is identically zero, so that $f(x) = \phi(x)$ on Γ, which proves the lemma. □

From the lemma we conclude that we can find all eigenfunctions corresponding to eigenvalues smaller than λ^0 among the solutions of the Dirichlet problems (5.11).

Let now f be a solution of (5.11) with $\lambda < \lambda^0$. Clearly, f is an eigenfunction with eigenvalue λ, if

$$(L-\lambda)f(x) = 0, \quad x \in \mathcal{M} \tag{5.13}$$

Thus we need to compute the left-hand side of (5.13). Now $f(y)$ can be represented in terms of the equilibrium potentials $h^\lambda_{x,\mathcal{M}\setminus x} \equiv h^\lambda_x(y)$ defined in (3.3) as $f(y) = \sum_{x\in\mathcal{M}} \phi_x h^\lambda_x(y)$. Thus

$$(L-\lambda)f(x) = \sum_{z\in\mathcal{M}} \phi_z (L-\lambda)h^\lambda_z = \sum_{z\in\mathcal{M}} \phi_z e^\lambda_{z,\mathcal{M}\setminus z}(x) \tag{5.14}$$

where we used (3.7). Let us denote by $\mathcal{E}_{\mathcal{M}}(\lambda)$ the $|\mathcal{M}| \times |\mathcal{M}|$- matrix with elements

$$(\mathcal{E}_{\mathcal{M}}(\lambda))_{xy} \equiv e^\lambda_{z,\mathcal{M}\setminus z}(x) \tag{5.15}$$

We can then conclude that:

Lemma 5.4 A number $\lambda < \lambda^0$ is an eigenvalue of the matrix $L = (1-P)$ if and only if

$$\det \mathcal{E}_{\mathcal{M}}(\lambda) = 0 \tag{5.16}$$

Anticipating that we are interested in small λ, we want to re-write the matrix $\mathcal{E}_{\mathcal{M}}$ in a more convenient form. To do so let us set

$$h_x^\lambda(y) \equiv h_x(y) + \psi_x^\lambda(y) \tag{5.17}$$

where $h_x(y) \equiv h_x^0(y)$ and consequently $\psi_x^\lambda(y$ solves the inhomogeneous Dirichlet problem

$$\begin{aligned} (L-\lambda)\psi_x^\lambda(y) &= \lambda h_x(y), \quad y \in \Gamma\backslash\mathcal{M} \\ \psi_x^\lambda(y) &= 0, \quad y \in \mathcal{M} \end{aligned} \tag{5.18}$$

Lemma 5.5

$$\begin{aligned} (\mathcal{E}_{\mathcal{M}}(\lambda))_{xz} = \mathbb{Q}(x)^{-1} \quad & \left(\frac{1}{2}\sum_{y\neq y'} \mathbb{Q}(y')p(y',y)[h_z(y')-h_z(y)][h_x(y')\right. \\ & \left. -h_x(y)] - \lambda\sum_y \mathbb{Q}(y)\left(h_z(y)h_x(y) + h_x(y)\psi_z^\lambda(y)\right)\right) \end{aligned} \tag{5.19}$$

Proof. Note that

$$\begin{aligned} (L-\lambda)h_z^\lambda(x) &= (L-\lambda)h_z(x) + (L-\lambda)\psi_z^\lambda(x) \\ &= Lh_z(x) - \lambda h_z(x) + (L-\lambda)\psi_z^\lambda(x) \end{aligned} \tag{5.20}$$

Now by adding a huge zero,

$$\begin{aligned} Lh_z(x) \quad &= \mathbb{Q}(x)^{-1}\sum_{y'\in\Gamma} \mathbb{Q}(y')h_x(y')Lh_z(y') \\ &= \mathbb{Q}(x)^{-1}\tfrac{1}{2}\sum_{y,y'\in\Gamma} \mathbb{Q}(y')p(y',y))[h_z(y')-h_z(y)][h_x(y')-h_x(y)] \end{aligned} \tag{5.21}$$

Similarly,

$$\begin{aligned} &(L-\lambda)\psi_z^\lambda(x) = \\ &\qquad \mathbb{Q}(x)^{-1}\sum_{y'\in\Gamma}\left(\mathbb{Q}(y')h_x(y')(L-\lambda)\psi_z^\lambda(y') - \lambda 1\!\!1_{y'\neq x}h_x(y')h_z(y')\right) \end{aligned} \tag{5.22}$$

Since $\psi_z^\lambda(y) = 0$ whenever $y \in \mathcal{M}$, and $Lh_x(y)$ vanishes whenever $y \notin \mathcal{M}$, using the symmetry of L, we get that the right-hand side of (5.22) is equal to

$$-\lambda\mathbb{Q}(x)^{-1}\sum_{y'\in\Gamma}\left(\mathbb{Q}(y')h_x(y')(\psi_z^\lambda(y') + 1\!\!1_{y'\neq x}h_x(y')h_z(y')\right) \tag{5.23}$$

Adding the left-over term $-\lambda h_z(x) = -\lambda h_x(x) h_z(x)$ from (5.1) to (5.22), we arrive at (5.19). $\square$

Remark. Note that we get an alternative probabilistic interpretation of $Lh_z(x)$ as

$$Lh_z(x) = \begin{cases} -\mathbb{P}_z[\tau_x \leq \tau_{\mathcal{M}}], & \text{if} \quad z \neq x \\ \mathbb{P}_x[\tau_{\mathcal{M}\setminus x} < \tau_x] & \text{if} \quad z = x \end{cases} \tag{5.24}$$

Note that the off-diagonal quantities can in turn be expressed via the renewal equations as

$$\mathbb{P}_z[\tau_x \leq \tau_{\mathcal{M}}] = \mathbb{P}_z[\tau_x < \tau_{\mathcal{M}\setminus x}]\mathbb{P}_z[\tau_{\mathcal{M}\setminus z} < \tau_z] = h_{x,\mathcal{M}\setminus x}(z) e_{z,\mathcal{M}\setminus z}(z) \tag{5.25}$$

We will see in Section 6 that the equilibrium potential $h_{x,\mathcal{M}\setminus x}(z)$ can be estimated along with the capacities rather well.

We are now in a position to relate the small eigenvalues of $(1-P)$ to the eigenvalues of the classical capacity matrix. Let us denote by $\|f\|_2$ the ℓ^2-norm with respect to the measure $\mathbb{Q}$, i.e. $\|f\|_2^2 = \sum_y \mathbb{Q}(y) f(y)^2$.

Theorem 5.1 *If $\lambda < \lambda^0$ is an eigenvalue of L, then there exists an eigenvalue μ of the $|\mathcal{M}| \times |\mathcal{M}|$-matrix $\mathcal{K}$ whose matrix elements are given by*

$$\mathcal{K}_{zx} = \frac{\frac{1}{2}\sum_{y \neq y'} \mathbb{Q}(y') p(y', y)[h_z(y') - h_z(y)][h_x(y') - h_x(y)]}{\|h_z\|_2 \|h_x\|_2} \tag{5.26}$$

such that $\lambda = \mu\,(1 + O(\rho(\epsilon))$.

Proof. The proof will rely on the following general fact.

Lemma 5.6 Let A be a finite dimensional self-adjoint matrix. Let $B(\lambda)$ a Lipshitz continuous family of bounded operators on the same space that satisfies the bound $\|B(\lambda)\| \leq \delta + \lambda C$, and $\|B(\lambda) - B(\lambda')\| \leq C\|\lambda - \lambda'\|$ for $0 \leq \delta \ll 1$, and $0 \leq C < \infty$. Assume that A has k eigenvalues $\lambda_1, \ldots, \lambda_k$ in an interval $[0, a]$ with $a < \delta/C$. Then

(i) Any solution λ_i' of the equation

$$\det(A - \lambda(\mathbb{1} + B(\lambda))) = 0 \tag{5.27}$$

satisfies $|\lambda_i' - \lambda_i| \leq 4\delta\lambda_i$, for some $i = 1, \ldots, k$.

(ii) There exists $\delta_0 > 0$ and $a_0 > 0$ such that for all $\delta < \delta_0$, and $a < a_0$, Equation 5.27 has exactly k solutions $l_1', \ldots, \lambda_k'$.

(iii) If the eigenvalue λ_i is simple and isolated with $\min_{j \neq i} |\lambda_i - \lambda_j| \geq 2\delta\lambda_i$, then, if λ_i' is a solution of 5.27 with $|\lambda_i' - \lambda_i| \leq 4\delta\lambda_i$, ther exists a unique solution c of the equation

$$(A - \lambda_i'(\mathbb{1}_i' + B(\lambda_i')))c = 0 \tag{5.28}$$

Moreover, if c_i denotes the normalized eigenfunction of A with eigenvalue λ_i, then

$$\|c - c_i\|_2 \leq 2\lambda_i' \tag{5.29}$$

A proof of this lemma can be found in the appendix of [11].

First we divide the row x of the matrix $\mathcal{E}(\lambda)$ by $\|h_x\|_2$ and multiply the columns by $\|h_z\|_2$ to obtain a matrix $\mathcal{G}(\lambda)$ with identical determinant that can be written as

$$\mathcal{G}(\lambda) = \mathcal{K} - \lambda\mathbb{1} - \lambda B(\lambda) \tag{5.30}$$

where

$$B(\lambda)_{zx} = \frac{\sum_y \mathbb{Q}(y) h_z(y) h_x(y) \left(\mathbb{1}_{x\neq y} + \psi_x^\lambda(y)/h_x(y)\right)}{\|h_z\|_2 \|h_x\|_2} \tag{5.31}$$

Note that $\mathcal{G}(\lambda)$ is symmetric. We must estimate the operator norm of $B(\lambda)$. We will use the corresponding standard estimator $\left(\sum_{x,z\in\mathcal{M}} B_{zx}^2\right)^{1/2}$.

We first deal with the off-diagonal elements that have no additional λ or other small factor in front of them.

Lemma 5.7 There is a constant $C < \infty$ such that

$$\max_{x\neq z\in\mathcal{M}} \frac{\sum_{y\in\Gamma} \mathbb{Q}(y) h_x(y) h_z(y)}{\|h_x\|_2 \|h_z\|_2} \leq C a_\epsilon^{-1} \max_{m\in\mathcal{M}} \mathbb{Q}(m)^{-1} \mathrm{cap}_m(\mathcal{M}\backslash m) \leq \rho(\epsilon) \tag{5.32}$$

Proof. Note first by the estimate (4.4) the equilibrium potentials $h_x(y)$ are essentially equal to one on $A(x)$. Thus the denominator in (5.32) is bounded from below by

$$\sqrt{\sum_{y\in A(x)} \mathbb{Q}(y) h_x^2(y) \sum_{y\in A(y)} \mathbb{Q}(y) h_z^2(y) h_i^2(y)} \geq \sqrt{\mathbb{Q}(A(x))\mathbb{Q}(A(z))} \tag{5.33}$$

To bound the numerators, we will use Lemma 4.3 in the special situation when $J = \mathcal{M}\backslash x$.

Lemma 5.8 For any $x \neq z \in \mathcal{M}$,

$$\sum_{y\in\Gamma} \mathbb{Q}(y) h_x(y) h_z(y) \leq C\rho(\epsilon)\sqrt{\mathbb{Q}(x)\mathbb{Q}(z)} \tag{5.34}$$

Proof. By (ii) of Lemma 4.4, if $y \in A(m)$, then

(i) If $m = z$, either

$$\mathbb{Q}(y) \leq \frac{3}{2}\mathbb{Q}(x)a_\epsilon^{-1}|\mathcal{M}|\frac{\mathrm{cap}_m(x)}{\mathbb{Q}(x)} \tag{5.35}$$

or

$$\mathbb{Q}(y)h_x(y)h_z(y) \leq \frac{3}{2}\mathbb{Q}(x)a_\epsilon^{-1}|\mathcal{M}|\frac{\mathrm{cap}_m(x)}{\mathbb{Q}(x)} \tag{5.36}$$

(ii) If $m = x$,

$$\mathbb{Q}(y) \leq \frac{3}{2}\mathbb{Q}(z)a_\epsilon^{-1}|\mathcal{M}|\frac{\mathrm{cap}_m(z)}{\mathbb{Q}(z)} \tag{5.37}$$

or

$$\mathbb{Q}(y)h_x(y)h_z(y) \leq \frac{3}{2}\mathbb{Q}(z)a_\epsilon^{-1}|\mathcal{M}|\frac{\mathrm{cap}_m(z)}{\mathbb{Q}(z)} \tag{5.38}$$

(iii) Let $m \notin \{x, z\}$, and assume w.r.g. that $\mathrm{cap}_m(x) \geq \mathrm{cap}_m(z)$. Then, if $\mathrm{cap}_m(y) > 3\mathrm{cap}_m(z)$, already

$$\mathbb{Q}(y)\sqrt{h_x(y)h_z(y)} \leq \frac{3}{2}\sqrt{\mathbb{Q}(x)\mathbb{Q}(z)}a_\epsilon^{-1}|\mathcal{M}|\sqrt{\frac{\mathrm{cap}_m(x)\mathrm{cap}_m(z)}{\mathbb{Q}(x)\mathbb{Q}(z)}} \tag{5.39}$$

while otherwise

$$\mathbb{Q}(y) \leq 3\mathbb{Q}(y)\frac{\mathrm{cap}_m(z)}{\mathrm{cap}_y(m)} \leq 3a_\epsilon^{-1}|\mathcal{M}|\sqrt{\mathrm{cap}_m(z)\mathrm{cap}_m(x)} \tag{5.40}$$

Summing over y yields e.g. in case (i)

$$\begin{aligned}\sum_{y\in A(m)} \mathbb{Q}(y)h_x(y)h_y(y) \leq \quad & C\left|\left\{y \in A(m) : \mathbb{Q}(y) \geq 32a_\epsilon^{-1}|\mathcal{M}|\mathrm{cap}_m(x)\right\}\right| \\ & \times a_\epsilon^{-1}|\mathcal{M}|\mathrm{cap}_m(x)\end{aligned} \tag{5.41}$$

and in case (ii) the same expression with x replaced by z. The case (iii) is concluded in the same way.

