THE SHARP SOBOLEV INEQUALITY ON METRIC MEASURE SPACES WITH LOWER RICCI CURVATURE BOUNDS

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Abstract. We show that the sharp Sobolev inequality as known for Riemannian manifolds with a positive lower bound on the Ricci curvature holds in the same form for metric measure spaces satisfying the $\mathrm{RCD}^*(K,N)$ condition for positive K.

1. Introduction and statement of result

Starting with the works of Sturm [S2, S3] and Lott & Villani [LV], analysis and geometry on metric measure spaces which satisfy some generalized Ricci curvature bounds became a very active and fruitful research area. Since then, many results known for Riemannian manifolds have been generalized to this setting, although there were still some drawbacks. The recent strengthening of the $\mathrm{CD}^*(K,N)$ condition (which is based on convexity properties of entropy functionals on the Wasserstein space) to the Riemannian curvature-dimension condition $\mathrm{RCD}^*(K,N)$ in metric measure spaces done in [AGS3] and [EKS] established a link to the earlier curvature-dimension condition for Dirichlet forms of Bakry & Émery introduced in [BÉ] (which is based on Bochner's inequality). In particular, a weak version of Bochner's inequality holds in these spaces. This makes it possible to redo proofs originally done in the abstract Dirichlet form setting with minor changes. Here we are concerned with functional inequalities and especially with the sharp Sobolev inequality. To be precise, the main result will be:

Theorem 1.1 (Sharp Sobolev inequality). *Let $(X,d,\mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ space with $K>0$, $N\in(2,\infty)$. Then for every $f\in W^{1,2}(X,d,\mathfrak{m})$ it holds:*

$$\|f\|^2_{\frac{2N}{N-2}} \le \|f\|_2^2 + \frac{4}{N(N-2)}\cdot\frac{N-1}{K}\||\nabla f|_w\|_2^2. \tag{1.1}$$

(The definitions for the weak gradient $|\nabla f|_w$ and the $\mathrm{RCD}^*(K,N)$ condition will be given below.) Sharpness here means that there is an $\mathrm{RCD}^*(K,N)$ space such that there exist non-constant functions which satisfy (1.1) with equality. This is the case for the standard unit sphere in $\mathbb{R}^N$ equipped with the Riemannian distance and the normalized Riemannian measure; see [H, Theorem 5.1] for this classical result. Sharp constants in Sobolev inequalities are not only interesting in themselves, but also a useful tool in the study of nonlinear PDEs involving critical Sobolev exponents (where the embedding fails to be compact) – see for instance the famous Yamabe problem [LP]. On $\mathbb{R}^n$, the sharp constant is due to Aubin [A3] and independently Talenti [T]. Aubin also dealt with the sphere [A2]. The general Riemannian manifold case requires a positive lower bound on the Ricci curvature just as in our case and is due to Ilias [I].

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Theorem 1.1 in particular provides a positive answer to the remark [V, Remark 30.24] which asked for the validity of this inequality in $\mathrm{CD}(K,N)$ spaces – though using the stronger $\mathrm{RCD}^*(K,N)$ condition. Apart from the L^1-Sobolev inequality in [V, Theorem 30.23], many other results concerning Sobolev inequalities on metric measure spaces are in a sense more abstract, like [HK, Theorem 2.1], which shows that a space supporting some other types of inequalities also satisfies some kind of Sobolev inequality; see also [HK, SC]. Of course, there are many other types of functional inequalities which hold on $\mathrm{CD}^e(K,N)$ or $\mathrm{RCD}^*(K,N)$ spaces, among them several sorts of Poincaré inequalities, logarithmic Sobolev inequalities and HWI-inequalities. Further, with the help of Bochner's inequality, one can hope to redo many proofs of the Dirichlet form setting to get even more inequalities.

For the proof we will follow chapter 6 of the recent book by Bakry, Gentil & Ledoux [BGL]. The idea is the following: In [EKS], the authors were able to adapt the transport based proofs of the classical HWI and logarithmic Sobolev inequalities to get finite-dimensional analogs. Bakry showed in [B] that for abstract Dirichlet forms such a finite dimensional Log-Sobolev inequality is already enough to prove the sharp Sobolev inequality. This can be done by showing a Nash inequality which yields a Sobolev inequality with non-optimal constants. Bringing into play various ingredients like Poincaré inequalities and regularity estimates, one can use a perturbed variational problem (in the spirit of the Yamabe problem) to obtain good extremizers which allow to perform the necessary computations (via so-called Γ-calculus) to optimize the constant. This idea is based on a series of papers by Rothaus [R2, R3, R4], who dealt with the logarithmic Sobolev inequality. We will collect many results without giving the proofs since in these cases the original proofs in the Dirichlet form setting work mostly literally; instead, we give precise references. These results are nevertheless new in the metric measure space setting.

2. Preliminaries

Here we briefly introduce notation and the basic objects. More details can be found in [AGS2], [AGS3] and [EKS], or, for a more compact overview, [AGS1], [G] or [A1]. Regarding the notation we should mention that, since we are only concerned with a single, fixed measure $\mathfrak{m}$, we omit writing $\mathfrak{m}$-almost everywhere, i.e. every pointwise looking (in)equality is meant $\mathfrak{m}$-a.e., unless otherwise stated!

To simplify the exposition, we assume from the beginning on that our metric measure spaces $(X,d,\mathfrak{m})$ consist of a complete, separable, compact, geodesic metric space (X,d) together with a Borel probability measure $\mathfrak{m}$ with full support. Not all of the properties of the metric space are necessary to define $\mathrm{CD}^e(K,N)$ spaces, but in the case $K>0$ all $\mathrm{CD}^e(K,N)$ spaces will have these properties anyway. For a function $f\colon X\to\mathbb{R}$ let $|\nabla f|(x) := \limsup_{y\to x}\frac{|f(y)-f(x)|}{d(y,x)}$ be the local Lipschitz constant. With this, the Cheeger energy is defined via a relaxation procedure as $\mathrm{Ch}\colon L^2(X,\mathfrak{m})\to[0,\infty]$,

$$\mathrm{Ch}\,(f) := \inf\left\{\liminf_{k\to\infty}\frac{1}{2}\int_X|\nabla f_k|^2\mathrm{d}\mathfrak{m}\;\middle|\; f_k\colon X\to\mathbb{R}\text{ Lipschitz, } f_k\to f \text{ in } L^2(X,\mathfrak{m})\right\}$$

with domain $W^{1,2}(X,d,\mathfrak{m}) := \{f\in L^2(X,\mathfrak{m}) \mid \mathrm{Ch}\,(f)<\infty\}$. It is a lower semi-continuous and convex functional on $L^2(X,\mathfrak{m})$. The element with minimal L^2-norm in the collection of weak gradients of f,

$$\{G\in L^2(X,\mathfrak{m}) \mid \exists\, f_k\colon X\to\mathbb{R}\text{ Lipschitz s.t. } f_k\to f, |\nabla f_k|\rightharpoonup G \text{ in } L^2(X,\mathfrak{m})\},$$

is the minimal weak gradient $|\nabla f|_w$ of f and provides an integral representation of the Cheeger energy, i.e. $\mathrm{Ch}\,(f)=\frac{1}{2}\int_X|\nabla f|_w^2\,\mathrm{d}\mathfrak{m}$. In this generality, the Cheeger energy need not be a quadratic form, so the Sobolev space equipped with the norm

$\|f\|_{W^{1,2}}^2 := \|f\|_2^2 + 2\,\mathrm{Ch}\,(f)$ is only a Banach space. With this norm, Lipschitz functions are dense in $W^{1,2}(X,d,\mathfrak{m})$.

