Institute for Applied Mathematics
Summer 2015
Prof. Dr. Karl-Theodor Sturm, Angelo Profeta

"Stochastic Processes"

Exercise sheet 9

Hand in Tue 16.6.2015 during lecture break

Exercise 1 (Knight on a chessboard) *[5 Pts.]*

Consider a knight on a chessboard. We assume that it moves by uniformly choosing one of the admissible moves from its current position. Calculate the expected number of moves the knight needs to come back to its starting point, if it starts in a corner.

Exercise 2 (Rate of convergence) *[5 Pts.]*

Let S be a finite state space and p aperiodic and irreducible. Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ starting in $x \in S$ and $(Y_n)_{n\in\mathbb{N}}$ starting in $y \in S, y \neq x$ (both with transition matrix p). Let $T := \inf\{n \geq 0 \,|\, X_n = Y_n\}$ be the first time the chains meet. Show that there is $r < 1$ and $C \in (0, \infty)$ such that $\mathbb{P}(T > n) \leq C \cdot r^n$.

Exercise 3 (Absorbing Markov chain) *[5 Pts.]*

Recall that a state $x \in S$ of a Markov chain is absorbing, if $p(x, x) = 1$. The chain is called absorbing Makov chain if there is at least one absorbing state and if it is possible to go from any state to at least one absorbing state in a finite number of steps. Assume that the state space is finite, $n := |S|$, and that there are r absorbing states. The transition matrix of such a chain can be written in the form

$$P = \begin{pmatrix} Q & R \\ 0 & \mathrm{id} \end{pmatrix},$$

where id is the $r \times r$ identity matrix, Q the $(n-r) \times (n-r)$-submatrix that describes the transitions from non-absorbing to non-absorbing states and R the $(n-r) \times r$-submatrix that describes the transitions from non-absorbing to absorbing states.

a) Show that $Q^n \to 0$ as $n \to \infty$, i.e. the probability that the chain will be absorbed is 1.

b) Conclude that $\mathrm{id} - Q$ is invertible.

c) Find an expression for the limiting distribution $\lim_{n\to\infty} P^n$.

Exercise 4 (Sorting your books by learning) [*5 Pts.*]

Given the approaching exam phase you start panicking and try to learn with a randomized strategy. Suppose you want to take exams in three lectures and for each you have one book with which you are learning (big ones like Evans' PDE book or Lang's Algebra book, so that you will use them over and over again). So you have three books, which we will denote by 1, 2 and 3, and every morning you take one of them from your shelf; the probability that you take book i is $\alpha_i \in (0,1), i = 1,2,3$. When you stop learning in the evening, you put the book you used to the left-hand end of the shelf. Let p_n be the probability that on day n you find the books in the order 1,2,3 (left-to-right) on your shelf. Show that p_n converges independently of the initial arrangement of the books as $n \to \infty$ and determine the limit.