This implies the statement of the lemma. □

Remark. Note that the estimates in the proof of Lemma 5.7 also imply that

$$\mathbb{Q}(A(x)(1 - O(\rho(\epsilon))\sum_y \mathbb{Q}(y)h_x(y)^2 \leq \mathbb{Q}(A(x)(1 + O(\rho(\epsilon)) \tag{5.42}$$

The remaining contribution to the matrix elements of $B(\lambda)$ are of order λ, and thus the crudest estimates will suffice:

Lemma 5.9 If λ^0 denotes the principal eigenvalue of the operator L with Dirichlet boundary conditions in $\mathcal{M}$, then

$$\left|\sum_{y\in\Gamma} \mathbb{Q}(y) \left(h_z(y)\psi_x^\lambda(y)\right)\right| \leq \frac{\lambda}{(\lambda^0-\lambda)}\|h_z\|_2\|h_x\|_2 \tag{5.43}$$

Proof. Recall that ψ_x^λ solves the Dirichlet problem (5.18). But the Dirichlet operator $L^{\mathcal{M}} - \lambda$ is invertible for $\lambda < \lambda^0$ and is bounded as an operator on $\ell^2(\Gamma, \mathbb{Q})$ by $1/(\lambda^0 - \lambda)$. Thus

$$\|\psi_x^\lambda\|_2^2 \leq \left(\frac{\lambda}{\lambda^0-\lambda}\right)^2 \|h_x\|_2^2 \tag{5.44}$$

The assertion of the lemma now follows from the Cauchy-Schwartz inequality. □

As a consequence of the preceding lemmata, we see that the matrix $B(\lambda)$ is indeed bounded in norm by

$$\|B(\lambda)\| \leq C\rho(\epsilon) + c\frac{\lambda}{\lambda^0-\lambda} \tag{5.45}$$

The theorem follows from Lemma 5.6. □

The computation of the eigenvalues of the capacity matrix is now in principle a finite, though in general not trivial problem. The main difficulty is of course the computation of the capacities and induction coefficients. Capacities can be estimated quite efficiently, as we will see in the next section, the off-diagonal terms however, pose in general a more serious problem, although in many practical cases exact symmetries may be very helpful. On the other hand, a particularly nice situation arises when *no* symmetries are present.

In fact we will prove the following theorem.

Theorem 5.2 *Assume that there exists $x \in \mathcal{M}$ such that for some $\delta \ll 1$*

$$\frac{\mathrm{cap}_x(\mathcal{M}\backslash x)}{\|h_x\|_2^2} \geq \delta \max_{z\in\mathcal{M}\backslash x} \frac{\mathrm{cap}_z(\mathcal{M}\backslash z)}{\|h_z\|_2^2} \tag{5.46}$$

Then the largest eigenvalue of L is given by

$$\lambda_x = \frac{\mathrm{cap}_x(\mathcal{M}\backslash x)}{\|h_x\|_2^2}(1+O(\delta)) \tag{5.47}$$

and all other eigenvalues of L satisfy

$$\lambda \leq C\delta\lambda_x \tag{5.48}$$

Moreover, the eigenvector, ϕ, corresponding to the largest eigenvalues normalized s.t. $\phi_x = 1$ satisfies $\phi_z \leq C\delta$, for $z \neq x$.

Proof. This is a simple perturbation argument. Note that we can write

$$\mathcal{K} = \hat{\mathcal{K}} + \check{\mathcal{K}} \tag{5.49}$$

where $\hat{\mathcal{K}}_{uv} = \mathcal{K}_{xx}\delta_{xv}\delta_{xv}$. Now we estimate the norm of $\check{\mathcal{K}}$.

By the Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\tfrac{1}{2}\sum_{y,y'} \mathbb{Q}(y')p(y',y)[h_x(y')-h_x(y)][h_z(y')-h_z(y)]\right| \\ \leq \sqrt{\mathrm{cap}_x(\mathcal{M}\backslash x)\mathrm{cap}_z(\mathcal{M}\backslash z)}\end{aligned} \tag{5.50}$$

Thus

$$\left|\frac{q_z}{q_x}\mathcal{K}_{zx}^2\right| \leq \mathcal{K}_{xx}\mathcal{K}_{zz} \tag{5.51}$$

Whence by assumption,

$$\|\check{\mathcal{K}}\| \leq \mathcal{K}_{xx}\sqrt{\delta|\mathcal{M}| + \delta^2|\mathcal{M}|^2} \tag{5.52}$$

Since obviously $\hat{\mathcal{K}}$ has one eigenvalue $\mathcal{K}_{xx}$ with the obvious eigenvector and all other eigenvalues are zero, the announced result follows from standard perturbation theory. $\square$

Theorem 5.2 has the following simple corollary, that allows in many situations a complete characterization of the small eigenvalues of L.

Corollary 5.1 *Assume that we can construct a sequence of metastable sets $\mathcal{M}_k \supset \mathcal{M}_{k-1} \supset \ldots \supset \mathcal{M}_2 \supset \mathcal{M}_1 = x_0$, such that for any i, $\mathcal{M}_i\backslash\mathcal{M}_{i-1} = x_i$ is a single point, and that each $\mathcal{M}_i$ satisfies the assumptions of Theorem 5.2. Then L has k eigenvalues*

$$\lambda_i = \frac{\mathrm{cap}_{x_i}(\mathcal{M}_{i-1})}{\mathbb{Q}(A(x_i))}\left(1+O(\delta)\right) \tag{5.53}$$

The corresponding normalized eigenfunction is given by

$$\psi_i(y) = \frac{h_{x_i}(y)}{\|h_{x_i}\|_2} + O(\delta) \tag{5.54}$$

6. Computation of Capacities

We have seen so far that in metastable dynamics we can largely reduce the computation of key quantities to the computation of *capacities*. The usefulness of all this thus depends on how well we can compute capacities.

While clearly the universality of our approach ends here, and model specific properties have to enter the game, it is rather surprising to what extent precise computations of capacities are possible in a multitude of specific systems.

6.1. GENERAL PRINCIPLES

The key to success is the variational representation of capacities through the Dirichlet principle, i.e. Eq. (3.11). The Dirichlet principle immediately yields two avenues towards bounds:

- Upper bounds via judiciously chosen test functions
- Lower bounds via monotonicity of the Dirichlet form in the transition probabilities via simplified processes.

These two principles are well-known and give rise to the so-called "Rayleigh's short-cut rules" in the language of electric networks (see e.g. [25] and references wherein). In the context of metastable systems, the usefulness of these principle can be enhanced by an iterative method.

The key idea of iteration is to get first of all control of the minimizer in the Dirichlet principle, i.e. the equilibrium potential. In metastable systems, when we are interested in computing e.g. the capacity $\mathrm{cap}_{B_x}(B_y)$ where B_x, B_y represent two metastable sets, our first goal will always be to identify domains where $h_{B_x,B_y}(z)$ is close to zero or close to one. This is done with the help of the renewal estimate of Lemma 4.1. While with looks cyclic at first glance (we need to know the capacities in order to estimate the equilibrium potential, which we want to use in order to estimate capacities....) it yields a tool to enhance "poor" bounds in order to get good ones. Thus the first step in the program is to get a first estimate on capacities of the form $\mathrm{cap}_z(B)$ for arbitrary z, B.

(i) Choose a roughly ok looking test function for the upper bound.
(ii) Dramatically simplify the state space of the process to obtain a system that can be solved exactly for the lower bound. In most examples, this leads to choosing a one-dimensional or quasi-one-dimensional system.
(iii) Insert the resulting bounds in (4.4) to obtain bounds on $h_{B_x,B_y}(z)$.

Using this bound we can now identify the set

- $D_x \equiv \{z : h_{B_x,B_y}(z) < \delta\}$
- $D_y \equiv \{z : h_{B_x,B_y}(z) > 1-\delta\}$ for $\delta \ll 0$ suitably chosen.
  If the complement of the set $D_x \cup D_y$ contains no further metastable set, we define
- $I \equiv \{z \in (D_x \cup D_y)^c : \mathbb{Q}(z) < \rho \sup_{w\in(D_x\cup D_y)^c} \mathbb{Q}(w)\}$ for $\rho \ll 1$ conveniently chosen.

Let us denote by $\mathcal{S} \equiv (D_x \cup D_y \cup I)^c$.

The idea is that the set I will be irrelevant for the value of the capacity, no matter what value $h_{B_x,B_y}(z)$ takes where, and that the sets D_x and D_y give no contribution to the capacity to leading order. The only problem is thus to find the equilibrium potential, or a reasonably good approximation to it on the set $\mathcal{S}$. We return to this problem shortly. Of course this idea can only make sense if the sets D_x and D_y can be connected through $\mathcal{S}$. If that is not the case, we will have to analyse the set $(D_x \cup D_y)^c$ more carefully.

$D_x \cup D_y$ contains further metastable sets, say w, then it will be possible to identify domains D_w on which $h_{B_x,B_y}(z)$ takes on a constant values c_w (to be determined later). Note that this can be done again with the help of the renewal bounds; The starting point (in the discrete case) is of course the observation that

$$\begin{aligned} h_{B_x,B_y}(z) &= \mathbb{P}_z[\tau_{B_x} < \tau_{B_y}] \\ &= \mathbb{P}_z[\tau_{B_x} < \tau_{B_y}, \tau_w < \tau_{B_x}] + \mathbb{P}_z[\tau_{B_x} < \tau_{B_y}, \tau_w \geq \tau_{B_x}] \\ &= \mathbb{P}_z[\tau_w < \tau_{B_x \cup B_y}]\mathbb{P}_w[\tau_{B_x} < \tau_{B_y}] \\ &= \mathbb{P}_z[\tau_w < \tau_{B_x \cup B_y}]c_w \end{aligned} \tag{6.1}$$

The problem, to be solved with the help of (4.4) and the a priori bounds on capacities is thus to determine the set of points for which $\mathbb{P}_z[\tau_w < \tau_{B_x \cup B_y}] > 1 - \delta$.

Once with is done, we proceed as in the former case, but increasing the set $D_x \cup D_y$ in their definition of I to $D \equiv D_x \cup D_y \cup D_{w_1} \cup \ldots \cup D_{W_k}$ if k such sets can be identified. It should now be the case that the set $D_x \cup D_y \cup D_{w_1} \cup \ldots \cup D_{W_k} \cup \mathcal{S}$ is connected. The remaining problem consists then in the determination of the equilibrium potential on the set $\mathcal{S}$ and of the values c_{w_i}.

At this stage we can then obtain upper and lower bounds in terms of variational problems that involve only the sets $\mathcal{S}$; to what extend these problems then can be solved depends on the problem at hand.

Upper Bound. To obtain the upper bound, we choose a test-function h^+ with the properties that

$$\begin{aligned} h^+(z) &= 1, \quad z \in D_x \\ h^+(z) &= 0, \quad z \in D_y \\ h^+(z) &= c_{w_i}, \quad z \in D_{w_i} \end{aligned} \tag{6.2}$$

where the constants c_{w_i} are determined only later. On I, the function h^+ can be chosen essentially arbitrarily, while on $\mathcal{S}$, we chose h^+ such that it optimizes the restriction of the Dirichlet form to $\mathcal{S}$ with boundary conditions implied by (6.2) on $\partial\mathcal{S} \cap \partial D$. Finally, the constants c_{w_i} are determined by minimizing the result as a function of these constants.

Lower Bound. For the lower bound, we use that if h^* denotes the true minimizer, then

$$\Phi(h^*) \geq \Phi_{\mathcal{S}}(h^*) \tag{6.3}$$

where $\Phi_{\mathcal{S}}$ is the restriction of the Dirichlet form to the subset $\mathcal{S}$, i.e.

$$\Phi_{\mathcal{S}}(h) = \frac{1}{2} \sum_{\substack{x \vee y \in \mathcal{S} \\ x \wedge y \in \mathcal{S} \cup D}} \mathbb{Q}(x) p(x,y) [h(x) - h(y)]^2 \tag{6.4}$$

Finally we minorise $\Phi_{\mathcal{S}}(h^*)$ by taking the infimum over all h on $\mathcal{S}$, with boundary conditions imposed by what we know a priori about the equilibrium potential. In particular we know that these boundary conditions are close to constants on the different components of D. Of course we do not really know the constants c_{w_i}, but taking the infimum over these, we surely are on the safe side.

Thus, if we can show that the minimizers in the lower bound differ little from the minimizers with constant boundary conditions, we get upper and lower bounds that coincide up to small error terms. Of course in general, it may remain difficult to actually compute these minimizers. However, the problem is greatly reduced in complexity with respect to the original problem, and in many instances this problem can be solved quite explicitly (see [8, 12]).

7. What to do when Points are too Small?

In the previous chapters we have mainly relied on the fact that by removing individual points from state space we already lifted the spectrum of the generator beyond the small eigenvalues corresponding to metastable transitions, or, in other terms, the fact that a set of points is reached in times much smaller that metastable transition times. This is not the case when the cardinality of phase space is too large, or even uncountable. Obvious examples are diffusion processes and spin systems at finite temperature.

Such situations require always some coarse-graining of state space. Metastable sets are then no longer collections of points, but collections of disjoint subsets. This coarse graining causes problems. One concerns the use of the fundamental relation (3.6) that cannot immediately be used to obtain a formula for the Green's function à la (3.25), if A cannot be chosen a single point. The solution must then be to find admissible sets A on those boundary $G_\Omega(x,y)$ is constant or varies very little. Of course such a procedure requires some a priori knowledge about the Green's function. A very similar problem arises in the use of renewal arguments to derive Lemma 4.1.

So far, we have no general rule for how to proceed in such situations. In the case of diffusion processes, it has turned out that general elliptic regularity theory (based on Harnack and Hölder inequalities) allows to obtain all desired results by replacing points with suitable ϵ-dependent balls around points. The details can be found in [10, 11], and we will not go into these problems.

A second method proved useful in the context of certain mean field spin systems and was successfully used in [8, 1] is "lumping" [40]. As this method is the basis of some of the estimates that are crucial in the analysis of ageing in the Random Energy Model (REM) that we will discuss later, we present this in some detail here, even though it is rather extensively presented in [8, 1].

Lumping. Let us suppose we have an ergodic reversible Markov chain those invariant measure is constant on the level set of some function $m : \Gamma \to \Sigma$. Let us further assume that we are interested only in events that can be expressed in terms of m. Of course the idea will always be that Σ should be a much smaller space than Γ. In such a situation it may appear natural to define metastable sets in terms of subsets of Σ rather than of Γ.

Example 7.1 Curie-Weiss Model. The simplest example of this type is furnished by the Curie-Weiss model. Here $\Gamma = \{-1, 1\}^N$, $\Sigma = \{-1, -1 + 2/N, \dots, 1-2/N, 1\}$ and $m(x) = N^{-1} \sum_{i=1}^{N} x_i$. The invariant measure is given by

$$\mathbb{Q}(x) = \mathbb{Q}_{\beta,N} \equiv \frac{e^{\beta N m(x)^2}}{Z_\beta} \tag{7.1}$$

and the transition probabilities are e.g. (for $x \neq y$)

$$p(x, y) = N^{-1} e^{-\mathbb{N}[m(y)-m(x)]_+} \mathbb{1}_{\|x-y\|_2=2} \tag{7.2}$$

where $[\cdot]_+$ denote the positive part of $\cdot$. If $\beta > 1$, the measure $\mathbb{Q}_{\beta,N}$ concentrates (for large N) near the points $\pm m^*(\beta)$ where $m^*(\beta) = \tanh(\beta m^*(\beta)) > 0$. Thus it would be natural to assume that a metastable set for our Markov chain could consist of the two subsets $M_\pm \equiv \{x \in \Gamma : m(x) \approx \pm m^*(\beta)$.

It is instructive to review our basic notions in this context. We see that the definition of metastability given in Section 4 may now not be very appropriate since it would involve ratios of quantities such as $\mathbb{P}_x[\tau_{M_+\cup M_-} < \tau_x]$ and $\mathbb{P}_x[\tau_{M_+} < \tau_x]$ which might tent to be close to one simply because of entropic reasons it is very difficult for a process starting in x to ever return to that point before an exponentially long time. Looking back at our tentative definition in Section 2, this now seems to be more promising, as we still expect the mean times for arrival in $M_- \cup M_+$ to be much shorter than than transition times between M_- and M_+. The question is whether and when we can re-express such mean times in terms of capacities?

