Institute for Applied Mathematics
Summer 2015
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“Stochastic Processes”

Exercise sheet 8

Hand in Tue 9.6.2015 during lecture break

Exercise 1 (Birth-and-death chain) *[5 Pts.]*

Consider the birth and death chain on $S := \{0, 1, 2 \dots\}$ defined by the transition probabilities

$$p(x, x+1) = p_x,\ p(x, x-1) = q_x,\ p(x, x) = r_x,\ q_0 = 0,$$
$$p_x, q_x, r_x \in [0, 1],\ p_x + q_x + r_x = 1.$$

Find a reversible measure.

Exercise 2 (Unique stationary distribution) *[5 Pts.]*

Let p be irreducible and assume that there exists a stationary distribution π. The aim of this exercise is to show that every other stationary measure is a multiple of π.

a) Let $\varphi\colon (0, \infty) \to \mathbb{R}$ be a strictly concave, bounded function and define the entropy of a measure μ as

$$\mathcal{E}(\mu) := \sum_{y \in S} \varphi\left(\frac{\mu(y)}{\pi(y)}\right) \pi(y).$$

Show that $\mathcal{E}(\mu p) \geq \mathcal{E}(\mu)$ and give an example of an admissible φ.

b) Assume that $p(x, y) > 0$ for all $x, y \in S$. Show that $\mu = \mu p$ if and only if $\mathcal{E}(\mu) = \mathcal{E}(\mu p)$.

c) Define $\tilde{p}(x, y) := \sum_{n=1}^{\infty} 2^{-n} p^n(x, y)$. Show that if μ is a stationary measure for p, then it is also a stationary measure for $\tilde{p}$, and conclude that every stationary measure is a multiple of π (without the extra assumption that $p(x, y) > 0$ for every $x, y \in S$!).

Exercise 3 (Ehrenfest chain) *[5 Pts.]*

Consider N particles which are distributed over two containers A and B and let X_n be the number of particles in container A at time n. In each time step, exactly one particle is chosen randomly (with uniform distribution) and changes the containers.

a) Specify the transition matrix and show that every state is recurrent.

b) Find a stationary distribution.

c) What is the mean return time for a state $i \in \{0, \dots, N\}$?

d) Is the chain reversible?

Exercise 4 (Reversible measure) *[5 Pts.]*

Let X_n be a Markov chain with transition matrix $P = (p(i,j))_{i,j \in S}$ on the countable state space S. Show that the Markov chain has a reversible measure if and only if the following two conditions are satisfied:

(i) If $p(i,j) > 0$ then $p(j,i) > 0$.

(ii) For every sequence $x_0, x_1, \dots, x_n$ with $x_n = x_0$ and $\prod_{i=1}^n p(x_i, x_{i-1}) > 0$ it holds:

$$\prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = 1.$$

Hinweis der Fachschaft: Die Fachschaft Mathematik feiert am 11.06 ihre Matheparty in der N8schicht. Der VVK findet am Mo. 8.06., Di. 9.06. und Mi. 10.06. vor der Mensa Poppelsdorf statt. Alle weiteren Infos auch auf fsmath.uni-bonn.de