Since the Cheeger energy is convex and lower semi-continuous, we can use the classical theory of gradient flows in Hilbert spaces (as presented e.g. in [AG, Section 3.1]). This gives us a Laplacian and a heat flow as the gradient flow of Ch in $L^2(X,\mathfrak{m})$. More precisely, consider the subdifferential of Ch in $f \in W^{1,2}(X,d,\mathfrak{m})$ defined by

$$\partial^- \mathrm{Ch}\,(f) := \left\{ v \in L^2(X,\mathfrak{m}) \middle| \forall g \in L^2(X,\mathfrak{m}) : \int_X v(g-f)\,\mathrm{d}\mathfrak{m} \le \mathrm{Ch}\,(g) - \mathrm{Ch}\,(f) \right\}.$$

This is a closed, convex subset of $L^2(X,\mathfrak{m})$, so that in case it is non-empty, there is a well-defined element of minimal L^2-norm, which we will call $-\Delta f$. The domain of the Laplacian is then $D(\Delta) := \{f \in L^2(X,\mathfrak{m}) \big| \partial^- \mathrm{Ch}\,(f) \neq \emptyset\}$. The general theory now yields:

Theorem 2.1 (Heat flow, [AG],[AGS2, Thm. 4.16])**.** *For every $f \in L^2(X,\mathfrak{m})$ there is a unique locally absolutely continuous curve $(0,\infty) \to L^2(X,\mathfrak{m}), t \mapsto \mathrm{H}_t f$, such that*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_t f \in \partial^- \mathrm{Ch}\,(\mathrm{H}_t f) & \forall t > 0, \\ \mathrm{H}_t f \to f \text{ in } L^2(X,\mathfrak{m}) & \text{as } t \to 0. \end{cases}$$

Moreover, $\mathrm{H}_t f \in D(\Delta)$ for every $t > 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_t f = \Delta \mathrm{H}_t f$ for a.e. $t > 0$, and we have the following properties:

i) *Comparison principle: Let $C \in \mathbb{R}$. If $f \ge C$, then $\mathrm{H}_t f \ge C$ for every $t > 0$. Analogously, if $f \le C$ then $\mathrm{H}_t f \le C$ for every $t > 0$.*
ii) *Mass preservation: For every $t > 0$: $\int_X \mathrm{H}_t f\,\mathrm{d}\mathfrak{m} = \int_X f\,\mathrm{d}\mathfrak{m}$.*
iii) *Contraction: For every $p \in [1,\infty]$ and every $f \in L^2(X,\mathfrak{m}) \cap L^p(X,\mathfrak{m})$ we have: $\|\mathrm{H}_t f\|_p \le \|f\|_p$.*

Remarks.

- One should be aware, that in this generality (i.e. when the Cheeger energy is not a quadratic form) the Laplacian and the heat flow might not be linear.
- By density of $L^2 \cap L^p$ in L^p one can extend the semigroup and the Laplacian to operators on $L^p(X,\mathfrak{m})$.
- In $\mathrm{CD}(K,\infty)$ spaces, the heat flow defined above coincides with the metric gradient flow of the relative entropy in $(\mathcal{P}_2, W_2)$, see [AGS2, Thm. 9.3].

If the Cheeger energy is a quadratic form, or, equivalently, if $W^{1,2}(X,d,\mathfrak{m})$ is a Hilbert space, $(X,d,\mathfrak{m})$ will be called infinitesimally Hilbertian. In this case one can define the "scalar product" for the gradients of two functions $f,g \in W^{1,2}(X,d,\mathfrak{m})$ by polarization:

$$\langle \nabla f, \nabla g \rangle := \frac{1}{4}\left(|\nabla(f+g)|_w^2 - |\nabla(f-g)|_w^2 \right),$$

This is a symmetric and bilinear operator $W^{1,2}(X,d,\mathfrak{m}) \times W^{1,2}(X,d,\mathfrak{m}) \to L^1(X,\mathfrak{m})$. Since we have to care about regularity, we collect here the most important calculus rules in the versions we will need later.

Lemma 2.2 (Calculus rules)**.** *Let $(X,d,\mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Then:*

i) *for $f,g \in W^{1,2}(X,d,\mathfrak{m})$ and every $c \in \mathbb{R}$ it holds: $|\nabla f|_w = |\nabla g|_w$ on $\{f - g = c\}$ (locality),*
ii) *for a Lipschitz function $\varphi \colon \mathbb{R} \to \mathbb{R}$ and $f,g \in W^{1,2}(X,d,\mathfrak{m})$ it holds: $\varphi \circ f \in W^{1,2}(X,d,\mathfrak{m})$ and $|\nabla(\varphi \circ f)|_w = |\varphi' \circ f||\nabla f|_w$ and $\langle \nabla(\varphi \circ f), \nabla g \rangle = \varphi' \circ f \langle \nabla f, \nabla g \rangle$; in particular, if φ is a contraction, then $|\nabla(\varphi \circ f)|_w \le |\nabla f|_w$ and $\mathrm{Ch}\,(\varphi \circ f) \le \mathrm{Ch}\,(f)$ (chain rule and contraction property),*

iii) *for* $f,g,h\in W^{1,2}(X,d,\mathfrak{m})\cap L^\infty(X,\mathfrak{m})$ *also* $fg\in W^{1,2}(X,d,\mathfrak{m})$ *and*

$$\langle\nabla(fg),\nabla h\rangle = f\langle\nabla g,\nabla h\rangle + g\langle\nabla f,\nabla h\rangle,$$

iv) *for* $f\in W^{1,2}(X,d,\mathfrak{m})\cap D(\Delta)$ *and* $g\in W^{1,2}(X,d,\mathfrak{m})$ *it holds:*

$$\int_X \langle\nabla f,\nabla g\rangle\,\mathrm{d}\mathfrak{m} = -\int_X g\Delta f\,\mathrm{d}\mathfrak{m},$$

v) *for* $\varphi\in C^2(\mathbb{R})$ *and* $f\in W^{1,2}(X,d,\mathfrak{m})\cap D(\Delta)\cap L^\infty(X,\mathfrak{m})$ *with* $|\nabla f|_w^2\in L^\infty(X,\mathfrak{m})$ *we have* $\varphi\circ f\in D(\Delta)$ *and* $\Delta(\varphi\circ f)=\varphi'\circ f\Delta f+\varphi''\circ f|\nabla f|_w^2$.