To understand this issue, recall that by (3.23), we have that (e.g.)

$$\mathbb{E}_x \tau_{M_+} = \frac{\sum_{y \notin M_+ \cup x} \mathbb{Q}(y) h_{x,M_+}(y)}{\mathrm{cap}_x(M_+)} \tag{7.3}$$

In fact, this formula is quite annoying since again the denominator is difficult to evaluate due to the fact that we anticipate some hard to control local behaviour of the equilibrium potential $h_{x,M_+}(y)$ near x.

In fact, it would be highly desirable if we could obtain a formula where only capacities of "fat" sets, e.g. sets that can be described in the form $\{x : m(x) \in A \subset [-1,1]\}$ enter. to do so let us go back to the original form of (3.6). If we multiply equation (3.22) on both side by $e_{M(x),M_+}(y)$, where $M(x) \equiv \{y : m(y) = m(x)\}$ and then sum over $M(x)$, we get

$$\sum_{y \in M(x)} \mathbb{Q}(x) \mathbb{E}_x \tau_{M_+} e_{M(x),M_+}(y) = h_{x,M_+}(y) \sum_{x \in M(x)} \sum_{y \notin M_+ \cup x} \mathbb{Q}(y) \tag{7.4}$$

Let us first consider the right-hand side. Since $\mathbb{Q}(y)$ depends only on $m(y)$, we can write

$$\sum_{x \in M(x)} \sum_{y \notin M_+ \cup x} \mathbb{Q}(y) h_{x,M_+}(y) = \sum_{y \notin M_+ \cup x} \mathbb{Q}(y) h_{M(x),M_+}(y) \tag{7.5}$$

Now if we knew that $h_{M(x),M_+}(y) = g_{M(x),M_-}(m(y))$ depended only on $m(y)$, then this would become simply

$$\sum_{m \neq m^*} \mathbb{Q}(M(m)) g_{x,M_+}(m) \tag{7.6}$$

On the other hand, if $\mathbb{E}_x \tau_{M_+} = f_{m(x)}(M_+)$ was a function of $m(x)$ only, the left hand side of (7.4) would reduce to

$$\mathrm{cap}_{M(x)}(M_+) \mathbb{E}_x \tau_{M_+} \tag{7.7}$$

and we would have the nice formula

$$\mathbb{E}_x \tau_{M_+} = \frac{1}{\mathrm{cap}_{M(x)}(M_+)} \sum_{m \neq m^*} \mathbb{Q}(M(m)) g_{M(x),M_+}(m) \tag{7.8}$$

In fact, it is easy to see that in our simple model, both properties are in fact verified. The reason for this is that the transition probabilities verify the property that for any $x, x' \in \Gamma$ such that $m(x) = m(x')$ and $m' \in [0,1]$,

$$\sum_{y: m(y) = m'} p(x,y) = \sum_{y: m(y) = m'} p(x',y) \tag{7.9}$$

In fact it is an old result due to Burke and Rosenblatt [13] that condition (7.9) is necessary and sufficient for the fact that the image process $m(t) \equiv m(x(t))$ is a Markov chain on $m(\Gamma)$ with transition rates given by

$$r(m, m') = \sum_{y: m(y) = m'} p(x, y) \tag{7.10}$$

where x is any point such that $m(x) = m$. We refer to the chain $m(t)$ as the *lumped chain*. Note also that in this case we have that

$$g_{M(x), M_+}(m) = \tilde{h}_{m(x), m^*}(m) \tag{7.11}$$

where $\tilde{h}$ is the equilibrium potential for the lumped Markov chain $m(t)$. Also,

$$\mathrm{cap}_{M(x)}(M_+) = \widetilde{\mathrm{cap}}_{m(x)}(m^*) \tag{7.12}$$

where $\widetilde{\mathrm{cap}}$ is the capacity for the lumped chain, and formula (7.8) can be derived directly in the context of the lumped chain. Thus, the study of the metastability problem in the high-dimensional Curie-Weiss model can be reduced readily to the study of a one-dimensional discrete problem.

The Curie-Weiss model is a particularly simple incident of the lumping technique. In general, if we are given some map m, the process $m(t) = m(x(t))$ will not be a Markov chain. In some cases it is however possible to construct a map into some higher-dimensional space that verifies property (7.9). In the context of spin systems when $\Gamma = \{-1, 1\}^N$ (and similar constructions work when $\{-1, 1\}$ is replaced by a general finite set), there is a natural class of such maps that proof often helpful.

Theorem 7.1 *Assume that according to some rule there is a partition of $\Lambda \equiv \{1, \dots, N\}$ into k subsets $\Lambda_1, \dots, \Lambda_k$. denote by m_i the maps*

$$m_i(x) = \frac{1}{|\Lambda_i|} \sum_{i \in \Lambda_i} x_i \tag{7.13}$$

and let m denote the k-dimensional vector $(m_1, \dots, m_k)$. Assume that $\mathbb{Q}(x)$ depends only in $m(x)$ and that the Markov chain $x(t)$ has transition rates that are of the form

$$p(x, x') = f(m(x), m(x'))\mathbb{1}_{\|x - x'\|_2^2 \leq 2} \tag{7.14}$$

Then $m(t) \equiv m(x(t))$ is a Markov chain on $m(\Gamma)$ with transition rates given by equation (7.10).

Proof. Consider two possible values of m and m' that may be connected by a simple transition, i.e. changing the sign of one component of $x \in M(m)$. Note that this has to happen in one of the boxes Λ_i, and changes the value of one component of m, namely m_i by plus or minus $2/|\Lambda_i|$. Suppose then that $m'_i = m_i + 2/|\Lambda_i|$. By (7.14),

$$\sum_{x':m(x')=m'} p(x,x') = f(m(x), m(x')) \sum_{j\in\Lambda_i} \mathbb{1}_{x_i=-1} \tag{7.15}$$

But $\sum_{j\in\Lambda_i} \mathbb{1}_{x_i=-1} = |\Lambda_i|(1 - m_i(x))/2$ depends only on $m(x)$, which proves our case. □

Remark. Lumping is a useful tool to treat some random mean field models, such as the random field Curie-Weiss model [8] and the Hopfield model (in the Hopfield model, the construction can be found in the context of large deviation theory in [44, 34]).

8. Simple Random Walk on the Hypercube

To illustrate the lumping procedure and to show what it can achieve, we turn to the ordinary random walk on $\{-1,1\}^N$. This will provide a preparation for the treatment of the Random Energy Model. This model has been studied in the past, mainly in view of convergence to equilibrium, see e.g. [41, 23, 55]. Problems that are more closely related to our questions were studied in [14, 20]. This section summarizes results obtained in [2, 4].

We consider a Markov chain $\sigma(t)$ in discrete time on $\mathcal{S}_N \equiv \{-1,1\}^N$ with transition probabilities

$$p(\sigma,\sigma') = 1/N, \quad \text{if} \quad \|\sigma - \sigma'\|_2^2 = 2 \tag{8.1}$$

and zero else. We will be interested in hitting probabilities on a certain subset (of moderate cardinality) $\mathcal{M} \subset \mathcal{S}_N$.

Proposition 8.1 *Set $M = |\mathcal{M}|$, $d = 2^M$. There exists a constant $c > 0$, such that*

i) For all $\eta \in \mathcal{M}$ and all $\sigma \in \Gamma\backslash\mathcal{M}$,

$$\left|\mathbb{P}_\sigma\left(\tau_\eta < \tau_{\mathcal{M}\backslash\eta}\right) - \tfrac{1}{M}\right| \le \tfrac{c}{N} \tag{8.2}$$

ii) For all $\eta \in \mathcal{M}$ and $\bar\eta \in \mathcal{M}$ with $\eta \neq \bar\eta$,

$$\left|\mathbb{P}_{\bar\eta}\left(\tau_\eta < \tau_{\mathcal{M}\backslash\{\eta,\bar\eta\}}\right) - \tfrac{1}{M-1}\right| \le \tfrac{c}{N} \tag{8.3}$$

iii) For all $\sigma \notin \mathcal{M}$,

$$\tfrac{M}{M+1}\left(1-\tfrac{c}{N}\right) \leq \mathbb{P}_\sigma\left(\tau_{\mathcal{M}} < \tau_\sigma\right) \leq \tfrac{M}{M+1} \tag{8.4}$$

iv) For all $\sigma \notin \mathcal{M}$ and all $\bar\sigma \notin \mathcal{M}\cup\sigma)$,

$$\left|\mathbb{P}_{\bar\sigma}\left(\tau_\sigma \leq \tau_{\mathcal{M}}\right) - \tfrac{1}{M+1}\right| \leq \tfrac{c}{N} \tag{8.5}$$

Proof. The key tool of the proof of this proposition is the construction of a lumped chain in the sense explained above. In constructing such a chain, we must take care that the events whose probabilities we are computing are mapped one-to-one into the reduced state space. To construct such a mapping, we consider a collection i of vectors $\xi^1, \dots, \xi^{|I|}$ as a $|I|\times N$ matrix ξ those rows are the vectors ξ^μ. We will denote by $\xi_i \in \{-1,1\}^M$ the column vectors of this matrix.

Next, let $\{e_1, \dots, e_k, \dots, e_d\}$ be an arbitrarily chosen labeling of all $d = 2^{|I|}$ elements of $\mathcal{S}_M$. Then ξ induces a partition of Λ into d disjoint (possibly empty) subsets, $\Lambda_k(I)$,

$$\Lambda_k(I) = \{i \in \Lambda \mid \xi_i = e_k\} \tag{8.6}$$

This is the partitioning we will use for the construction of the lumped chain according to the construction given above. We will write

$$\mathcal{P}_I(\Lambda) = \{\Lambda_k(I),\, 1 \leq k \leq d\} \tag{8.7}$$

m_I, that maps the elements of $\mathcal{S}_N$ into d-dimensional vectors,

$$m_I(\sigma) = \left(m_I^1(\sigma), \dots, m_I^k(\sigma), \dots, m_I^d(\sigma)\right), \quad \sigma \in \mathcal{S}_N \tag{8.8}$$

where, for all $k \in \{1, \dots, d\}$,

$$m_I^k(\sigma) = \frac{1}{|\Lambda_k(I)|}\sum_{i\in\Lambda_k(I)} \sigma_i \tag{8.9}$$

A few elementary properties of m_I are listed in the lemma below.

Lemma 8.1 i) The range of m_I, $m_{N,d}(I) \equiv m_I(\mathcal{S}_N)$, is a discrete subset of the d-dimensional cube $[-1,1]^d$ and may be described as follows. Let $\{u_k\}_{k=1}^d$ be the canonical basis of $\mathbb{R}^d$. Then,

$$x \in m_{N,d}(I) \iff x = \sum_{k=1}^d \frac{n_k}{|\Lambda_k(I)|} u_k \tag{8.10}$$

where, for all $1 \leq k \leq d$, $|n_k| \leq |\Lambda_k(I)|$ has the same parity as $|\Lambda_k(I)|$.

ii)

$$|\{\sigma \in \mathcal{S}_N \mid m_I(\sigma) = x\}| = \prod_{k=1}^{d} \binom{|\Lambda_k(I)|}{|\Lambda_k(I)|\frac{1+x_k}{2}}, \quad \forall x \in m_{N,d}(I) \quad (8.11)$$

In particular, the restriction of m_I to I is a one-to-one mapping from I onto $m_I(I)$.

iii) The elements of I are mapped onto corners of $[-1,1]^d$: for all $\sigma \in I$

$$m_I(\sigma) = (\sigma_{i_1}, \dots, \sigma_{i_k} \dots, \sigma_{i_d}), \quad \text{for any choice of indices} \quad i_k \in \Lambda_k(I) \quad (8.12)$$

iv) Let $\sigma \in \mathcal{S}_N$ be such that $\inf_{\eta \in I \setminus \sigma} \|\sigma - \eta\|_2 \geq \sqrt{\varepsilon N}$ for some $\varepsilon > 0$. Set $x \equiv m_I(\sigma)$ and $\mathcal{I} \equiv m_I(I)$. Then

$$\inf_{y \in \mathcal{I} \setminus x} \|x - y\|_2 \geq \frac{\varepsilon N}{2\sqrt{d} \max_k |\Lambda_k(I)|} \quad (8.13)$$

Proof of Lemma 8.1. Assertions i), ii), and iii) result from elementary observations. To prove assertion iv) note that for any $\eta \in I \setminus \sigma$, setting $y \equiv m_I(\eta)$ and using (8.12), we have:

$$\begin{aligned} \varepsilon N \leq \sum_{i=1}^{N} (\sigma_i - \eta_i)^2 &= \sum_{k=1}^{d} \sum_{i \in \Lambda_k} (\sigma_i - y_k)^2 \\ &= 2 \sum_{k=1}^{d} |\Lambda_k(I)|(1 - y_k x_k) \leq 2 \max_k |\Lambda_k(I)|(y, y - x) \end{aligned} \quad (8.14)$$

where we used in the last line that $1 - y_k x_k = y_k(y_k - x_k)$. But $(y, y - x) \leq \|y\|_2 \|y - x\|_2 = \sqrt{d} \|y - x\|_2$, so that

$$\|x - y\|_2 \geq \frac{\varepsilon N}{2\sqrt{d} \max_k |\Lambda_k(I)|} \quad (8.15)$$

which, together with assertion ii) yields (8.13). □

Note in particular that $\{\sigma_N(t)\}$ is reversible w.r.t. the measure

$$\mu_N(\sigma) = 2^{-N}, \quad \sigma \in \mathcal{S}_N \quad (8.16)$$

We will denote by $\{X_{I,N}(t)\}_{t \in \mathbb{N}}$ and call the *I-lumped chain* or the *lumped chain induced by I*, the chain defined through

$$X_{I,N}(t) \equiv m_I(\sigma_N(t)), \quad \forall t \in \mathbb{N} \quad (8.17)$$

To $m_{N,d}(I)$ we associate an undirected graph, $\mathcal{G}(m_{N,d}(I)) = (V(m_{N,d}(I)), E(m_{N,d}(I)))$, with set of vertices $V(m_{N,d}(I)) = m_{N,d}(I)$ and set of edges:

$$\begin{aligned} &E(m_{N,d}(I)) \\ &\quad = \left\{(x,x') \in m_{N,d}(I) \mid \exists_{k\in\{1,\dots,d\}}, \exists_{s\in\{-1,1\}} \, : \, x'-x = s\tfrac{2}{|\Lambda_k(I)|}u_k\right\} \end{aligned} \tag{8.18}$$

The properties of $\{X_{I,N}(t)\}$ are summarized in the lemma below.

