New Trends in Optimal Transport

2–6 March 2015

Abstracts

Luigi Ambrosio (*SNS Pisa*)

Extended metric measure spaces with Ricci lower bounds

I will report on some work in progress with M.Erbar and G.Savare'. The aim of the work is to extend to the class of extended metric measure spaces (i.e. spaces whose distance can attain the value infinity) some of the results obtained in collaboration with N.Gigli and G.Savare' for the class of metric measure spaces. In particular, independently of curvature assumptions, we want to relate the metric to the differentiable structure and to identify metric gradient flows of the entropy w.r.t. various (extended) distances. In connection with curvature, we relate gradient contractivity to contractivity of these distances. Finally, preliminary results about stability and connections with EVI are discussed.

Mathias Beiglböck (*Vienna*)

Towards time continuous martingale optimal transport

Motivated by problems in model-independent finance, different Monge-type martingale transport plans have been discovered by Hobson – Neuberger, Beiglbck – Julliet, Henry-Labordere – Touzi. We will present a new, unifying approach through a variational problem in continuous time. This is based on the mass transport viewpoint of the Skorokhod embedding problem established in collaboration with Cox and Huesmann.

Jean-David Benamou (*INRIA Rocquencourt*)

Iterative Bregman projections for regularized transportation problems

The idea which goes back to Schroedinger is to introduce an entropic regularization of the initial linear program version of optimal transportation. This regularized problem corresponds to a Kullback-Leibler Bregman divergence projection of a vector (representing some initial joint distribution) on the polytope of constraints. We show that for many problems related to optimal transport, the set of linear constraints can be split in an intersection of a few simple constraints, for which the projections can be computed in closed form. This allows us to make use of iterative Bregman projections (when there are only equality constraints) or more generally Bregman-Dykstra iterations (when inequality constraints are involved). We illustrate the usefulness of this approach on several variational problems related to optimal transport: barycenters for the optimal transport metric, tomographic reconstruction, multimarginal optimal transport and in particular its application to Brenier's relaxed solutions of incompressible Euler equations, partial un-balanced optimal transport and optimal transport with capacity constraints. This is joint work with Guillaume Carlier, Marco Cuturi, Luca Nenna and Gabriel Peyre.

Yann Brenier (*École Polytechnique*)

Monge-Ampere gravitation viewed as a large deviation principle for clouds of particles

On a Riemannian manifold, it is known from large deviation theory that brownian bridges between two given points concentrate aroung minimizing geodesics as the level of noise goes to zero. When considering the motion of clouds of indistinguishable points, instead of a single one, one may recover in a related way the model of Monge-Ampere gravitation which departs from classical Newtonian gravitation by substituting the Monge-Ampere equation for the Poisson equation.

Guillaume Carlier (*Paris Dauphine*)

Optimal transport and kinetic models of granular media

In this talk I will describe some kinetic models for granular media as first introduced by Benedetto, Caglioti, and Pulvirenti in 1997. The spatially inhomogeneous case is not very well-understood and, contrary to the spatially homogeneous case, the question of global existence or explosion in finite time is essentially open. Focusing on the one-dimensional case, I will first discuss some global estimates and in particular a global entropy bound when the interaction kernel is not too strong (in the sense that its second derivative is subquadratic close to zero). I will then discuss the special case of the quadratic kernel (for which the entropy bound is not valid) which has a very special structure: using a simple first integral of motion, generalized solutions of the initial kinetic equation can be defined. I will explain how these generalized solutions can be seen as Wasserstein gradient flows (but not on measures on the phase space but rather on suitable families of measures on the physical space). This is based on joint works with Martial Agueh and Reinhard Illner (UVIC, Victoria).

Maria Colombo (*SNS Pisa*)

Maximal flows of non-smooth vector fields and applications to PDEs

In 1989, Di Perna and Lions showed that Sobolev regularity for vector fields in R^d, with bounded divergence and a growth assumption, is sufficient to establish existence, uniqueness and stability of a generalized notion of flow, consisting of a suitable selection among the trajectories of the associated ODE. Their theory relies on a growth assumption on the vector field which prevents the trajectories from blowing up in finite time; in particular, it does not apply to fast-growing, smooth vector fields.

In this seminar we present a notion of maximal flow for non-smooth vector fields which allows for finite-time blow up of the trajectories. We show existence and uniqueness under only local assumptions on the vector field and we apply the result to a kinetic equation, the Vlasov-Poisson system, where we describe the solutions as transported by a suitable flow in the phase space. This allows, in turn, to prove existence of weak solutions for general initial data. (Joint work with Luigi Ambrosio and Alessio Figalli).

Dario Cordero-Erausquin (*Paris 6*)

A transport inequality on the sphere obtained by mass transport

Using the geometric properties of optimal transport (McCanns map), we establish a new transport inequality on compact manifolds with positive Ricci curvature that contains (after linearization) the sharp spectral comparison inequality.