These results are spread over different articles and can be found e.g. in [AGS1], [AGS3], [G], [S1, (2.4)]

Now we turn to the curvature-dimension condition. We assume the reader to be familiar with the basics of optimal transport. They can be found in [V] or [AG]. Let $(\mathscr{P}_2(X),W_2)$ be the L^2-Wasserstein space, i.e. $\mathscr{P}_2(X)$ is the space of Borel probability measures over X with finite second moment and W_2 the Wasserstein distance. The relative entropy functional is $\mathrm{Ent}\colon\mathscr{P}_2(X)\to(-\infty,\infty]$, $\mathrm{Ent}\,(\mu):=\int_X f\log f\,\mathrm{d}\mathfrak{m}$ if $\mu=f\mathfrak{m}$ and $\mathrm{Ent}\,(\mu):=+\infty$ otherwise; the domain will be denoted by $D(\mathrm{Ent}):=\{\mu\in\mathscr{P}_2(X)\mid \mathrm{Ent}\,(\mu)<\infty\}$. By abuse of notation, for an absolutely continuous measure $\mu=f\mathfrak{m}$ we will write $\mathrm{Ent}\,(f)$. For $N\in(0,\infty)$, the entropy power functional is $U_N\colon\mathscr{P}_2(X)\to[0,\infty]$, $U_N(\mu):=e^{-\frac{1}{N}\mathrm{Ent}(\mu)}$. We will also need the distortion coefficients: for $K\in\mathbb{R}$, $N\in(0,\infty)$, $\vartheta\in[0,\infty)$ and $t\in[0,1]$ we define

$$\sigma_{K,N}^{(t)}(\vartheta):=\begin{cases}\frac{\sinh(\sqrt{-K/N}\vartheta t)}{\sinh(\sqrt{-K/N}\vartheta)}, & \text{if } \frac{K}{N}\vartheta^2<0,\\ t, & \text{if } \frac{K}{N}\vartheta^2=0,\\ \frac{\sin(\sqrt{K/N}\vartheta t)}{\sin(\sqrt{K/N}\vartheta)}, & \text{if } 0<\frac{K}{N}\vartheta^2<\pi^2,\\ +\infty, & \text{if } \frac{K}{N}\vartheta^2\geq\pi^2.\end{cases}$$

Definition 2.3 (Curvature-dimension conditions)**.** Let $K\in\mathbb{R}$, $N\in(0,\infty)$. A metric measure space $(X,d,\mathfrak{m})$ satisfies the entropic curvature-dimension condition $\mathrm{CD}^e(K,N)$ if for every pair $\mu,\nu\in D(\mathrm{Ent})$ there is a constant speed geodesic $(\mu_t)_{t\in[0,1]}\subset D(\mathrm{Ent})$ with $\mu_0=\mu,\mu_1=\nu$ such that for all $t\in[0,1]$:

$$U_N(\mu_t)\geq\sigma_{K,N}^{(1-t)}\left(W_2(\mu,\nu)\right)U_N(\mu)+\sigma_{K,N}^{(t)}\left(W_2(\mu,\nu)\right)U_N(\nu). \tag{2.1}$$

It is a *strong* $\mathrm{CD}^e(K,N)$ space, if (2.1) holds for *every* geodesic.
The space satisfies the Riemannian curvature-dimension condition $\mathrm{RCD}^*(K,N)$, if it is a $\mathrm{CD}^e(K,N)$ space and additionally infinitesimally Hilbertian.

Remark. Since there is a whole jungle of curvature-dimension conditions, let us mention a few relations – at least for some of them. Already the original definitions of Sturm on the one hand, and Lott & Villani on the other hand differ; while the convexity property only has to be fulfilled for one particular entropy functional in Sturm's setting, it has to be satisfied for a whole family of entropy functionals in the setting of Lott & Villani. So the latter is a priori a stronger condition, but they are for instance equivalent in non-branching spaces (see [V, Theorem 30.32]). Later, Bacher & Sturm [BS] introduced the *reduced* curvature-dimension condition $\mathrm{CD}^*(K,N)$ to get a local-to-global result which was not available before. This condition is obtained by changing the original distortion coefficients to slightly smaller ones. For $K>0$ the following relations hold true between the original definition of Sturm and the reduced curvature-dimension condition:

$$\mathrm{CD}(K,N)\Rightarrow\mathrm{CD}^*(K,N)\Rightarrow\mathrm{CD}(K^*,N),$$

with $K^* := \frac{N-1}{N}K$. (From this follows for instance, that the reduced condition implies the same geometric and analytic consequences, but with slightly worse constants.) Concerning the new *entropic* curvature-dimension condition $\mathrm{CD}^e(K,N)$, we mention the result of Erbar, Kuwada & Sturm, that in essentially non-branching metric measure spaces (see [EKS, Def. 3.10] for a definition), $\mathrm{CD}^*(K,N)$ and $\mathrm{CD}^e(K,N)$ are equivalent [EKS, Thm. 3.12]. Finally, the *Riemannian* curvature-dimension condition (i.e. $\mathrm{CD}^e(K,N)$ plus inf. Hilbertian) is equivalent to an EVI formulation of the heat flow, which implies that $\mathrm{RCD}^*(K,N)$ spaces are in particular strong $\mathrm{CD}(K,\infty)$ spaces and thus the $\mathrm{RCD}^*(K,N)$ condition is also equivalent to $\mathrm{CD}^*(K,N)$ plus inf. Hilbertianity [EKS, Thm. 3.17] (since strong $\mathrm{CD}(K,\infty)$ spaces are essentially non-branching [RS, Thm. 1.1]).

3. N-Log-Sobolev, Nash and a first Sobolev inequality

Originally, the N-Log-Sobolev inequality was proven in [B, Proposition 6.7] under a curvature-dimension condition via heat flow monotonicity, i.e. by deriving a differential inequality for $\mathrm{Ent}\,(\mathrm{H}_t\, f)$. But recently, Erbar, Kuwada and Sturm rediscovered it with a transport based proof via an analogue of the HWI inequality. We just rewrite their result in a convenient form.

Proposition 3.1 (N-Log-Sobolev inequality, [EKS, Corollary 3.29])**.** *Let $(X,d,\mathfrak{m})$ be a $\mathrm{CD}^e(K,N)$ space with $K>0$, $N\in(0,\infty)$. Then for every $f\in W^{1,2}(X,d,\mathfrak{m})$ with $\int_X f^2\,\mathrm{d}\mathfrak{m}=1$ we have*

$$\mathrm{Ent}\left(f^2\right) \le \frac{N}{2}\log\left(1+\frac{8}{KN}\,\mathrm{Ch}\,(f)\right).$$

Proof. Use the result in [EKS] with $\mu=f^2\mathfrak{m}$ and observe that $8\,\mathrm{Ch}\,(f)=I(\mu)$, where I denotes the Fisher information functional. ☐

Proposition 3.2 (Nash inequality)**.** *Let $(X,d,\mathfrak{m})$ be a $\mathrm{CD}^e(K,N)$ space with $K>0$, $N\in(0,\infty)$. Then for every $f\in W^{1,2}(X,d,\mathfrak{m})$ it holds*

$$\|f\|_2^{N+2} \le \left[\|f\|_2^2+\frac{8}{KN}\,\mathrm{Ch}\,(f)\right]^{N/2}\|f\|_1^2.$$