Lemma 8.2 Given any subset $I \in \mathcal{S}_N$:

i) The process $\{X_{I,N}(t)\}$ is Markovian no matter how the initial distribution π° of $\{\sigma_N(t)\}$ is chosen.

ii) Set $\mathbb{Q}_N = \mu_N m_\xi^{-1}$. Then $\mathbb{Q}_N$ is the unique reversible invariant measure for the chain $\{X_{I,N}(t)\}$. In explicit form, the density of $\mathbb{Q}_N$ reads:

$$\mathbb{Q}_N(x) = \frac{1}{2^N}|\{\sigma \in \mathcal{S}_N \mid m_I(\sigma) = x\}|, \quad \forall x \in m_{N,d}(I) \tag{8.19}$$

iii) The transition probabilities $r_N(\,.\,,\,.\,)$ of $\{X_{I,N}(t)\}$

$$r_N(x,x') = \begin{cases} \frac{|\Lambda_k(I)|}{N}\frac{1-sx_k}{2} & \text{if } \ (x,x') \in E(m_{N,d}(I))) \\ & \quad\text{and } \ x'-x = s\frac{2}{|\Lambda_k(I)|}u_k \\ 0, & \text{otherwise} \end{cases} \tag{8.20}$$

Proof. These results follow from the general Theorem 7.1 and explicit calculations. □

8.1. MAIN INGREDIENTS OF THE PROOF OF PROPOSITION 8.1

Observe that the entropy produced by the lumping procedure gives rise through (8.19) to a potential, $F_N(x) \equiv -\frac{1}{N}\ln \mathbb{Q}_N(x)$. It moreover follows from assertions ii) and iii) of Lemma 8.1 that this potential is convex and takes on its global minimum at 0 and its global maximum at the corners of the cube $[-1,1]^d$. Thus the key idea in all the computations will be that the potential will have the tendency to drive the lumped process quickly to zero, before it does anything else; in other words, with overwhelming probability all the events we are interested in are realized in such a way that the process passes through zero.

Note. The diligent reader may have observed that depending on the particular properties of the vectors ξ^μ, the point 0 may or may not be in the range of m_I. To avoid notational complications, in the sequel 0 will be understood to stand for one of the points in $m_I(\mathcal{S}_N)$ closest to zero.

The next two Lemmata quantify this statement.

Lemma 8.3 Let $x \in m_I(I)$ and $y \in m_I(\mathcal{S}_N)\backslash\{x\}$ Then

$$\mathbb{R}^\circ_y(\tau_x < \tau_0) \leq \frac{c}{N} \tag{8.21}$$

Lemma 8.4 There exists a constant $c > 0$ such that, for all N large enough,

$$\mathbb{R}^\circ_x(\tau_0 < \tau_x) \geq \left(1 - \frac{c}{N}\right)^2, \quad \text{for all} \quad x \in m_I(I) \tag{8.22}$$

A consequence of the previous two Lemmata will be that the process starting from 0 hits the set of corners of the hypercube $[-1,1]^d$ with essentially uniform probability, more precisely:

Lemma 8.5 For all $J \subseteq m_I(I)$, $x \in J$,

$$\frac{\vartheta}{|J|} \leq \mathbb{R}^\circ_0(\tau_x \leq \tau_J) \leq \frac{1}{\vartheta|J|}, \tag{8.23}$$

where

$$\vartheta = \left(1 - \frac{c}{N}\right)^2 \tag{8.24}$$

The basis for the proofs of the preceding Lemmata is the following a priori estimate.

Lemma 8.6 There is a constant $c > 0$ such that, if $\mathbb{Q}(y) \leq e^{-\delta N}$,

$$\mathbb{R}^\circ_y[\tau_0 < \tau_y] \geq c\delta^2 \tag{8.25}$$

while otherwise

$$\mathbb{R}^\circ_y[\tau_0 < \tau_y] \geq cN^{-1} \tag{8.26}$$

Proof. Our general strategy for getting a priori lower bounds on such capacity type probabilities is the use of dramatically simplified chains. An L-steps path ω on $m_{N,d}(I)$, beginning at x and ending at y is defined as sequence of L sites $\omega = (\omega_0, \omega_1, \ldots, \omega_L)$, with $\omega_0 = x$, $\omega_L = y$, and $\omega_l = (\omega_l^k)_{k=1,\ldots,d} \in V(m_{N,d}(I))$ for all $1 \leq l \leq L$, that satisfies:

$$(\omega_l, \omega_{l-1}) \in E(m_{N,d}(I)), \qquad \text{for all} \quad l = 1, \ldots, L \tag{8.27}$$

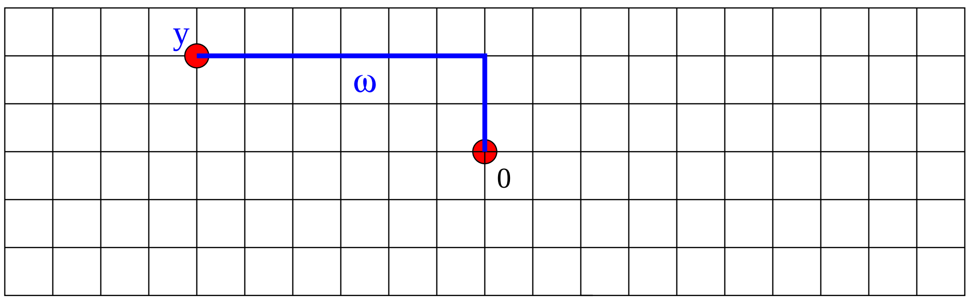
(We may also write $|\omega| = L$ to denote the length of ω.) If ω is such a path with $\omega_0 = y$ and $\omega_L = 0$, it is clear that

$$\begin{aligned} \text{cap}_y(0) &\equiv \inf_{h:h(0)=1, h(y)=0} \Phi(h) \\ &\geq \inf_{h:h(0)=1, h(y)=0} \frac{1}{2} \sum_{z,z' \in \omega} \mathbb{Q}(z) r_N(z,z')(h(z) - h(z'))^2 \end{aligned} \tag{8.28}$$

This lower bound corresponds to a simple one-dimensional problem involving the process restricted to the path ω and its solution is of course well known. In fact, if we enumerate sites of the path ω by $y = \omega_0, \dots, \omega_L = 0$, (8.28) yields

$$\mathrm{cap}_y(0) \geq \frac{1}{\sum_{k=0}^{L-1} \frac{1}{\mathbb{Q}(\omega_k) r_N(\omega_k, \omega_{k+1})}} \tag{8.29}$$

We optimize this bound by choosing the path ω in a more or less optimal way. Assume w.r.g. that $|\Lambda_1| y_1^2 \geq |\Lambda_2| y_2^2 \geq \dots > |\Lambda_d| y_d^2$ and that $y_\mu \geq 0$, for all μ. Then our path will consist of a sequence of straight pieces along the coordinate axis, starting with the first and ending with the last one.



This allows a very explicit representation of the denominator in (8.29) in the form

$$\begin{aligned} &\sum_{\mu=1}^{d} 2^N \prod_{k=1}^{\mu-1} \binom{|\Lambda_k|}{|\Lambda_k|/2}^{-1} \prod_{k=\mu+1}^{d} \binom{|\Lambda_k|}{|\Lambda_k| \frac{1+y_k}{2}}^{-1} \\ &\qquad \sum_{n=1}^{[y_\mu |\Lambda_\mu|/2]} \binom{|\Lambda_\mu|}{|\Lambda_\mu| \frac{1+y_k}{2} - n}^{-1} \frac{N}{|\Lambda_\mu|} \frac{2}{1+y_\mu - 2n/|\Lambda_\mu|} \\ &= \mathbb{Q}(y)^{-1} \sum_{\mu=1}^{d} \prod_{k=1}^{\mu-1} \frac{\binom{|\Lambda_k|}{|\Lambda_k| \frac{1+y_k}{2}}}{\binom{|\Lambda_k|}{|\Lambda_k|/2}} \\ &\qquad \sum_{n=1}^{[y_\mu |\Lambda_\mu|/2]} \frac{\binom{|\Lambda_\mu|}{|\Lambda_\mu| \frac{1+y_\mu}{2}}}{\binom{|\Lambda_\mu|}{|\Lambda_\mu| \frac{1+y_\mu}{2} - n}} \frac{N}{|\Lambda_\mu|} \frac{2}{1+y_\mu - 2n/|\Lambda_\mu|} \end{aligned} \tag{8.30}$$

From this formula one can in derive reasonably good bounds for all possible choices of y. Here our concern is only the situation when y is somewhat far from 0. In fact, assume that for some $\delta > 0$, there exists no k such that $|\Lambda_k| y_k^2 \geq \delta N$. Then it is easy to see that $\mathbb{Q}(y) \geq e^{-c\delta N}$. Otherwise, let by convention $|\Lambda_1|y_1^2 > \delta N$. Note that this implies in particular $\frac{|\Lambda_1|}{N} > \delta$. Then all terms in the sum (8.30) with $\mu > 1$ are bounded by $e^{-\delta N}$ while the term with $\mu = 1$ is bounded by

$$\frac{1}{\mathbb{Q}(y)} \sum_{n=1}^{[y_1|\Lambda_1|/2]} \frac{\binom{|\Lambda_1|}{|\Lambda_1|\frac{1+y_1}{2}}}{\binom{|\Lambda_1|}{|\Lambda_1|\frac{1+y_1}{2}-n}} \delta^{-1} \leq C\delta^{-2} \tag{8.31}$$

The bound (8.26) is trivial. □

Proof of Lemma 8.3. Note first that

$$\mathbb{R}_y^\circ(\tau_x < \tau_0) = \frac{\mathbb{R}_y^\circ(\tau_x < \tau_{y\cup 0})}{\mathbb{R}_y^\circ(\tau_{x\cup 0} < \tau_y)} \tag{8.32}$$

Let us first consider the case when

$$2^{-N}\mathbb{Q}(y)^{-1} \leq \delta N^{-1} \tag{8.33}$$

By reversibility its numerator may be rewritten as

$$\mathbb{R}_y^\circ(\tau_x < \tau_{y\cup 0}) = \frac{\mathbb{Q}(x)}{\mathbb{Q}(y)} \mathbb{R}_x^\circ(\tau_y < \tau_{x\cup 0}) \leq 2^{-N}/\mathbb{Q}(y) \tag{8.34}$$

Then by Lemma 8.6, the denominator of (8.32) obeys the bound

$$\mathbb{R}_y^\circ(\tau_{x\cup 0} < \tau_y) \geq cN^{-1} \tag{8.35}$$

if $\mathbb{Q}(y) > e^{-\delta N}$ (with, say,. $\delta = 0.1$). Thus in this case we get that

$$\mathbb{R}_y^\circ(\tau_x < \tau_0) \leq 2^{-N} e^{\delta N} N \tag{8.36}$$

which is exponentially small. On the other hand, if $\mathbb{Q}(y) < e^{-\delta N}$ then

$$\mathbb{R}_y^\circ(\tau_{x\cup 0} < \tau_y) \geq c\delta^{-2} \tag{8.37}$$

and we are done if (8.33) is satisfied.

Otherwise (8.37) always holds, and we can use that (if $y \neq x$),

$$\begin{aligned} \mathbb{R}_y^\circ(\tau_x < \tau_0) &= \sum_{y'} p(y,y')\mathbb{R}_{y'}^\circ(\tau_x < \tau_0) \\ &\leq C\sum_{y'} p(y,y')\mathbb{Q}(y')^{-1}2^{-N} \\ &= \sum_{\mu=1:|y_\mu|\neq 1}^{N} \sum_{\pm} \tfrac{|\Lambda_\mu(I)|}{N} \tfrac{1\pm y_\mu}{2} 2^{-N}\mathbb{Q}(y)^{-1} \\ &\quad + \sum_{\mu=1:y_\mu=\pm 1}^{N} \tfrac{|\Lambda_\mu(I)|}{N} \tfrac{1\pm y_\mu}{2} 2^{-N}\mathbb{Q}(y \mp 2u_\mu/|\Lambda_\mu|)^{-1} \end{aligned} \tag{8.38}$$

Using the explicit representation of $\mathbb{Q}$ in (8.19) shows that all terms in the sums are smaller than cN^{-1}, which concludes the proof of the lemma. $\square$

Remark. Note that we have striven to obtain only the crudest uniform upper bound, and this cannot be improved. of course we can get much sharper bounds as functions of y.

Proof of Lemma 8.4. To prove Lemma 8.4, we use that $\mathbb{R}_x^\circ(\tau_0 < \tau_x) = \mathbb{Q}(x)^{-1}\mathrm{cap}_0(x)$, while

$$\begin{aligned} \mathrm{cap}_0(x) &= \frac{1}{2}\sum_{z,z'}\mathbb{Q}(z)r_N(z,z')[h^*(z)-h^*(z')]^2 \\ &\geq \sum_{k=1}^{d}\mathbb{Q}(x)r_N(x,x-2u_k/|\Lambda_k|)\left[h^*(x)-h^*(x-2u_k/|\Lambda_k|)\right]^2 \end{aligned} \tag{8.39}$$

where $h^*(z) = \mathbb{R}_z^\circ(\tau_0 < \tau_x)$ if $z \notin \{x,0\}$, and $h^*(x) = 0$. Thus

$$\mathrm{cap}_0(x) \geq \sum_{k=1}^{d}\mathbb{Q}(x)r_N(x,x-2u_k/|\Lambda_k|)\left(1-cN^{-1}\right)^2 = \mathbb{Q}(x)\left(1-cN^{-1}\right)^2 \tag{8.40}$$

This yields the claimed estimate. $\square$

Proof of Lemma 8.5. Again using renewal,

$$\mathbb{R}_0^\circ(\tau_x \leq \tau_J) = \frac{\mathbb{R}_0^\circ(\tau_x \leq \tau_{J\cup 0})}{\mathbb{R}_0^\circ(\tau_J < \tau_0)} = \frac{\mathbb{R}_0^\circ(\tau_x \leq \tau_{J\cup 0})}{\sum_{y\in J}\mathbb{R}_0^\circ(\tau_y \leq \tau_{J\cup 0})} \tag{8.41}$$

so that we are left to bound a term of the form $\mathbb{R}_0^\circ(\tau_y \leq \tau_{J\cup 0})$, $y \in J$. To do so observe that

$$\mathbb{R}_0^\circ(\tau_y \leq \tau_{J\cup 0}) = \mathbb{R}_0^\circ(\tau_y < \tau_0) - \mathbb{R}_0^\circ\left(\tau_{J\setminus y} < \tau_y < \tau_0\right) \tag{8.42}$$

and that

$$\mathbb{R}_0^\circ\left(\tau_{J\setminus y} < \tau_y < \tau_0\right) = \sum_{z\in J\setminus y}\mathbb{R}_0^\circ(\tau_z \leq \tau_{J\cup 0})\,\mathbb{R}_z^\circ(\tau_y < \tau_0) \tag{8.43}$$

By assumption, the probabilities $\mathbb{R}_z^\circ(\tau_y < \tau_0)$ in the r.h.s. above obey the bound (8.21) of Lemma 8.3. Thus

$$\begin{aligned} \mathbb{R}_0^\circ\left(\tau_{J\setminus y} < \tau_y < \tau_0^\circ\right) &\leq \tfrac{c}{N}\sum_{z\in J\setminus y}\mathbb{R}_0^\circ(\tau_z \leq \tau_{J\cup 0}) \\ &\leq \tfrac{c}{N}\mathbb{R}_0^\circ(\tau_J < \tau_0) \end{aligned} \tag{8.44}$$

From (8.42) and (8.44) we deduce that

$$\mathbb{R}_0^\circ(\tau_y < \tau_0) - \frac{c}{N}\mathbb{R}_0^\circ(\tau_J < \tau_0) \leq \mathbb{R}_0^\circ(\tau_y \leq \tau_{J\cup 0}) \leq \mathbb{R}_0^\circ(\tau_y < \tau_0) \tag{8.45}$$

and, summing over $y \in J$,

$$\sum_{y\in J}\mathbb{R}_0^\circ(\tau_y \leq \tau_0) - |J|\frac{c}{N}\mathbb{R}_0^\circ(\tau_J < \tau_0) \leq \mathbb{R}_0^\circ(\tau_J \leq \tau_0) \leq \sum_{y\in J}\mathbb{R}_0^\circ(\tau_y < \tau_0) \tag{8.46}$$

Inserting the bounds (8.45) and (8.46) into (8.41), and using that

$$\frac{\mathbb{R}_0^\circ(\tau_J \leq \tau_0)}{\sum_{y\in J}\mathbb{R}_0^\circ(\tau_y < \tau_0)} \leq 1 \tag{8.47}$$

we arrive at:

$$R - \frac{c}{N} \leq \mathbb{R}_0^\circ(\tau_x \leq \tau_J) \leq R\left(\frac{1}{1-|J|\frac{c}{N}}\right) \tag{8.48}$$

where

$$R \equiv \frac{\mathbb{R}_0^\circ(\tau_x \leq \tau_0)}{\sum_{y\in J}\mathbb{R}_0^\circ(\tau_y \leq \tau_0)} \tag{8.49}$$

To estimate the above ratio we use first that, by reversibility,

$$R = \frac{\mathbb{Q}_N(x)\mathbb{R}_x^\circ(\tau_0 \leq \tau_x)}{\sum_{y\in J}\mathbb{Q}_N(y)\mathbb{R}_y^\circ(\tau_0 \leq \tau_y)} \tag{8.50}$$

and next that, by Lemma 8.4,

$$\vartheta\overline{R} \leq \mathbb{R}_0^\circ(\tau_x \leq \tau_J) \leq \frac{\overline{R}}{\vartheta} \tag{8.51}$$

where ϑ is defined in (8.24) and

$$\overline{R} \equiv \frac{\mathbb{Q}_N(x)}{\sum_{y\in J}\mathbb{Q}_N(y)} \tag{8.52}$$

Now since $J \subseteq m_I(I)$, and since $\mathbb{Q}_N(y) = 2^{-N}$ for all $y \in m_I(I)$,

$$\overline{R} = \frac{1}{|J|} \tag{8.53}$$

and (8.23) is proven. □

8.2. PROOF OF PROPOSITION 8.1

We are now ready to prove Proposition 8.1.

Notation. The following notation will be used throughout: $\mathcal{T} = m_{\mathcal{M}}(\mathcal{M})$, $y = m_{\mathcal{M}}(\sigma)$, $x = m_{\mathcal{M}}(\eta)$ and $\bar{x} = m_{\mathcal{M}}(\bar{\eta})$.

Proof of Proposition 8.1, i), and ii). Firstly

$$\mathbb{P}_\sigma\left(\tau_\eta < \tau_{\mathcal{M}\setminus\eta}\right) = \mathbb{R}_y^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) \tag{8.54}$$

Defining

$$\begin{aligned} R_1 &\equiv \mathbb{R}_y^\circ\left(\{\tau_x < \tau_{\mathcal{T}\setminus x}\} \cap \{\tau_0 < \tau_x\}\right) \\ R_2 &\equiv \mathbb{R}_y^\circ\left(\{\tau_x < \tau_{\mathcal{T}\setminus x}\} \cap \{\tau_x < \tau_0\}\right) \end{aligned} \tag{8.55}$$

$\mathbb{R}_y^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right)$ may be decomposed as

$$\mathbb{R}_y^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) = R_1 + R_2 \tag{8.56}$$

Obviously

$$0 \leq R_2 \leq \mathbb{R}_y^\circ\left(\tau_x < \tau_0\right) \leq \frac{c}{N} \tag{8.57}$$

while

$$\begin{aligned} R_1 &= \mathbb{R}_y^\circ\left(\tau_0 < \tau_x < \tau_{\mathcal{T}\setminus x}\right) \\ &= \mathbb{R}_y^\circ\left(\tau_0 < \tau_{\mathcal{T}}\right) \mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) \\ &= \left[1 - \mathbb{R}_y^\circ\left(\tau_{\mathcal{T}} < \tau_0\right)\right] \mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) \end{aligned} \tag{8.58}$$

which, together with the bound

$$\mathbb{R}_y^\circ\left(\tau_{\mathcal{T}} < \tau_0\right) \leq \sum_{x'\in\mathcal{T}} \mathbb{R}_y^\circ\left(\tau_{x'} < \tau_0\right) \tag{8.59}$$

yields

$$\mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right)\left[1 - M \sup_{x'\in\mathcal{T}} \mathbb{R}_y^\circ\left(\tau_{x'} < \tau_0\right)\right] \leq R_1 \leq \mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) \tag{8.60}$$

We are thus left to bound the quantities $\sup_{x'\in\mathcal{T}} \mathbb{R}_y^\circ\left(\tau_{x'} < \tau_0\right)$ and $\mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right)$, which will be done by means of, respectively, Lemma 8.3 and Lemma 8.5.

$$\left|\mathbb{R}_0^\circ\left(\tau_x < \tau_{\mathcal{T}\setminus x}\right) - \frac{1}{M}\right| \leq \frac{c}{N} \tag{8.61}$$

for some constant $c_0 > 0$.

Collecting the previous estimates we obtain that, for large enough N,

$$\left| \mathbb{R}_y^\circ \left(\tau_x < \tau_{\mathcal{T}\backslash x}\right) - \frac{1}{M} \right| \leq \frac{c}{N} \tag{8.62}$$

for some constant $c > 0$. This yields the claim of assertion i). The proof of assertion ii) is very similar to that of assertion i). □

In order to study the probabilities appearing in (iii) and (iv) we must construct the lumped chain based on the vectors from $\mathcal{M}$ and σ. Otherwise, there is little difference. The following notation will be used throughout: $I \equiv \mathcal{M} \cup \sigma$, $\mathcal{I} \equiv m_I(I)$, $y \equiv m_I(\sigma)$, and $\bar{y} \equiv m_I(\bar{\sigma})$. It will moreover be assumed that $\omega \in \mathcal{E}_N$.

Proof of Proposition 8.2, iii) and iv). With the notation introduced above

$$\mathbb{P}_\sigma \left(\tau_{\mathcal{M}} < \tau_\sigma\right) = \mathbb{R}_y^\circ \left(\tau_{\mathcal{I}\backslash y} < \tau_y\right) \tag{8.63}$$

and

$$\mathbb{P}_{\bar{\sigma}} \left(\tau_\sigma \leq \tau_{\mathcal{M}}\right) = \mathbb{R}_{\bar{y}}^\circ \left(\tau_y < \tau_{\mathcal{I}\backslash y}\right) \tag{8.64}$$

Let us first consider the capacity-like quantity (8.63). To prove an upper bound, we will chose as a test function in the formula for the capacity $\mathrm{cap}_{\mathcal{I}\backslash y}(y)$ the function

$$h(z) = \begin{cases} \frac{1}{M+1}, & \text{if} \quad z \notin \mathcal{I} \\ 1, & \text{if} \quad z \in \mathcal{I}\backslash y \\ 0, & \text{if} \quad z = y \end{cases} \tag{8.65}$$

This gives that

$$\begin{aligned} \mathbb{R}_y^\circ \left(\tau_{\mathcal{I}\backslash y} < \tau_y\right) \leq \mathbb{Q}(\mathcal{I}\backslash y)^{-1} \Bigg(& \mathbb{Q}(y) \sum_{k=1}^{d} r_N(y, y - 2 \text{ sign } (y_k) u_k/|\Lambda_k|) \tfrac{1}{(M+1)^2} \\ & + \sum_{x \in \mathcal{I}\backslash y} \mathbb{Q}(x) \sum_{k=1}^{d} r_N(x, x - 2 \text{ sign } (x_k) u_k/|\Lambda_k|) \left(1 - \tfrac{1}{M+1}\right)^2 \Bigg) \\ = & \frac{M}{M+1} \end{aligned} \tag{8.66}$$

To get the corresponding lower bound we have only to show that in fact

$$\mathbb{R}_z \left(\tau_{\mathcal{I}\backslash y} < \tau_y\right) = \frac{1}{M+1} \left(1 + O(1/N)\right) \tag{8.67}$$

for $z \notin \mathcal{I}$. But Lemma 8.5,

$$\mathbb{R}_0\left(\tau_{\mathcal{I}\setminus y} < \tau_y\right) = \frac{1}{M+1}\left(1 + O(1/N)\right) \tag{8.68}$$

while

$$\mathbb{R}_z\left(\tau_0 < \tau_{\mathcal{I}}\right) \geq 1 - \frac{c}{N} \tag{8.69}$$

Combining these facts in the by now familiar way, we see that (8.67) holds, and hence assertion (iii) of the proposition is proven. Assertion (iv) is then proven in just the same way as assertion (i). □

9. Dynamics of the REM

We will now see show that the results from the last section allow us to analyse the dynamics of a much more complicated model, namely the random energy model.

The REM is a very simple, but instructive model for spin glasses that was introduced by B. Derrida [De1,De2] in the '80ies and that has been studied extensively since then [27, 32, 52, 54]. The configuration space again the hypercube $\mathcal{S}_N = \{-1, 1\}^N$. On an abstract probability space $(\Omega, \mathcal{F}, P)$ we define the family of i.i.d. standard normal random variables $\{X_\sigma\}_{\sigma \in \mathcal{S}_N}$. We set $E_\sigma \equiv [X_\sigma]_+ \equiv (X_\sigma \vee 0)$. We define a random (Gibbs) probability measure on $\mathcal{S}_N$, $\mu_{\beta,N}$, by setting

$$\mu_{\beta,N}(\sigma) \equiv \frac{e^{\beta\sqrt{N}E_\sigma}}{Z_{\beta,N}} \tag{9.1}$$

where $Z_{\beta,N}$ is the normalizing partition function. For our purposes it is enough to know that if $\beta > \sqrt{2\ln 2}$, then the Gibbs measure is asymptotically concentrated on the random set of vertices σ for which E_σ is maximal. I.e. if

$$E_{\sigma^{(1)}} \geq E_{\sigma^{(2)}} \geq E_{\sigma^{(3)}} \geq \ldots \geq E_{\sigma^{(2^N)}} \tag{9.2}$$

(note that this ordering depends of course on N) then for any finite k,

$$\lim_{k\uparrow\infty} \lim_{N\uparrow\infty} \mu_{\beta,N}\left(\{\sigma^{(1)}, \ldots, \sigma^{(k)}\right) = 1, \quad \text{a.s.} \tag{9.3}$$

This fact suggests that for any Markov chain that is reversible with respect to the Gibbs measure (9.1), the states $\sigma^{(1)}, \ldots, \sigma^{(k)}$ are good candidates for metastable states. It will be important to have a precise understanding of their respective energy values. This information is contained in a classical result of extreme value theory that states that:

Proposition 9.1 *Define*

$$u_N(x) \equiv \sqrt{2N\ln 2} + \frac{x}{\sqrt{2N\ln 2}} - \frac{1}{2}\frac{\ln(N\ln 2)+\ln 4\pi}{\sqrt{2N\ln 2}} \tag{9.4}$$

and define the point process

$$\mathcal{P}_N \equiv \sum_{\sigma\in\{-1,1\}^N} \delta_{u_N^{-1}(X_\sigma)} \tag{9.5}$$

Moreover, let $\mathcal{P}$ denote the Poisson point process on $\mathbb{R}$ with intensity measure $\mathcal{P}$. Then,

$$\mathcal{P}_N \xrightarrow{\mathcal{D}} \mathcal{P} \tag{9.6}$$

Let us now define a particular Glauber dynamics for this model. We will construct a Markov chain $\sigma(t)$ with state space $\mathcal{S}_N$ and discrete time $t \in \mathbb{N}$ by prescribing transition probabilities $p_N(\sigma,\eta) = \mathbb{P}[\sigma(t+1) = \eta|\sigma(t) = \sigma]$ by

$$p_N(\sigma,\eta) = \begin{cases} \frac{1}{N}e^{-\beta\sqrt{N}E_\sigma}, & \text{if} \quad \|\sigma-\eta\|_2 = \sqrt{2} \\ 1 - e^{-\beta\sqrt{N}E_\sigma}, & \text{if} \quad \sigma = \eta \\ 0, & \text{otherwise} \end{cases} \tag{9.7}$$

Note that the dynamics is also random, i.e. the law of the Markov chain is a measure valued random variable on Ω that takes values in the space of Markov measures on the path space $\mathcal{S}_N^{\mathbb{N}}$. We will mostly take a pointwise point of view, i.e. we consider the dynamics for a given fixed realization of the disorder parameter $\omega \in \Omega$.

One may now think of the sets

$$T_N(E) \equiv \{\sigma \in \mathcal{S}_N | E_\sigma \geq u_N(E)\} \tag{9.8}$$

with $E \in \mathbb{R}$ as candidates for sets of metastable states, as they are the "deepest minima". The difficulty here is, however, that there is no good separation between the states in $T_N(E)$ and those outside: in fact the difference in depth is only of the order $N^{-1/2}$ between the most shallow minimum within $T_N(E)$ and the deepest one without. This will lead us to ask rather different questions when before, but as we will see, our tools are still of good use. Note that this is also related to the fact, observed by Fontes, Isopi, Kohayakawa, and Picco [30], that the phase transition in this model is not visible in terms of the behaviour of the spectral gap of the generator (see also [47] for an analysis of the dynamical phase transition in this model).

9.1. AGEING

In systems like the REM, physicist have discovered a novel concept characterizing long term dynamics that is called "ageing" (we refer to the reviews [6, 16] for an overview and further references. [5] contains a short review from a mathematical perspective). This phenomenon is typically characterized in terms of the behaviour of an autocorrelations function, that could for instance be taken as

$$C_N(t,s) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i(t)\sigma_i(t+s) \tag{9.9}$$

Ageing is then said to occur whenever the $C_N(t,s)$ does not become independent of t as both s and t tend to infinity. In fact in many cases of ageing dynamics it turns out that asymptotically, the correlation function tends to a limit that is a function of s/t only.

The idea that the long-time dynamics of the model can be described effectively in terms of metastable transitions between the states in $T_N(E)$ gave rise to the ad hoc definition of an effective "trap model" that should effectively represent this dynamics. This model introduced by Bouchaud and Dean [7] can easily be described as follows: the state space is reduced to M points, representing the elements of $T_N(E)$. Each of the states is assigned a positive random energy E_k which is taken to be exponentially distributed with rate one, which is justified from the Poisson convergence result Proposition 9.1. Of course this does not represent the actual energy of the state, but the properly rescaled one, more precisely $\sqrt{N}u_N^{-1}(X_\sigma)$.