Proof. See [BGL, Proposition 6.2.3(ii)] for the simple proof. One can easily see that the proof does not really require a quadratic form. The N-Log-Sobolev inequality and the fact that the Cheeger energy is in any case 2-homogeneous is enough to do the necessary computations. The basic idea is that by Hölder's inequality the function $r\mapsto\varphi(r):=\log\|f\|_{1/r}$ is convex and

$$-\,\mathrm{Ent}\left(\frac{|f|^2}{\|f\|_2^2}\right)=\varphi'(1/2)\le\frac{\varphi(1)-\varphi(1/2)}{1/2}=-\log\frac{\|f\|_2^2}{\|f\|_1^2},$$

so that the N-Log-Sobolev inequality finally gives the result. ☐

Proposition 3.3 (Sobolev inequality)**.** *Let $(X,d,\mathfrak{m})$ be a $\mathrm{CD}^e(K,N)$ space with $K>0$, $N\in(2,\infty)$. Then there are $A\ge1$, $B>0$ (depending only on K and N), such that for every $f\in W^{1,2}(X,d,\mathfrak{m})$:*

$$\|f\|_{2^*}^2 \le A\,\|f\|_2^2+B\,\mathrm{Ch}\,(f)\,,$$

where $2^*:=\frac{2N}{N-2}$ is the critical Sobolev exponent.

Proof. We omit the quite technical proof via the slicing method introduced in [BCLSC]. It can be found in [BGL, Proposition 6.2.3(iii)] for Dirichlet forms. Let us just say that one considers the slices, where f takes values between 2^k and $2^{k+1}, k\in\mathbb{Z}$, and then estimates them via the Nash inequality and many calculations. In the end one sums it up. The only critical ingredients are the Nash inequality above and basic properties of the Cheeger energy like the contraction property and the locality. Again, we do not need Ch to be quadratic; all necessary properties are always valid. □

Remarks.
- Out of the proof one gets explicit constants, namely:
$$A = \frac{1}{3} 2^{\frac{2(3N-2)}{N-2}} \quad \text{and} \quad B = \frac{1}{KN} 2^{\frac{7N-6}{N-2}}.$$
- This Sobolev inequality yields the Sobolev embedding $W^{1,2}(X, d, \mathfrak{m}) \subset L^{2^*}(X, \mathfrak{m})$. Later, under the additional assumption that the Cheeger energy is quadratic, we get the compact Sobolev embedding $W^{1,2}(X, d, \mathfrak{m}) \Subset L^q(X, \mathfrak{m})$ for every $q \in [1, 2^*)$.

4. Some applications

In this section we collect a few consequences of the preceding inequalities which we will need later on. From this section on we are assuming that our spaces satisfy the $\mathrm{RCD}^*(K, N)$ condition, so that the heat flow and the Laplacian are linear.

Let us first mention that the heat flow is ultracontractive, meaning that for every $p, q \in [1, \infty]$ with $p < q$ there is a constant $C > 0$ (only depending on K and N) such that for every $t \in (0, 1]$ and every $f \in L^p(X, \mathfrak{m})$:

$$\| \mathrm{H}_t f\|_q \leq C t^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_p. \tag{4.1}$$

For a proof see [BGL, Theorem 6.3.1] which is based on [CKS]. It is a classical "heat flow monotonicity" proof, i.e. you differentiate some object (in this case the squared L^2-norm) along the heat flow and then get a differential inequality you can easily solve. It is the same here because we have the Nash inequality and the identity $\frac{\mathrm{d}}{\mathrm{d}t} \| \mathrm{H}_t f\|_2^2 = -4 \, \mathrm{Ch}\,(\mathrm{H}_t f)$. There are several different formulations of ultracontractivity, for instance by not specifying the t behaviour; out of the proof one actually gets an estimate which is valid for all $t > 0$, namely

$$\| \mathrm{H}_t f\|_\infty \leq \left(\frac{1}{1 - e^{-Kt}} \right)^{\frac{N}{2}} \|f\|_1.$$

(By the Riesz-Thorin interpolation theorem you can recover the estimate for general $p < q$.) The above version (4.1) implies a useful bound on the resolvent $R_\lambda := (\lambda \,\mathrm{id} - \Delta)^{-1}, \lambda > 0$, which we will need later.

Lemma 4.1. *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}^*(K, N)$ space with $K > 0$, $N \in (2, \infty)$. If $1 \leq p \leq \frac{N}{2}$, then the resolvent as operator*

$$R_\lambda \colon L^p(X, \mathfrak{m}) \to L^q(X, \mathfrak{m})$$

is bounded for every $q < \frac{pN}{N-2p}$, whereas for $p > \frac{N}{2}$

$$R_\lambda \colon L^p(X, \mathfrak{m}) \to L^\infty(X, \mathfrak{m})$$

is bounded.

Proof. See [BGL, Corollary 6.3.3] for the proof. Here, we only sketch the argument. If we use, that the resolvent can be expressed by the heat semigroup via a Laplace transform, then (4.1) yields the following:

$$\|R_\lambda f\|_q \leq \int_0^\infty e^{-\lambda t} \| \mathrm{H}_t f\|_q \,\mathrm{d}t \leq C \int_0^1 t^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_p \,\mathrm{d}t + \int_1^\infty e^{-\lambda t} \| \mathrm{H}_t f\|_q \,\mathrm{d}t.$$

The first integral converges for the range of p, q stated in the Lemma and the second integral can be bounded since by contractivity of H_t in every L^p the norm $\| \mathrm{H}_t f\|_q$ is non-increasing in t. □

The following results demonstrate how useful various forms of Poincaré inequalities are. Let us first show the compact Sobolev embedding and then turn to an improvement of the Sobolev inequality.

Proposition 4.2 (Rellich-Kondrachov)**.** *Let $(X,d,\mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ space with $K>0$, $N\in(2,\infty)$. Let $f\colon\mathbb{N}\to W^{1,2}(X,d,\mathfrak{m})$ be a bounded sequence, i.e. such that $\sup_{k\in\mathbb{N}}\|f_k\|^2_{W^{1,2}}<\infty$. Then there is $f_*\in W^{1,2}(X,d,\mathfrak{m})$ and a subsequence f_{k_ℓ}, such that for every $q\in[1,2^*)$,*

$$f_{k_\ell}\to f_* \text{ in } L^q(X,\mathfrak{m}).$$

Proof. As observed in [EKS, Theorem 4.22], the result [HK, Theorem 8.1] can be applied to yield the compact embedding. Since this is not obvious, let us carry out the argument in more detail. Their theorem shows that the compactness for a sequence such that $\|f_k\|_1+\||\nabla f_k|_w\|_2$ is bounded can be deduced if the space supports a Poincaré inequality of the form

$$\fint_{B_r(z)}\left|f-\fint_{B_r(z)}f\,\mathrm{d}\mathfrak{m}\right|\mathrm{d}\mathfrak{m}\le Cr\left(\fint_{B_{2r}(z)}|\nabla f|_w^2\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}}, \tag{4.2}$$