The dynamics is now a continuous time Markov chain $Y(t)$ taking values in $S_M \equiv \{1,\dots,M\}$. If the process is in state k, it waits an exponentially distributed time with mean proportional to $e^{E_k\alpha}$ where $\alpha = \beta/\beta_c$, and then jumps with equal probability in one of the other states $k' \in S_M$. This process is then analyzed using essentially techniques from *renewal* theory. The essential point is that if one starts the process from the uniform distribution, it is possible to show that if one only considers the times, T_i, at which the process changes its state, then the counting process, $c(t)$, that counts the number of these jumps in the time interval $(0,t]$ is a classical renewal (counting) process; moreover, as $n \uparrow \infty$, this renewal process converges to a renewal process with a *deterministic* law for the renewal time with a heavy-tailed distribution (in the sense that the mean is infinite[1]) whose density is proportional to $t^{-1-1/\alpha}$. It is the emergence of such *non-Markovian* limit processes that is ultimately responsible for all the ageing

[1] This is clearly due to the fact that the average of the waiting time $e^{\alpha E_i}$ over the disorder is infinite.

phenomena observed in the abundant literature on this and related models. The correlation function to be studied in the trap model is then simply the probability, $\Pi_N(t,s)$, that no jump occurs in a time interval $[t, t+s]$. One sees easily that this quantity satisfies a renewal equation

$$\Pi_N(t,s) = 1 - F_N(s+t) + \int_0^t \Pi_N(t-u,s)dF_N(u) \tag{9.10}$$

where F_N is is just the mean waiting time distribution, i.e. $F_N(t) = \sum_{i=1}^N \left(1 - e^{-t/\tau_i}\right)$ where $\tau_i \equiv \tau_0 \exp(E_i/\alpha)$ and E_i are exponentially distributed random variables. The key point is that F_N converges a.s. to the deterministic function

$$F_\infty(t) \equiv \alpha \int_1^\infty dx e^{-t/x} x^{-1-\alpha} \tag{9.11}$$

and consequently Π_N converges to the solution Π_∞ of the renewal equation

$$\Pi_\infty(t,s) = 1 - F_\infty(s+t) + \int_0^t \Pi_\infty(t-u,s)dF_\infty(u) \tag{9.12}$$

The particular behaviour of the solution is due to the fact that the kernel F_∞ of the equation is singular in the sense that the mean renewal time is infinite. It is, however, not hard to analyse the asymptotics of the solution using Laplace transform methods. It turns out that to leading order (as $t, s \uparrow \infty$),

$$\Pi_\infty(t,s) \sim \frac{1}{\pi \mathrm{cosec}(\alpha\pi)} \int_{s/t}^\infty dx \frac{1}{(1+x)x^\alpha} \equiv H_0(s/t) \tag{9.13}$$

i.e. is indeed a function of s/t.

The purpose of the analysis presented in [1, 2, 3] is to justify the predictions of this trap model in a rigorous way. We will briefly review the main aspects of this analysis.

9.2. JUSTIFYING THE TRAP MODEL

Three assumptions entering in the definition of the trap model that need to be justified: 1) the uniform distribution of the jump distribution, 2) the distribution of the random mean exit time, and 3) the exponential distribution of the transition times. We will show that the first two assumptions can be derived. The last assumption will in fact not hold true strictly speaking, which will be the cause of a lot of trouble.

To prove the uniformity of the distribution of the jumps over $T_N(E)$, we have to show that

$$\mathbb{P}_\eta[\tau_{\eta'} = \tau_{T_N(E)}] \sim \frac{1}{|T_N(E)|} \tag{9.14}$$

for any $\eta \neq \eta' \in T_N(E)$. The nice thing is that this follows from Proposition 8.1 due to the simple fact that

$$\mathbb{P}_\eta[\tau_{\eta'} = \tau_{T_N(E)}] = e^{-\beta\sqrt{N}E_\eta}\mathbb{P}^\circ_\eta[\tau_{\eta'} = \tau_{T_N(E)}] \tag{9.15}$$

where $\mathbb{P}^\circ$ denotes the law of the simple random walk studied in Section 7. The reason for this is very simple: the probability in our REM process to jump to any neighboring site, *conditioned to jump* is the same as in the ordinary random walk, while $e^{-\beta\sqrt{N}E_\eta}$ is the probability not to move away from η.

So this was easy (but recall that we had to work a little in Section 7!). Next we turn to the mean values $\mathbb{E}_\eta \tau_{T_N(E)\setminus\eta}$. Not surprisingly, we will draw on our formulas for mean hitting times of Section 4. In fact, (4.2) reads here

$$\begin{aligned}
\mathbb{E}_\eta \tau_{T_N(E)\setminus\eta} &= \frac{1}{\operatorname{cap}_\eta(T_N(E)\setminus\eta)} \sum_{\sigma\notin T_N(E)\setminus\eta} \mu_{\beta,N}(\sigma) h_{\eta,T_N(E)\setminus\eta}(\sigma) \\
&= \frac{1}{\mu_{\beta,N}(\eta)\mathbb{P}_\eta[\tau_{T_N(E)}<\tau_\eta]} \left(\mu_{\beta,N}(\eta) + \sum_{\sigma\notin T_N(E)} \mu_{N,\beta}(\sigma)\mathbb{P}_\sigma[\tau_\eta < \tau_{T_N(E)\setminus\eta}]\right) \\
&= \frac{1}{e^{\beta\sqrt{N}E_\eta}\mathbb{P}_\eta[\tau_{T_N(E)}<\tau_\eta]} \left(e^{\beta\sqrt{N}\eta} + \sum_{\sigma\notin T_N(E)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}_\sigma[\tau_\eta < \tau_{T_N(E)\setminus\eta}]\right)
\end{aligned} \tag{9.16}$$

as in (9.15) we have that

$$e^{\beta\sqrt{N}E_\eta}\mathbb{P}_\eta[\tau_{T_N(E)} < \tau_\eta] = \mathbb{P}^\circ_\eta[\tau_{T_N(E)} < \tau_\eta] \tag{9.17}$$

while

$$\mathbb{P}_x[\tau_\eta < \tau_{T_N(E)\setminus\eta}] = \mathbb{P}^\circ_\sigma[\tau_\eta < \tau_{T_N(E)\setminus\eta}] \tag{9.18}$$

This allows to express our mean time entirely in terms of probabilities computed in the simple random walk:

$$\begin{aligned}
&\mathbb{E}_\eta \tau_{T_N(E)\setminus\eta} \\
&= \frac{1}{\mathbb{P}^\circ_\eta[\tau_{T_N(E)} < \tau_\eta]} \left(e^{\beta\sqrt{N}E_\eta} + \sum_{\sigma\notin T_N(E)} e^{\beta\sqrt{N}E_\sigma}\mathbb{P}^\circ_\sigma[\tau_\eta < \tau_{T_N(E)\setminus\eta}]\right)
\end{aligned} \tag{9.19}$$

and those have all been computed in Proposition 8.1. With $|T_N(E)| \equiv M$, this gives

$$\begin{aligned} \mathbb{E}_\eta \tau_{T_N(E)\setminus\eta} &= \frac{1}{1-\frac{1}{M}} \left(e^{\beta\sqrt{N}E_\eta} + \sum_{\sigma \notin T_N(E)} e^{\beta\sqrt{N}E_\sigma} \frac{1}{M} \right) (1+O(1/N)) \\ &= \frac{e^{\beta\sqrt{N}E_\eta}}{1-\frac{1}{M}} \left(1 + M^{-1} e^{-\beta\sqrt{N}E_\eta} Z_{\beta,N}(T_N(E)^c)\right)(1+O(1/N)) \end{aligned} \tag{9.20}$$

where

$$Z_{\beta,N}(T_N(E)^c) \equiv \sum_{\sigma \notin T_N(E)} e^{\beta\sqrt{N}E_\sigma} \tag{9.21}$$

Thus we are reduced to computing a single, purely equilibrium quantity, the restricted partition function $Z_{\beta,N}(T_N(E)^c)$. Note that the two terms in the bracket correspond rather naturally to the time it takes the process to leave for the first time η and to that time it takes then to travel from η to $T_N(E)$. The restricted partition function $Z_{\beta,N}(T_N(E)^c)$ was studied in the context of analysing the equilibrium measure of the REM, and we have very good control over it. In fact,

Lemma 9.1 The partition function $Z_{\beta,N}(T_N(E)^c)$ can be written as

$$Z_{\beta,N}(T_N(E)^c) = \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{\alpha-1}\left(1 + \mathcal{V}_{N,E} e^{E/2} \frac{\alpha-1}{\sqrt{2\alpha-1}}\right) \tag{9.22}$$

where $\mathcal{V}_{N,E}$ is a random variable of mean zero and variance one, all of those moments are finite.

Using this information, we can express our mean time as follows:

Lemma 9.2 With the preceding notation

$$\begin{aligned} \mathbb{E}_\eta \tau_{T_N(E)\setminus\eta} = \frac{e^{\beta\sqrt{N}u_N(0)+\alpha u_N^{-1}(E_\eta)}}{1-\frac{1}{M}} \\ \left(1 + \frac{e^{-\alpha u_N^{-1}(E_\eta)+(\alpha-1)E}}{M(\alpha-1)}\left(1+\mathcal{V}_{N,E} e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right)\right)(1+O(1/N)) \end{aligned} \tag{9.23}$$

Notice that the second term in the bracket tends to zero as $E \downarrow -\infty$ for any "fixed" η, but this convergence is not uniform. Note that $M \sim e^{-E}$. Modulo this non-uniformity, this result does however support the assumption in the trap model that modulo an overall factor $(\exp(\beta\sqrt{N}u_N(0)))$, the mean exit time from η converges to a random variable of the form $\exp(\sigma e_\eta)$ where e_η is exponentially distributed with mean one.

Next we should investigate the exponential distribution of these times. Our usual way is to look at Laplace transforms. To do so, we have to get some control on the spectrum. What corresponds to our previous a priory estimates is now the bound on the maximal mean time it takes to hit $T_N(E)$. Form the analysis above we deduce readily

Lemma 9.3 With the notation from above, let

$$\begin{aligned}\widehat{\Theta}(E) \quad &\equiv \quad (1-\tfrac{1}{|T(E)|})^{-1} e^{\beta\sqrt{N}u_N(0)+\alpha E} \\ &\left[1+\frac{e^{-E}}{|T(E)|(\alpha-1)}\left(1+\mathcal{V}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right)\right](1+O(1/N))\end{aligned} \tag{9.24}$$

Then

$$\Theta(E) \equiv \max_{\sigma\in\mathcal{S}_N} \mathbb{E}_\sigma \tau_{T(E)} \leq \widehat{\Theta}(E) \tag{9.25}$$

Using this a priori estimate, by using renewal equations and Taylor expansions exclusively for the ensuing Laplace transforms of times that terminate on arrival at $T_N(E)$, we can then proof the rather detailed estimate on $G^\sigma_{T_N(E)\backslash\sigma}(u) \equiv \mathbb{E}_\sigma e^{u\tau_{T_N(E)\backslash\sigma}}$:

Theorem 9.1 *For any $\sigma \in T(E)$, the Laplace transform $G^\sigma_{T(E)\backslash\sigma}(u)$ can be written as*

$$G^\sigma_{T(E)\backslash\sigma}(u) = \frac{a_\sigma}{1-(1-e^{-u})\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma} b_\sigma} + R_\sigma(u) \tag{9.26}$$

where

$$a_\sigma = 1 + O(\widehat{\Theta}(E)/\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma}) \tag{9.27}$$

$$b_\sigma = 1 + O(\widehat{\Theta}(E)/\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma}) \tag{9.28}$$

and $R_\sigma(u)$ is analytic in the half-plane $\Re(u) < 1/\widehat{\Theta}(E)$, periodic with period 2π in the imaginary direction, and satisfies
(i) for all $|u| \leq a/\widehat{\Theta}(E)$,

$$|R_\sigma(u)| \leq C(a)\left(e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)\right)^2 \tag{9.29}$$

and
(ii) for all u with $\Re(u) < (1-\epsilon)\widehat{\Theta}(E)$ and $|1-e^{-u}| \geq 2\epsilon^{-1}e^{-\beta\sqrt{N}E_\sigma}$

$$|R_\sigma(u)| \leq 2\frac{e^{-\beta\sqrt{N}E_\sigma}}{|1-e^{-u}|(1-\Re(u)\widehat{\Theta}(E))} \tag{9.30}$$

Moreover,

$$a_\sigma + R_\sigma(0) = 1 \tag{9.31}$$

This proposition allows in fact to prove very good estimates on the distribution function of $\tau_{T(E)\backslash\sigma}$. Note first that if

$$\mathcal{L}(u) \equiv \sum_{n=0}^{\infty} e^{un} \mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} > n] \tag{9.32}$$

then

$$\mathcal{L}(u) = \frac{G^\sigma_{T(E)\backslash\sigma}(u) - 1}{e^u - 1} \tag{9.33}$$

Corollary 9.1 *With the notation of Theorem 9.1, for any $\epsilon > 0$ and for any positive integer $n \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} = n] = \;& \frac{a_\sigma}{\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma} b_\sigma} e^{-n/\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma} b_\sigma} \\ & + O\left(e^{-n(1-\epsilon)/\widehat{\Theta}(E)} e^{-\beta\sqrt{N}E_\sigma} \epsilon^{-1} \ln\left(\widehat{\Theta}(E)\epsilon\right)\right)\end{aligned} \tag{9.34}$$

and (for $n > 0$)

$$\begin{aligned}\mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} > n] = \;& a_\sigma e^{-n/\mathbb{E}_\sigma \tau_{T(E)\backslash\sigma} b_\sigma} \\ & + O\left(e^{-n(1-\epsilon)/\widehat{\Theta}(E)} e^{-\beta\sqrt{N}E_\sigma} \widehat{\Theta}(E)\epsilon^{-1}\right)\end{aligned} \tag{9.35}$$

For the proofs of these assertions, see [9].

We see that the deviations from the exponential distribution are substantial, when E_σ is close to E. This means that the assumptions leading to the trap model are not fulfilled, and that we cannot expect that the trap model is a true "limit" of our dynamics. On the other hand, when E_σ is fixed and $E \downarrow -\infty$, the distribution of the exit time converges indeed to the exponential distribution. Thus, we may still hope that with regard to the long-term asymptotics, the trap model yields the correct predictions.

9.3. THE RENEWAL EQUATIONS

The first step now is to define a correlation function that has a good chance to resemble the one used in the trap model. A good choice seems the probability that the process does not jump from a state in the top to another state in the top during a time interval of the form $[n, n+m]$. To define this precisely, we introduce the following random times. For any $k \in \mathbb{N}$, let k_- denote the last time before k at which the process has visited the top, i.e.

$$k_- \equiv \sup\{l < k \mid \sigma(l) \in T_N(E)\} \tag{9.36}$$

Then set

$$\Pi(n,m,N,E) \equiv \mathbb{P}\left[\forall_{k\in[n+1,n+m]}\, \sigma(k) \notin T_N(E)\backslash\sigma(k_-)\right] \tag{9.37}$$

To be as close as possible to Bouchaud, the natural choice is the uniform distribution on $T_N(E)$ that we will denote by π_E. However, we will also need to introduce the respective functions with starting point in an arbitrary state σ. Thus we set

$$\Pi_\sigma(m,n,N,E) \equiv \mathbb{P}\left[\forall_{k\in[n+1,n+m]}\, \sigma(k) \notin T_N(E)\backslash\sigma(k_-) \mid \sigma(0)=\sigma\right] \tag{9.38}$$

and

$$\Pi(m,n,N,E) \equiv \frac{1}{|T_N(E)|}\sum_{\sigma\in T_N(E)} \Pi_\sigma(m,n,N,E) \tag{9.39}$$

We will also use vector notation and write $\underline{\Pi}(n,m,N,E)$ for the M dimensional vector with components $\Pi_\sigma(n,m,N,E)$, $\sigma \in T_N(E)$.

Note that it is easy to derive a renewal equation for the quantities (9.38). Just observe that event in the probability in (9.38) occurs either

(i) $\sigma(k) \notin T(E)\backslash\sigma$, for all $k \in [0, n+m]$, or
(ii) there is $0 < l \leq n$, s.t. $l = \inf\{k \leq n \mid \sigma(k) \in T(E)\backslash\sigma\}$, and $\forall_{k\in[n+1,n+m]}\, \sigma(k) \notin T_N(E)\backslash\sigma(k_-)$.

Since this decomposition is disjoint, it implies system of renewal equations:

$$\begin{aligned}
\Pi_\sigma(m,n,E) &= \mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} > m+n] \\
&+ \sum_{k=1}^{n}\sum_{\sigma'\in T(E)\backslash\sigma} \mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} = k, X_k=\sigma', X_l \notin T(E)\backslash X_{l_-}, \forall n \leq l \leq m+n] \\
&= \mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} > m+n] \\
&\quad + \sum_{k=1}^{n}\sum_{\sigma'\in T(E)\backslash\sigma} \mathbb{P}_\sigma[\tau_{\sigma'} = \tau_{T(E)\backslash\sigma} = k]\Pi_{\sigma'}(m, n-k, E)
\end{aligned} \tag{9.40}$$

Now it would be nice to be able to transform this into a single equation for Π by summing over σ. This would work if we had the relation

$$\mathbb{P}_\sigma[\tau_{\sigma'} = \tau_{T(E)\backslash\sigma} = k] = \frac{\pi_E(\sigma')}{1-\pi_E(\sigma)}\mathbb{P}_\sigma[\tau_{T(E)\backslash\sigma} = k] \tag{9.41}$$

sadly this is not true, and we cannot even proof a reasonable approximate version of this. All we can show, that this relation holds averaged over k. This leaves us for the time being with no alternative but to analyse the full system of equations (9.40).

The method of choice for doing this are Laplace transforms.

We set

$$\Pi^*_\sigma(m,u,E) \equiv \sum_{n=0}^{\infty} e^{nu}\Pi_\sigma(m,n,E) \tag{9.42}$$

for $u \in \mathbb{C}$ whenever this sum converges. Let us define

$$F^*_\sigma(m,u) \equiv \sum_{n=0}^{\infty} e^{nu}\mathbb{P}_\sigma[\tau_{T(E)\setminus\sigma} > m+n] \tag{9.43}$$

Then it follows from (9.40) that for any $\sigma \in T(E)$,

$$\Pi^*_\sigma(m,u,E) = F^*_\sigma(m,u) + \sum_{\sigma' \in T(E)\setminus\sigma} G^\sigma_{\sigma',T(E)\setminus\sigma}(u)\Pi^*_{\sigma'}(m,u,E) \tag{9.44}$$

Let us denote by $K^*_E(u)$ the $|T(E)| \times |T(E)|$ matrix with elements[2]

$$(K^*_E(u))_{\sigma,\sigma'} \equiv \begin{cases} G^\sigma_{\sigma',T(E)\setminus\sigma}(u) & \text{if} \quad \sigma \neq \sigma' \\ 0, & \text{if} \quad \sigma = \sigma' \end{cases} \tag{9.45}$$

Then clearly the solution of equation (9.44) can be written as

$$\underline{\Pi}^*(m,u,E) = \left(\left[1\!\!1 - K^*_E(u) \right]^{-1} K^*_E(u) + 1\!\!1 \right) \underline{F}^*(m,u) \tag{9.46}$$

where $\underline{\Pi}^*$ and $\underline{F}^*$ denote the vectors with components Π^*_σ, and F^*_σ.

The matrix

$$M^*_E(u) \equiv \left[1\!\!1 - K^*_E(u) \right]^{-1} K^*_E(u) \tag{9.47}$$

is known as the Laplace transform of the resolvent of the system of renewal equations.

Our task is to compute the inverse Laplace transform of the right hand side of (9.46). This requires estimates in the complex u-plane. Basically, there are two difficulties:

i) Inversion of the matrix $1\!\!1 - K^*_E(u)$. This is in general quite difficult, and we will not be able to do this for all values of u. However, we are greatly helped by the fact that at $u = 0$, the matrix $K^*_E(0)$ has a very simple form in that it has almost constant columns. Thus the vector

[2] We will often write $K^*_{\sigma,\sigma'}(u)$ instead of $(K^*_E(u))_{\sigma,\sigma'}$ whenever no confusion is possible

1 is an eigenvector with eigenvalue zero, and all other eigenvalues are close to zero. This property can be carried over perturbatively to small values of $|u|$. This will allow us to compute on a small neighborhood of the origin the inverse to leading orders in $1/u$, which will be responsible for the leading long time behaviour of the inverse Laplace transform.

ii) For the use of the Laplace inversion formula, we need reasonable control of the Laplace transform also away from the origin. We expect that these not to give important contributions, but this requires both a judicious choice of the integration contour in the complex plane, and some bounds on the integrands on these contours. Let us recall that

$$\Pi(n,m,E) = \frac{1}{2\pi i}\int_{-i\pi}^{i} \pi du e^{-un}\left[(\mathbb{1}, M_E^*(u)\underline{F}^*(m,u)) + (\mathbb{1}, F^*(m,u))\right] \tag{9.48}$$

We will deform the integration path to the contour $\mathcal{C}$ given as follows: consisting of the three parts

$$\mathcal{A} \equiv \left\{u \in \mathbb{C} \,:\, \Re z = 1/2,\, |\Im z| \in [1/\sqrt{2\kappa}, \pi\widehat{\Theta}]\right\} \tag{9.49}$$

$$\mathcal{B} \equiv \left\{u \in \mathbb{C} \,:\, \Re z \in [1/\tilde{t}, 1/2],\, \Re z = \kappa|\Im z|^2\right\} \tag{9.50}$$

and

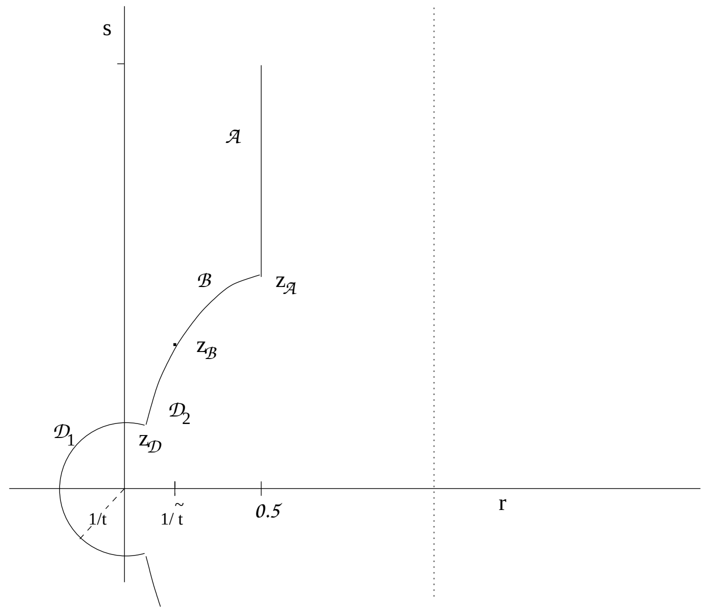
$$\mathcal{D} \equiv \mathcal{D}_1 \cup \mathcal{D}_2 \tag{9.51}$$

where

$$\begin{aligned} \mathcal{D}_1 &\equiv \left\{u \in \mathbb{C} \,:\, |z| = 1/t\,,\, \Re z < c|\Im z|^2\right\} \\ \mathcal{D}_2 &\equiv \Big\{u \in \mathbb{C} \,:\, \Re z \in [\sqrt{1/(4\kappa^2) + 1/t^2} - 1/(2\kappa), 1/\tilde{t}], \\ &\qquad \Re z = \kappa|\Im z|^2\Big\} \end{aligned} \tag{9.52}$$

Here t and κ are positive parameters that are assumed to be chosen such that $\mathcal{C}$ lies in the domain of validity of certain estimates. In what follows, t must be thought of as very large compared with one. At this stage no constraint is imposed on the parameter $\tilde{t}$; it will be chosen as $\tilde{t} = t^\eta$, for suitable $0 < \eta < 1$, later. For future reference let us define the points:

$$\begin{array}{lll} z_A = r_A + is_A, & z_B = r_B + is_B, & z_D = r_D + is_D \\ r_A = 1/2, & r_B = 1/\tilde{t}, & r_D = \sqrt{1/(4\kappa^2) + 1/t^2} - 1/(2\kappa) \\ s_A = 1/\sqrt{2\kappa}, & s_B = 1/\sqrt{\kappa\tilde{t}}, & s_D = ((\sqrt{1 + (2\kappa/t)^2} - 1)/2\kappa^2)^{1/2} \end{array}$$



The contour C in the variables r and s

The main contribution of the integral will come from the integral along $\mathcal{D}$.

Our main result is then the following

Theorem 9.2 *Let $\beta > \sqrt{2\ln 2}$. Then there is a sequence $c_N \sim \exp(\beta\sqrt{N}u_N(E))$ such that for any $\epsilon > 0$*

$$\lim_{t,s\uparrow\infty}\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty} P\left[\left|\frac{\Pi([c_N s],[c_N t],N,E)}{\Pi_\infty(s,t)} - 1\right| > \epsilon\right] = 0 \tag{9.53}$$

where $\Pi_\infty(s,t)$ is the limiting correlation function of the trap model.

In the sequel I will indicate the main steps of the necessary estimates, while for details I have to refer to [3].

9.4. UNIFORM ESTIMATES ON M_E^*

The first thing we want to do is to show that the resolvent is small as soon as we go away from zero along our path $\mathcal{C}$. It is almost a miracle that we are able to do this, because for this we need to show that the norm of $K_E^*(u)$ is smaller than 1. But at $u=0$, K_E^* has an eigenvalue 1, so we are in a delicate situation where we really need to get quite precise estimates. In particular, we need to use a norm that gives the right values 1 at zero.

Fortunately, operator norm in $L_\infty(\mathbb{C}^M)$, which is given as

$$\|K\| \equiv \max_{\sigma\in T(E)} \sum_{\sigma'\in T(E)} |K_{\sigma,\sigma'}| \tag{9.54}$$

has the right property and is well suited for our matrices. First, its value is one at $u = 0$. Second, we can show that it decays along the imaginary axis up to the value $\pm i\pi$ at rate $v\bar{\Theta}^{-1}$. More precisely, we have

Lemma 9.4 Let $v \in [-\pi, \pi]$. Then (for N large enough),

$$\|K_E^*(iv)\| \leq \frac{1}{\sqrt{2(1-\cos v)\bar{\Theta}^2\left(1 - O(\bar{\Theta}^{-1})\right) + 1} - \frac{4}{M-1}\left(1 + O(d/N)\right)} \tag{9.55}$$

The proof of the Lemma is somewhat tedious, but essentially makes use of the simple idea to use renewal to represent

$$K^*_{\sigma,\sigma'}(u) = \frac{G^\sigma_{\sigma',T(E)}(u)}{1 - G^\sigma_{\sigma,T(E)}(u)} \tag{9.56}$$

Here the numerator is always small, if u is purely imaginary. On the other hand, the Laplace transform in the denominator is dominated by the process realising the event to go from σ to σ in a single step. This part is easily computed and gives, if $u = iv$, $1 - e^{iv}p(\sigma,\sigma)$. This yields roughly the behaviour

$$\|K_E^*(iv)\| \sim \frac{\mathbb{P}_\sigma[\tau_{T(E)\setminus\sigma} < \tau_{T(E)}]}{|1 - e^{iv}|} \tag{9.57}$$

which explains the estimate given in the lemma.

The estimate (9.55) can now be extended a little bit off the imaginary axis in the positive real direction. This uses simply Taylor expansions about $u = iv$; of course, the trick is to use again the representation (9.56) and to exploit the fact that the functions appearing in the numerator and denominator are analytic in the the half-space $\Re u < 1/\bar{\Theta}$, well beyond the first pole of K_E^* itself.

To state these estimates, we need some notation: First, let

$$z = \widehat{\Theta}(E)u \tag{9.58}$$

The real and imaginary parts of z will always be called r and s:

$$z = r + is \tag{9.59}$$

Thus

$$\begin{aligned} r &= \widehat{\Theta}(E)w \\ s &= \widehat{\Theta}(E)v \end{aligned} \tag{9.60}$$

Definition 9.1 $0 < C_1, C_2 < \infty$, and $0 < \gamma < 1$ be numerical constants. With the above notation we define the sets:

$$\begin{aligned} D_1(C_1) &\equiv \left\{u \in \mathbb{C} : \sqrt{r^2+s^2} \geq C_1/\sqrt{M}\right\} \\ D_2(C_2,\gamma) &\equiv \left\{u \in \mathbb{C} : 0 \leq r < \min\left(\tfrac{\gamma s^2}{C_2\sqrt{1+s^2}}, 1-\gamma\right),\ \ v \in [-\pi,\pi]\right\} \\ D_3 &\equiv \{u \in \mathbb{C} : -1 \leq r < 0,\ |s| < 1\} \\ D_4 &\equiv \{u \in \mathbb{C} : |r| < 1,\ |s| < 1\} \end{aligned} \tag{9.61}$$

Lemma 9.5 There exist constants $0 < C, C' < \infty$ such that, for all $0 < \gamma < 1$ and all $u \in D_2(C',\gamma)$,

$$\|K_E^*(u)\| \leq \frac{1 + C\gamma^{-1}r}{\sqrt{1+\widehat{\Theta}^2 2(1-\cos v)(1-O(\bar{\Theta}^{-1}))} - \frac{4}{M-1}(1+O(d/N)) - C'\gamma^{-1}r} \tag{9.62}$$

Consequently, for all $0 < \gamma < 1$ there exists a constant $0 < L < \infty$ (depending on C, C' and γ) such that, for all $u \in D_1(4) \cap D_2(L,\gamma)$,

$$\|K_E^*(u)\| < 1 \tag{9.63}$$

and

$$\|M_E^*(u)\| \leq \frac{1 + C\gamma^{-1}r}{\sqrt{1+\widehat{\Theta}^2 2(1-\cos v)(1-O(\bar{\Theta}^{-1}))} - 1 - \frac{4}{M-1}(1+O(d/N)) - (C+C')\gamma^{-1}r} \tag{9.64}$$

The last estimate of this kind we will need concerns the case when $|u|$ is very small and $w \leq 0$. Its derivation is very similar to that of the preceding ones.