together with a Sobolev inequality of the form

$$\|u\|_{2^*}\le C(\|u\|_1+\||\nabla u|_w\|_2). \tag{4.3}$$

First observe that we can replace (4.3) by our Sobolev inequality in Proposition 3.3, if we also replace the boundedness assumption by the boundedness of $\|f_k\|^2_{W^{1,2}}$, since this is only used to get a weakly converging subsequence in L^{2^*}. Concerning the Poincaré inequality, we want to use that in spaces such that the Rényi entropy functional $S_N\colon\mathcal{P}_2(X)\to\mathbb{R}, S_N(f\mathfrak{m}):=-\int_X f^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}$ is strongly convex we have a suitable weak local Poincaré inequality by [R1, Corollary 1]. By [EKS, Rem. 3.18, Cor. 3.13] and consistency in K, $\mathrm{RCD}^*(K,N)$ spaces with $K>0$ are also strong $\mathrm{CD}^*(0,N)$ spaces, i.e. the Rényi entropy is strongly convex in the Wasserstein space $(\mathcal{P}_2,W_2)$ (recall that "strongly" means that the convexity inequality holds for *every* geodesic in $\mathcal{P}_2$). Thus we can apply the result of [HK]. □

Let us also mention the following improvement of the Sobolev inequality by "tightening" the constants to $A=1$.

Proposition 4.3. *Let $(X,d,\mathfrak{m})$ be an $\mathrm{RCD}^*(K,N)$ space with $K>0$, $N\in(2,\infty)$. Then there is a constant $\tilde{B}$ (depending on A,B in Prop. 3.3 and on the Poincaré constant) such that for every $f\in W^{1,2}(X,d,\mathfrak{m})$:*

$$\|f\|^2_{2^*}\le\|f\|_2^2+\tilde{B}\,\mathrm{Ch}\,(f)\,. \tag{4.4}$$

Proof. We only sketch the proof which can be found in [BGL, Proposition 6.2.2]. First let us show that for $f\in L^{2^*}(X,\mathfrak{m})$

$$\|f\|^2_{2^*}\le\left(\int_X f\,\mathrm{d}\mathfrak{m}\right)^2+(2^*-1)\left\|f-\int_X f\,\mathrm{d}\mathfrak{m}\right\|^2_{2^*}. \tag{4.5}$$

Observe that this inequality is true if $\int_X f\,\mathrm{d}\mathfrak{m}=0$, and it is homogeneous in the sense that if it holds for f, then it also holds for $cf, c\in\mathbb{R}$. Thus it suffices to assume that $\int_X f\,\mathrm{d}\mathfrak{m}=1$. Then we can decompose it as $f=1+\tilde{r}g$ with $\tilde{r}\in\mathbb{R}$ and a function g such that $\int_X g\,\mathrm{d}\mathfrak{m}=0$ and $\int_X g^2\,\mathrm{d}\mathfrak{m}=1$. If we consider the function $\psi(r):=\|1+rg\|^2_{2^*}$, then differentiating twice in r and using Hölder's inequality yields the following differential inequality

$$\psi''(r)\le 2(2^*-1)\|g\|^2_{2^*}$$
$$\psi(0)=1,\ \psi'(0)=0.$$

By integrating this we get the desired inequality

$$\psi(r) \leq 1 + (2^* - 1) r^2 \|g\|_{2^*}^2 .$$

Now we need the space to support a global Poincaré inequality of the form

$$\left\| f - \int_X f \, \mathrm{d}\mathfrak{m} \right\|_2^2 \leq C_P \, \mathrm{Ch}\,(f) .$$

This is true in spaces satisfying the curvature-dimension condition of Lott & Villani (which is a priori stronger than the original $\mathrm{CD}(K,N)$ condition of Sturm), see [LV, Theorem 6.18]. As [RS, Cor. 1.3] proved, $\mathrm{RCD}^*(K,N)$ spaces satisfy the condition of Lott & Villani, so that we indeed have the Poinaré inequality. Let us now finish the proof. Applying Jensen's inequality on the first term and using the Sobolev and the Poincaré inequality on the second term of the right-hand-side of (4.5), we have our result. □

Remark. Actually, by [RS] we would already get the curvature-dimension condition of Lott & Villani for *essentially non-branching* $\mathrm{CD}(K,N)$ spaces (see [RS] or [EKS] for a definition). This means that the Proposition is also true for essentially non-branching $\mathrm{CD}^e(K,N)$ spaces instead of $\mathrm{RCD}^*(K,N)$ spaces.

5. The sharp constant

Now we can begin with the proof of the main result. It will be divided into two parts. First we have to establish the existence of good "almost" extremal functions, which will be extremal for a perturbed inequality. Subsequently we will use the associated Euler-Lagrange equation with some "Γ-calculus" and Bochner's inequality to get an upper bound for the sharp constant. This is based on [BGL, Theorem 6.8.3].

5.1. **Extremal functions.** We have to perturb the tight inequality (4.4) to get good extremizers via the direct method of the calculus of variations. Thus the inequality we consider now is

$$\|f\|_q^2 \leq (1+\varepsilon) \|f\|_2^2 + \tilde{B} \, \mathrm{Ch}\,(f) \quad \text{for } q \in [2, 2^*),\ \varepsilon > 0. \tag{5.1}$$

Replacing $\tilde{B}$ by the (strictly positive) sharp constant

$$C(q,\varepsilon) := \inf\{C > 0 \big| \, \forall f \in W^{1,2}(X,d,\mathfrak{m}) : \|f\|_q^2 \leq (1+\varepsilon) \|f\|_2^2 + C \, \mathrm{Ch}\,(f)\}, \tag{5.2}$$

we can pose a variational problem. To this end, let

$$F_{q,\varepsilon}(f) := (1+\varepsilon) \|f\|_2^2 - \|f\|_q^2 + C(q,\varepsilon) \, \mathrm{Ch}\,(f) .$$

This functional is non-negative on $W^{1,2}(X,d,\mathfrak{m})$, since the Sobolev inequality (5.1) is still valid with $C(q,\varepsilon)$ in place of $\tilde{B}$. Then finding an extremal function for the inequality corresponds to finding a minimizer for the functional $F_{q,\varepsilon}$. The following lemma assures the existence of a minimizing function and lists some first properties of it, which will be needed later.