Lemma 9.6 For M large enough,
(i) for all $u \in D_3$,

$$\|K_E^*(u)\| \leq \frac{1}{\sqrt{1+r^2+s^2} - \frac{5}{M}} \tag{9.65}$$

(i) for all $u \in D_1(4) \cap D_3$, $\|K_E^*(u)\| < 1$ and

$$\|M_E^*(u)\| \leq \frac{1}{\sqrt{1+r^2+s^2} - 1 - \frac{5}{M}} \tag{9.66}$$

The estimates in this subsection suffice to show that the the contributions from the integration excepting the little circle around the origin do not give a significant contribution. To extract the leading behaviour, we need of course far more precise control on our kernels in this domain.

9.5. PERTURBATIVE ESTIMATES FOR SMALL u

So far we have been able to avoid the problem of inverting the matrix $1-K^*(u)$. Fortunately, for small u, this matrix is very close to a matrix with constant columns whose inverse we can of course compute easily. The idea is thus to use perturbation theory.

The basis is the expansion

$$\begin{aligned} K^*_{\sigma,\sigma'}(u) &= \tfrac{1}{1-G^{\sigma}_{\sigma,T}(u)}\left(G^{\sigma}_{\sigma',T}(0)+u\tfrac{d}{du}G^{\sigma}_{\sigma',T}(0)+\tfrac{u^2}{2}\tfrac{d^2}{du^2}G^{\sigma}_{\sigma',T}(\tilde u)\right)\\ &= \tfrac{1}{1-G^{\sigma}_{\sigma,T}(u)}\left(\mathbb{P}[\tau^{\sigma}_{\sigma'}\le\tau^{\sigma}_{T}]+u\mathbb{E}\tau^{\sigma}_{\sigma'}1\!\!1_{\{\tau^{\sigma}_{\sigma'}\le\tau^{\sigma}_{T}\}}+\tfrac{u^2}{2}\tfrac{d^2}{du^2}G^{\sigma}_{\sigma',T}(\tilde u)\right) \end{aligned} \tag{9.67}$$

We define

$$\begin{aligned} \mathcal{K}^{*(0)}_{\sigma,\sigma'}(u)\equiv\ & \tfrac{1}{1-G^{\sigma}_{\sigma,T}(u)}\left(\tfrac{1}{M}\mathbb{P}[\tau^{\sigma}_{T\backslash\sigma}<\tau^{\sigma}_{T}]\left(1+u\mathbb{E}[\tau^{\sigma}_{T\backslash\sigma}|\tau^{\sigma}_{T\backslash\sigma}=\tau^{\sigma}_{T}]\right)\right),\\ & \forall\sigma,\sigma'\in T(E) \end{aligned} \tag{9.68}$$

as the leading part. We then prove a norm estimate, valid for

$$\left\{u\in\mathbb{C}\mid r\le s^2/4\right\}\subseteq D_2(L,\gamma)\cap D_4 \tag{9.69}$$

that states that

$$\left|K^*(u)-\mathcal{K}^{*(0)}(u)\right|\le\frac{C\gamma^{-1}(s^2+r^2)+(1+\sqrt{s^2+r^2})O(1/(M-1))}{\sqrt{1+(s^2+r^2)/2}-5/M} \tag{9.70}$$

Now $\mathcal{K}^{*(0)}(u)$ has a unique non-zero eigenvalues

$$\lambda(u)\equiv\sum_{\sigma\in T}\mathcal{K}^{*(0)}_{\sigma,\sigma'}(u) \tag{9.71}$$

The corresponding left eigenvector is proportional to $(1,1,\dots,1)$. Based on this, we define

$$M^{*(0)}(u)\equiv[1\!\!1-\mathcal{K}^{*(0)}(u)]^{-1}\mathcal{K}^{*(0)}(u) \tag{9.72}$$

and decompose the Laplace transform of the resolvent (defined in (9.47)) into

$$M^*(u)\equiv M^{*(0)}(u)+M^{*(1)}(u) \tag{9.73}$$

It is a simple matter to see that

Lemma 9.7 Set

$$\begin{aligned} R(u) &\equiv [\mathbb{1} - \mathcal{K}^{*(0)}(u)]^{-1} \\ \rho(u) &\equiv \max\left(|1-\lambda(u)|^{-1}, 1\right) \end{aligned} \tag{9.74}$$

Then,

$$M^{*(1)}(u) = R(u)\mathcal{K}^{*(1)}(u)R(u)\frac{1}{\mathbb{1} - R(u)\mathcal{K}^{*(1)}(u)} \tag{9.75}$$

and, if $\|R(u)\mathcal{K}^{*(1)}(u)\| < 1$,

$$\|M^{*(1)}(u)\| \leq \frac{\|\mathcal{K}^{*(1)}(u)\|\rho(u)^2}{1 - \|\mathcal{K}^{*(1)}(u)\|\rho(u)} \tag{9.76}$$

It will turn out that this estimate is good enough to show that $M^{*(1)}(u)$ can be neglected. The main remaining issue is now to compute the eigenvalue $\lambda(u)$. Explicitly, we have

$$1 - \lambda(u) = \frac{1}{|T|}\sum_{\sigma\in T}\left[1 - \frac{G^{\sigma}_{T\backslash\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)}\left(1 + u\mathbb{E}[\tau^{\sigma}_{T\backslash\sigma} | \tau^{\sigma}_{T\backslash\sigma} = \tau^{\sigma}_{T}]\right)\right] \tag{9.77}$$

This is still quite a complicated expression, and we need to do some more simplification. The key is the following lemma.

Lemma 9.8 Recall that $u = z/\widehat{\Theta}(E)$ and set

$$z_\sigma \equiv \left(1 - \tfrac{1}{M}\right) e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E) \tag{9.78}$$

If u belongs to the set

$$D_\delta \equiv \left\{u \in \mathbb{C} \mid r < s^2/4\,,\, |z| \leq \delta\right\}, \quad 0 < \delta < 1 \tag{9.79}$$

then, for N large enough,

$$\left|1 - \frac{G^{\sigma}_{T\backslash\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)}\left(1 + u\mathbb{E}[\tau^{\sigma}_{T\backslash\sigma} | \tau^{\sigma}_{T\backslash\sigma} = \tau^{\sigma}_{T}]\right) - \frac{z}{z - z_\sigma}\right| \leq C(\delta)|z| \tag{9.80}$$

for some constant $0 < C(\delta) < \infty$ that only depends on δ.

The point of this lemma is that it shows that the summands in (9.77) can be replaced by $z/(z - z_\sigma)$, since this is dominating the error of order $|z|$. The nice thing is that the sum over these leading terms can be expressed as integrals with respect to our Poisson point process. Also, we see that these are now Laplace transforms of exponentially distributed random variables.

Thus, we are approaching something the looks like the trap model. Note, however, that the proof of Lemma 9.8 is quite involved.

9.6. POISSON CONVERGENCE AND SELF AVERAGING

The next step is now to represent the sum over the $z/(z-z_\sigma)$ as an integral with respect to the point process $\sum_\sigma \delta_{u_N^{-1}(E_\sigma)}$. Then we want to use to facts: first, that this point process converges to a Poisson point process, and second that the integral over that Poisson point process converges, as we take the cut of E to minus infinity, to a deterministic integral. Both facts are of course well known.

Let us write

$$z_\sigma = (1-1/M)e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E) \equiv \frac{1}{e^{\alpha(u_N^{-1}(E_\sigma)-E)}\tau_{E,N}} \tag{9.81}$$

It will also be convenient to define

$$\mathcal{N}^*_{N,E} \equiv \sum_{\sigma\in\{-1,1\}^N} \delta_{\exp\{\alpha(-E+u_N^{-1}(E_\sigma))\}} = \sum_{\sigma\in\{-1,1\}^N} \delta_{1/(z_\sigma\tau_{N,E})} \tag{9.82}$$

It is easy to see that this process converges weakly to the Poisson point process $\mathcal{N}^*_E$ on $[1,\infty)$ with intensity measure $\alpha^{-1}e^E x^{-1-1/\alpha}dx$. On the other hand, one can show without difficulty that

$$\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}\tau_{N,E} = 1-1/\alpha \equiv \tau_\infty, \quad \text{in Probability.} \tag{9.83}$$

The following Lemma yields all we need to analyse sums like (9.77).

Lemma 9.9 Let g be a bounded continuous function on $\mathbb{R}^+$, such that $\left|\int_0^\infty \frac{dx}{x^{1+1/\alpha}}g(x)\right| < +\infty$, and let X_N be a family of positive random variables that converge in distribution to the positive random variable X. Then for any $b>0$,

(i) $\int_b^\infty \mathcal{N}^*_{N,E}(dx)g(xX_N)$ converges, as $N\uparrow\infty$, to the random variable $\int_b^\infty \mathcal{N}^*_E(dx)g(xX)$.

(ii) If X_E is a family of random variables such that, as $E\downarrow-\infty$, $X_E\to a\in\mathbb{R}^+$ almost surely, then

$$\lim_{E\downarrow-\infty} e^{+E}\int_1^\infty \mathcal{N}^*_E(dx)g(xX_E) = \alpha^{-1}\int_1^\infty \frac{dx}{x^{1+1/\alpha}}g(xa), \quad \text{a.s.} \tag{9.84}$$

(iii) If g is a complex valued function on $\mathbb{C}$, and if for some domain $B\subset\mathbb{C}$, for all $x\in\mathbb{R}^+$, $z\in B$, $g(zx)$ is bounded, and for all $z\in B$,

$$\left|\int_0^\infty \frac{dx}{x^{1+1/\alpha}}g(zx)\right| < \infty \tag{9.85}$$

holds, then

$$\lim_{E\downarrow-\infty} P\left[\lim_{N\uparrow\infty}\sup_{z\in B}\left| e^E\int_1^\infty \mathcal{N}_E^*(dx)g(zxX_E)- (az)^{1/\alpha}\alpha^{-1}\int_{az}^\infty \frac{dx}{x^{1+1/\alpha}}g(x)\right| > \epsilon\right] = 0 \tag{9.86}$$

Applying this lemma to the sum in (9.77) yields the

Corollary 9.2 *Uniformly in* $\Re(z) < \max(|\Im(z)|, 1/2)$,

$$\begin{aligned}\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty} & (1-\lambda(u)) \\ &= \alpha^{-1}\int_1^\infty \frac{dx}{x^{1+1/\alpha}}\frac{xz\tau_\infty}{xz\tau_\infty - 1} + O(|z|) \\ &= (-z\tau_\infty)^{1/\alpha}\pi\mathrm{cosec}(\pi/\alpha) + O(|z|), \quad \textit{in Probability.}\end{aligned} \tag{9.87}$$

9.7. LAPLACE INVERSION

Let us now show how roughly how to go back to real space. The term that will give the main contribution is of the form

$$\left(1\!\!1, M^{*(0)}(u)\underline{F}^*(m,u)\right) = \frac{\lambda(u)}{1-\lambda(u)}\left(1\!\!1, \underline{F}^*(m,u)\right) \equiv h_{N,E}(u,m) \tag{9.88}$$

It turns out the the limit of this expression has the following nice representation:

Proposition 9.2 *For* u *on* $\mathcal{C}$, *we have that*

$$\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty} h_{N,E}(u,m) = H_0^*(s,z)(1+O(|z|^{1-1/\alpha}, |z|^{1/\alpha})) + O(z^{-1/\alpha}e^{-s/\tau_\infty}) \tag{9.89}$$

where $H_0^*(s,u) \equiv \int_0^\infty dte^{zt}\int_{s/t}^\infty \frac{dx}{x^{1/\alpha}(1+x)}$ *is the Laplace transform of the function* H_0 *defined in (9.13).*

Proof. As we already know $\lambda(u)$, the main work goes into the analysis of $(1\!\!1, \underline{F}^*(m,u))$. This goes largely parallel to the analysis of $M^*(u)$. It turns out that the leading contribution can be written in the form

$$(1\!\!1, \underline{F}^*(m,u)) \sim \frac{1}{|T(E)|}\sum_{\sigma\in T(E)} e^{-me^{-\sqrt{N}E_\sigma}}\frac{\widehat{\Theta}(E)}{z_\sigma - z} \tag{9.90}$$

and

Lemma 9.10 Set $s \equiv m/\widehat{\Theta}$. Then, uniformly on $\Re z < \max(\Im z, 1/2)$, and $\Re(u) \leq |\Im u|$,

$$\lim_{E\downarrow -\infty} \lim_{N\uparrow\infty} \frac{1}{|T(E)|} \sum_{\sigma\in T(E)} e^{-me^{-\beta\sqrt{N}E_\sigma}} \frac{1}{z_\sigma - z}$$

$$= (-z\tau_\infty)^{1/\alpha} \left(z^{-1}\pi\mathrm{cosec}(\pi/\alpha) - \int_0^\infty dte^{zt} \int_0^{s/t} \frac{dx}{x^{1/\alpha}(1+x)} \right) \tag{9.91}$$

$$+O(e^{-s/\tau_\infty}) \quad \text{in Probability.}$$

Thus

$$\lim_{E\downarrow -\infty} \lim_{N\uparrow\infty} \frac{\lambda(u)}{1-\lambda(u)} \frac{1}{|T(E)|} \sum_{\sigma\in T(E)} e^{-me^{-\sqrt{N}E_\sigma}} \frac{\widehat{\Theta}(E)}{z_\sigma - z}$$

$$= u^{-1} - \frac{\int_0^\infty dte^{ut} \int_0^{s/t} \frac{dx}{x^{1/\alpha}(1+x)}}{\pi\mathrm{cosec}(\pi/\alpha)} \left(1 + O(|z|^{1-1/\alpha}, |z|^{1/\alpha})\right) \tag{9.92}$$

$$+O\left(z^{-1/\alpha} e^{-s/\tau_\infty}\right)$$

The leading term is readily identified as the Laplace transform of

$$H_0(s/t) \equiv 1 - \frac{\int_0^{s/t} \frac{dx}{x^{1/\alpha}(1+x)}}{\pi\mathrm{cosec}(\pi/\alpha)} \tag{9.93}$$

which we recognise as precisely the function that appeared as the leading asymptotic contribution in the trap model.

It remains the rather painstaking task to show that indeed all other contributions can be neglected. The interested reader will find them in [3]. In any case the logic of these terms is simple: First off all, only the behaviour of a term near $u = 0$ matters, since the a priori estimates show that the contributions to the inverse Laplace transform from the path off zero are small. Then, whenever a term has a higher power in u than the leading term, its inverse Laplace transform decays faster in time. Since this is the case for all error terms we have produced, we can indeed conclude that our Main Theorem holds.

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