Lemma 5.1. *It holds*

$$\inf\{F_{q,\varepsilon}(f) \big| f \in W^{1,2}(X,d,\mathfrak{m}) \ \text{s.t.}\ \|f\|_q = 1\} = 0,$$

and there is a minimizer $f_* \in \{f \in W^{1,2}(X,d,\mathfrak{m}) \big| \|f\|_q = 1\}$ *of* $F_{q,\varepsilon}$ *that satisfies:*

i) f_* *is not constant,*

ii) $f_* \in D(\Delta)$ *and it satisfies the Euler-Lagrange equation*

$$f_*^{q-1} = (1+\varepsilon) f_* - \frac{1}{2} C(q,\varepsilon) \Delta f_* . \tag{5.3}$$

iii) $f_* \in L^\infty(X,\mathfrak{m})$ *and there is a positive constant such that* $f_* \geq c > 0$, *Proof.* To gain existence we first consider the minimization problem

$$\inf\{F_{q,\varepsilon}(f)\big|f \in W^{1,2}(X,d,\mathfrak{m}) \text{ s.t. } \|f\|_{W^{1,2}} = 1\}.$$

In this class of functions one can show existence of a minimizer $\tilde{f}_*$ by an application of the direct method of the calculus of variations in a standard way, since the Cheeger energy is L^2-lower semicontinuous and we already have the compact Sobolev embedding. Because of the optimality of the constant $C(q,\varepsilon)$ the minimum of $F_{q,\varepsilon}$ is zero. By the contraction property $\mathrm{Ch}\,(|f|) \leq \mathrm{Ch}\,(f)$ one can take a minimizing sequence of non-negative functions, so that the minimizer is non-negative, too. Next, observe that $F_{q,\varepsilon}$ is 2-homogeneous, i.e. for every $c \geq 0$ it holds $F_{q,\varepsilon}(cf) = c^2 F_{q,\varepsilon}(f)$. This allows us to scale our minimizer $\tilde{f}_*$: if we take $f_* := \tilde{f}_*/\|\tilde{f}_*\|_q$, then $\|f_*\|_q = 1$ and

$$F_{q,\varepsilon}(f_*) = F_{q,\varepsilon}\left(\tilde{f}_*/\|\tilde{f}_*\|_q\right) = \frac{1}{\|\tilde{f}_*\|_q^2} F_{q,\varepsilon}(\tilde{f}_*) = 0,$$

as desired. So f_* is also a minimizer of $F_{q,\varepsilon}$ (but this time with the constraint claimed in the lemma). This scaling makes the Euler-Lagrange equation more convenient.

Since the minimizer is not the zero function, it can also not be a constant $c > 0$, because otherwise $0 = F_{q,\varepsilon}(c) = \varepsilon c^2$, which is a contradiction. The Euler-Lagrange equation follows in the standard way by computing $\frac{\mathrm{d}}{\mathrm{d}\,\tau}\big|_{\tau=0} F_{q,\varepsilon}(f_* + \tau v)$ for $v \in W^{1,2}(X,d,\mathfrak{m})$. The term with the Cheeger energy exactly results in the scalar product for weak gradients:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} \mathrm{Ch}\,(f_* + \tau v) = \int_X \langle \nabla f_*, \nabla v\rangle \,\mathrm{d}\mathfrak{m}.$$

That way we get the weak form of the Euler-Lagrange equation:

(5.4)
$$\int_X \left(f_*^{q-1} - (1+\varepsilon) f_*\right) v \,\mathrm{d}\mathfrak{m} = \frac{1}{2} C(q,\varepsilon) \int_X \langle \nabla f_*, \nabla v\rangle \,\mathrm{d}\mathfrak{m} \;\; \forall\, v \in W^{1,2}(X,d,\mathfrak{m}).$$

Considering iii), observe that the equation (5.3) can be rewritten in terms of the resolvent of the Laplacian:

$$f_* = \frac{2}{C(q,\varepsilon)} \left(\frac{2(1+\varepsilon)}{C(q,\varepsilon)}\,\mathrm{id} - \Delta\right)^{-1} f_*^{q-1} = \frac{2}{C(q,\varepsilon)} R_{\lambda^*}(f_*^{q-1}), \tag{5.5}$$

with $\lambda^* := \frac{2(1+\varepsilon)}{C(q,\varepsilon)}$. In view of Lemma 4.1, to get boundedness of the minimizer we only have to show that $f_*^{q-1} \in L^r(X,\mathfrak{m})$ for some $r > \frac{N}{2}$. We know that $f_* \in L^q(X,\mathfrak{m})$, so that $f_*^{q-1} \in L^{\frac{q}{q-1}}(X,\mathfrak{m})$. If $\frac{q}{q-1} \leq \frac{N}{2}$, then one has to use an iteration procedure based on Lemma 4.1 and the identity (5.5) to arrive at an exponent larger than $\frac{N}{2}$.

The strictly positive lower bound on f_* can be achieved by using the Euler-Lagrange equation in the form (5.5) and recalling that the resolvent is linked to the heat semigroup by a Laplace transform. If we choose some $T > 1$, this yields the following:

(5.6)
$$f_* = \frac{2}{C(q,\varepsilon)} R_{\lambda^*}(f_*^{q-1}) = \frac{2}{C(q,\varepsilon)} \int_0^\infty e^{-\lambda^* t} \mathrm{H}_t\, f_*^{q-1} \,\mathrm{d}t \geq \frac{2}{C(q,\varepsilon)} \int_1^T e^{-\lambda^* t} \mathrm{H}_t\, f_*^{q-1} \,\mathrm{d}t,$$

where in the inequality we used that $\mathrm{H}_t\, f_*^{q-1} \geq 0$ since the heat semigroup is positivity preserving. So we only have to estimate the heat semigroup. To do this we will use the following Harnack inequality of Garofalo and Mondino [GM, Theorem

1.4]: If $(X,d,\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space with $K>0, N\geq 1$, then for all $f\in L^1(X,\mathfrak{m})$ with $f\geq 0$ and for every $x,y\in X$ and $0<s<t$ it holds:

$$\mathrm{H}_t\, f(x)\geq \mathrm{H}_s\, f(y)\exp\left(-\frac{d(x,y)^2}{4(t-s)e^{\frac{2Ks}{3}}}\right)\left(\frac{e^{\frac{2Ks}{3}}-1}{e^{\frac{2Kt}{3}}-1}\right)^{\frac{N}{2}}.$$

We want to get a strictly positive lower bound on the right-hand-side. By fixing $s=1/2$ and taking the supremum over $y\in X$ we are essentially done: By [AGS3, Prop 6.4] and the ultracontractivity (4.1), the semigroup is a bounded operator $L^1(X,\mathfrak{m})\to \mathrm{Lip}(X,d)$, hence $\sup_{y\in X}\mathrm{H}_{1/2}\, f(y)>0$ is well-defined and by mass preservation of H_t strictly positive, if $\|f\|_1\neq 0$. Now, due to our restricted time intervall $t\in(1,T)$, we have an explicit strictly positive lower bound on the heat flow:

$$\mathrm{H}_t\, f(x)\geq \inf_{t\in(1,T)}\left[\left(\sup_{y\in X}\mathrm{H}_{1/2}\, f(y)\right)\exp\left(-\frac{(\mathrm{diam}\, X)^2}{4(t-\frac{1}{2})e^{\frac{K}{3}}}\right)\left(\frac{e^{\frac{K}{3}}-1}{e^{\frac{2Kt}{3}}-1}\right)^{\frac{N}{2}}\right]=: c_*>0.$$

Thus, setting $f:=f_*^{q-1}$, we see that f_* is strictly positive by (5.6).

□

5.2. **Γ-calculus.** In this subsection we conclude Theorem 1.1 by providing the necessary computations. To justify them we have to ensure that all involved functions have the right regularity to apply Lemma 2.2. For convenience we rather work with $g:=\ln f_*$ instead of f_*.

Lemma 5.2. *The function g defined as above satisfies:*

i) $g\in W^{1,2}(X,d,\mathfrak{m})\cap D(\Delta)\cap L^\infty(X,\mathfrak{m})$, $|\nabla g|_w^2\in W^{1,2}(X,d,\mathfrak{m})\cap L^\infty(X,\mathfrak{m})$ *and additionally* $\Delta g\in W^{1,2}(X,d,\mathfrak{m})\cap L^\infty(X,\mathfrak{m})$,

ii) $e^{bg}\in W^{1,2}(X,d,\mathfrak{m})\cap D(\Delta)\cap L^\infty(X,\mathfrak{m})$ *and* $\Delta e^{bg}\in W^{1,2}(X,d,\mathfrak{m})\cap L^\infty(X,\mathfrak{m})$ *for every* $b\in\mathbb{R}$.

Proof. i) One easily sees that $g\in L^\infty(X,\mathfrak{m})$ because $f_*\in L^\infty(X,\mathfrak{m})$. To see the other properties, we use the chain rules for the weak gradient and the Laplacian. Let $c_1,c_2>0$ such that $c_1\leq f_*\leq c_2$ and observe that $\ln\colon (c_1,c_2)\to\mathbb{R}$ is an increasing, Lipschitz continuous and twice differentiable function. Therefore the chain rule yields $g\in W^{1,2}(X,d,\mathfrak{m})$ and

$$|\nabla g|_w=|\nabla(\ln f_*)|_w=\frac{1}{f_*}|\nabla f_*|_w. \tag{5.7}$$

Because of the Euler-Lagrange equation (5.3) we know that $\Delta f_*\in L^\infty(X,\mathfrak{m})$. By the gradient interpolation [AMS, Theorem 3.1] we then see that $|\nabla f_*|_w^2\in L^\infty(X,\mathfrak{m})$; then (5.7) in particular implies $|\nabla g|_w^2\in L^\infty(X,\mathfrak{m})$. So we can apply the chain rule of the Laplacian to see that $g\in D(\Delta)$ and

$$\Delta g=\frac{1}{f_*}\Delta f_*-\frac{1}{f_*^2}|\nabla f_*|_w^2. \tag{5.8}$$

Thus also $\Delta g\in L^\infty(X,\mathfrak{m})$.
It remains to show that $\Delta g\in W^{1,2}(X,d,\mathfrak{m})$. This can be done by looking at the right-hand-side of (5.8). The chain rule tells us that $f_*^{q-1}\in W^{1,2}(X,d,\mathfrak{m})$, such that again by the Euler-Lagrange equation we see that $\Delta f_*\in W^{1,2}(X,d,\mathfrak{m})$. Let us briefly summarize what we have right now: $f_*\in D(\Delta)$ with $|\nabla f_*|_w^2\in L^\infty(X,\mathfrak{m})$ and $\Delta f_*\in W^{1,2}(X,d,\mathfrak{m})$. Therefore we can use [S1, Lemma 3.2] which shows, that $|\nabla f_*|_w^2\in W^{1,2}(X,d,\mathfrak{m})$. Hence $|\nabla g|_w^2\in W^{1,2}(X,d,\mathfrak{m})$ and also $\Delta g\in W^{1,2}(X,d,\mathfrak{m})$.

ii) Follows exactly as i). □

Now we are ready for the calculations. Let us state the precise result as a lemma:

Lemma 5.3. *For $q \in (2, 2^*)$ and $\varepsilon > 0$ it holds:*

$$C(q,\varepsilon) \le \frac{2(N-1)(q-2)(1+\varepsilon)}{NK}. \tag{5.9}$$

Proof. We proceed in two steps: First we deal with the Euler-Lagrange equation and then with Bochner's inequality. In both we have to insert composed functions to introduce enough parameters to match the calculations.

In order not to get lost in the lengthy computations, let us briefly see what we will show and how this proves the lemma. From the Euler-Lagrange equation we will get the following identity, valid for every $b \in \mathbb{R}$:

$$\begin{aligned}\int_X e^{bg}(\Delta g)^2 \,\mathrm{d}\mathfrak{m} = \frac{2(q-2)(1+\varepsilon)}{C(q,\varepsilon)} \int_X e^{bg}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} - (q-2+b)\int_X e^{bg}|\nabla g|_w^4 \,\mathrm{d}\mathfrak{m} \\ - (q-1+b)\int_X e^{bg}\Delta g|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m},\end{aligned} \tag{5.10}$$

while on the other hand the Bochner inequality will yield, for every $a, d \in \mathbb{R}$:

$$\begin{aligned}\int_X e^{(2a+d)g}(\Delta g)^2 \,\mathrm{d}\mathfrak{m} \ge \frac{KN}{N-1}\int_X e^{(2a+d)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} \\ - \left(a^2 + \frac{N}{N-1}ad + \frac{N}{2(N-1)}d^2\right)\int_X e^{(2a+d)g}|\nabla g|_w^4 \,\mathrm{d}\mathfrak{m} \\ - \left(2a + \frac{3N}{2(N-1)}d\right)\int_X e^{(2a+d)g}\Delta g|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{5.11}$$

Putting those two expressions together and comparing the coefficients results in the following system of equations:

$$\begin{aligned} b &= 2a + d \\ q - 1 + b &= 2a + \frac{3N}{2(N-1)}d \\ q - 2 + b &= a^2 + \frac{N}{(N-1)}ad + \frac{N}{2(N-1)}d^2. \end{aligned}$$

One solution to this is

$$\begin{aligned} d &= \frac{2(q-1)(N-1)}{N+2}, \\ b &= \frac{2(N-q+3)}{N+2} + 2\sqrt{\frac{(q-1)N(2N+2q-Nq)}{(N+2)^2}}, \\ a &= \frac{1}{2}b - \frac{1}{2}d = \frac{N-q+3}{N+2} - \frac{(q-1)(N-1)}{N+2} + \sqrt{\frac{(q-1)N(2N+2q-Nq)}{(N+2)^2}}. \end{aligned}$$

Observe that this is real-valued, because $2N + 2q - Nq > 0$ if $q < \frac{2N}{N-2} = 2^*$. Therefore we get an estimate between the two first terms on the right-hand-sides of (5.10) and (5.11):

$$\frac{2(q-2)(1+\varepsilon)}{C(q,\varepsilon)}\int_X e^{(2a+d)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} \ge \frac{KN}{N-1}\int_X e^{(2a+d)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m},$$

Since f_* – and thus also g – is non-constant, we get $\int_X e^{(2a+d)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} > 0$, so that we can divide by it and eventually get:

$$C(q,\varepsilon) \le \frac{2(q-2)(1+\varepsilon)(N-1)}{KN}.$$

This proves the lemma. So let us give some hints for the calculations.

Step 1: Euler-Lagrange equation
We will use the weak form of the Euler-Lagrange equation (5.4) with the representation $f_* = e^g$ and with the test function $v = e^{(b-1)g}\Delta g$. As we have seen in Lemma 5.2, together with a further use of the Leibniz rule, v is indeed in $W^{1,2}(X,d,\mathfrak{m})$. To keep the exposition clear, we proceed step by step. Starting with the left-hand-side of (5.4) and performing an integration by parts we get

$$\begin{aligned} I &:= \int_X \left(e^{(q-1)g} - (1+\varepsilon)e^g\right) e^{(b-1)g}\Delta g \,\mathrm{d}\mathfrak{m} \\ &= -\int_X \langle \nabla g, \nabla((e^{(q-1)g} - (1+\varepsilon)e^g)e^{(b-1)g})\rangle \,\mathrm{d}\mathfrak{m} \\ &= -(q-2+b)\int_X e^{(q-1)g}e^{(b-1)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} + (1+\varepsilon)b\int_X e^{bg}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{5.12}$$

To get rid of $e^{(q-1)g}$, we have to use the Euler-Lagrange equation again – this time with the test function $v = e^{(b-1)g}|\nabla g|_w^2 \in W^{1,2}(X,d,\mathfrak{m})$:

$$\begin{aligned} &\int_X e^{(q-1)g}e^{(b-1)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} \\ &= (1+\varepsilon)\int_X e^{bg}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} - \frac{C(q,\varepsilon)}{2}\int_X \Delta(e^g)e^{(b-1)g}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} \\ &= (1+\varepsilon)\int_X e^{bg}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} - \frac{C(q,\varepsilon)}{2}\int_X e^{bg}\Delta g|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} - \frac{C(q,\varepsilon)}{2}\int_X e^{bg}|\nabla g|_w^4 \,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Inserting this in (5.12), we finally arrive at

$$\begin{aligned} I = &-(1+\varepsilon)(q-2)\int_X e^{bg}|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} + (q-2+b)\frac{C(q,\varepsilon)}{2}\int_X e^{bg}\Delta g|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m} \\ &+ (q-2+b)\frac{C(q,\varepsilon)}{2}\int_X e^{bg}|\nabla g|_w^4 \,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{5.13}$$

With a similar calculation the right-hand-side can be written as

$$\begin{aligned} II &:= \frac{C(q,\varepsilon)}{2}\int_X \langle \nabla e^g, \nabla(e^{(b-1)g}\Delta g)\rangle \,\mathrm{d}\mathfrak{m} \\ &= -\frac{C(q,\varepsilon)}{2}\int_X e^{bg}(\Delta g)^2 \,\mathrm{d}\mathfrak{m} - \frac{C(q,\varepsilon)}{2}\int_X e^{bg}\Delta g|\nabla g|_w^2 \,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{5.14}$$

With (5.13) and (5.14), the Euler-Lagrange equation $I = II$ simplifies to (5.10).
Step 2: Bochner's inequality
The other part is based on the Bochner inequality in [EKS, Theorem 4.8]:
For every $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X,d,\mathfrak{m})$ and every $u \in D(\Delta) \cap L^\infty(X,\mathfrak{m})$ with $u \geq 0$ and $\Delta u \in L^\infty(X,\mathfrak{m})$ it holds

$$\frac{1}{2}\int_X \Delta u|\nabla f|_w^2 \,\mathrm{d}\mathfrak{m} - \int_X u\langle \nabla f, \nabla \Delta f\rangle \,\mathrm{d}\mathfrak{m} \geq K\int_X u|\nabla f|_w^2 \,\mathrm{d}\mathfrak{m} + \frac{1}{N}\int_X u(\Delta f)^2 \,\mathrm{d}\mathfrak{m}.$$

One just uses it with $f = e^{ag}$ and $u = e^{dg}$, $a, d \in \mathbb{R}$ and executes chain rules along with some integration by parts to get (5.11).
This completes the proof of Lemma 5.3. □

Finally, this easily finishes the proof of our main theorem.

Proof of Theorem 1.1. We will show that the sharp constant $C(q,\varepsilon)$ depends continuously on ε and q. Then we can take the limits $\varepsilon \to 0$ and $q \to 2^*$ in (5.9) to get the desired result. Let us start with the dependence on ε.
From the monotonicity of $C(q,\varepsilon)$ in ε and the bound $0 < C(q,\varepsilon) \leq C(q,0)$ we know

that $\lim_{\varepsilon\to 0} C(q,\varepsilon)$ exists. We now have to show that it equals $C(q,0)$. We proceed by contradiction: Assume $\lim_{\varepsilon\to 0} C(q,\varepsilon) \neq C(q,0)$. Then there is a sequence $\varepsilon_k \to 0$ s.t.

$$\lim_{k\to\infty} C(q,\varepsilon_k) < C(q,0).$$

For every $k \in \mathbb{N}$ we have the following Sobolev inequality:

$$\forall\, f \in W^{1,2}(X,d,\mathfrak{m}) : \quad \|f\|_q^2 \le (1+\varepsilon_k)\|f\|_2^2 + C(q,\varepsilon_k)\,\mathrm{Ch}\,(f)\,.$$

Taking the limit $k\to\infty$ on both sides yields that *for every* $f \in W^{1,2}(X,d,\mathfrak{m})$ we have

$$\begin{aligned}\|f\|_q^2 &\le \lim_{k\to\infty} \left[(1+\varepsilon_k)\|f\|_2^2 + C(q,\varepsilon_k)\,\mathrm{Ch}\,(f)\right] \\ &= \|f\|_2^2 + \lim_{k\to\infty} C(q,\varepsilon_k)\,\mathrm{Ch}\,(f)\end{aligned}$$

This is a contradiction to the optimality of $C(q,0)$.

The constant $C(q,\varepsilon)$ is also monotone in q so that the same proof works for the limit $q\to 2^*$ by noting that $\|f\|_q^2$ depends continuously on q. Therefore the bound (5.9) implies the sharp constant in Theorem 1.1. □

As one can see from the proof of Lemma 5.3, for $q < 2^*$ one even gets a slightly better constant. For more on this improved inequality, see [BGL, Remark 6.8.4, Remark 6.8.5].

Corollary 5.4. *For a subcritical exponent $q \in [2,2^*]$ we have the Sobolev inequality*

$$\forall\, f \in W^{1,2}(X,d,\mathfrak{m}) : \quad \|f\|_q^2 \le \|f\|_2^2 + \frac{2(N-1)(q-2)}{NK}\,\mathrm{Ch}\,(f)\,. \tag{5.15}$$

Also, rewriting (5.15) *as* $\frac{\|f\|_q^2 - \|f\|_2^2}{q-2} \le \frac{2(N-1)}{NK}\,\mathrm{Ch}\,(f)$ *and taking the limit $q\to 2$, we get the following sharp logarithmic Sobolev inequality originally due to* [B]*:*

$$\forall\, f \in W^{1,2}(X,d,\mathfrak{m}) \text{ with } \int_X f^2\,\mathrm{d}\mathfrak{m} = 1 : \quad \mathrm{Ent}\left(f^2\right) \le \frac{4(N-1)}{NK}\,\mathrm{Ch}\,(f)\,.$$

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