Katy Craig (*UCLA*)

The discrete gradient flow for omega-convex functions in the Wasserstein metric

In recent years, there has been significant interest in interaction energies involving Newtonian attraction, as in the Keller-Segel equation for chemotaxis, and Newtonian repulsion, as in models of vortex densities in superconductors. When these energies are restricted to bounded densities, they satisfy a generalized convexity property known as omega-convexity, where omega is a log-Lipschitz modulus of convexity. In this talk, I will discuss the discrete gradient flow of omega-convex functions, including new quantitative estimates on the rate of convergence in the Wasserstein metric.

Marco Cuturi (*Kyoto*)

An overview of Wasserstein barycenter algorithms

We discuss in this talk practical approaches to solve the Wasserstein barycenter problem, both on measures supported on a discrete support in an Euclidean space and for those supported on a finite set. We present first the original formulations of Agueh and Carlier (2011), including the multi-marginal form and the LP form for measures supported on a discrete set, and review the different approaches that have been proposed after, namely sliced approaches, smoothed primal descent schemes, smooth and non-smooth dual schemes and Bregman iterative projections.

Nicolas Fournier (*Paris 6*)

Rate of convergence in Wasserstein distance of the empirical measure

We study how the empirical measure of N i.i.d. d-dimensional random variables is close to their common law, when measured in Wasserstein distance W_p. We provide some more or less optimal non-asymptotic L^p-bounds and concentration inequalities. This is a joint work with Arnaud Guillin.

Ivan Gentil (*Lyon*)

An optimal form of the Li-Yau inequality under a curvature bounded from below

The Li-Yau inequality gives regularity properties for the heat equation on a Riemannian manifold with a non-negative Ricci curvature. In particular, it gives contraction for the heat equation with respect to the Wasserstein distance. In this talk I will give an optimal form of the Li-Yau inequality on a Riemannian manifold with a Ricci curvature bounded from below.

Paola Gori-Giorgi (*VU Amsterdam*)

Optimal transport meets electronic density functional theory

Electronic structure calculations are at the very heart of predictive material science, chemistry and biochemistry. Their goal is to solve, in a reliable and computationally affordable way, the many-electron problem, a complex combination of quantum-mechanical and many-body effects. The most widely used approach, which achieves a reasonable compromise between accuracy and computational cost, is Kohn-Sham (KS) density-functional theory (DFT), for which Walter Kohn was awarded the Nobel Prize in Chemistry in 1998. Although exact in principle, practical implementations of KS-DFT must heavily rely on approximations for the so-called exchange-correlation functional. After a basic introduction on KS-DFT, I will show that an important piece of exact information on the exchange-correlation functional can be formulated as a multimarginal optimal transport problem with cost function given by the Coulomb repulsive potential. The implications for physics and chemistry will be illustrated with some applications to model quantum wires and quantum dots.

Bangxian Han (*Paris Dauphine*)

The continuity equation on metric measure spaces

Aim of this talk is to show that it makes sense to write the continuity equation on a metric measure space (X, d, m) and that absolutely continuous curves (μ_t) w.r.t. the distance W_2 can be completely characterized as solutions of the continuity equation itself, provided we impose the condition $\mu_t \leq Cm$ for every t and some $C > 0$. We will also see that some classical results such as Brenier-Benamou formula can be obtained in non-smooth case. This is a joint work with N. Gigli.

Nicolas Julliet (*Strasbourg*)

On the martingale transport problem

We consider optimal transport plans on $\mathbb{R}$ that enjoy the additional constraint that for almost every x, given the starting position x, the barycenter of the transported mass is x. We explain how cyclical monotonicity and duality can be adapted to this case. We also study the stability and relate our results to the theory of martingales. The results are based on a joint work with Mathias Beiglbck (Vienna).

Kazumasa Kuwada (*Tokyo Tech*)

Coupling by reflection of Brownian motions on $\mathrm{RCD}(K, \infty)$ spaces

In this talk, I will construct a certain coupling of two Brownian motions on a $\mathrm{RCD}(K, \infty)$ space satisfying the following property: The distribution of the coupling time of this coupling is estimated by that of a 1-dimensional Ornstein Uhlenbeck process. As a result, when $K \geq 0$, we can show that this coupled particles meet in a finite time with probability one. For the proof, I show a monotonicity of a time-dependent, concave transportation cost for heat flows. We have shown this monotonicity on a Riemannian manifold by employing a coupling by reflection constructed by means of stochastic differential geometry. Though the same method does not seem to work on RCD spaces, I will provide an alternative proof based on the (reverse) Gaussian isoperimetric inequalities for the heat semigroup.

Bruno Lévy (*INRIA Nancy*)

Optimal transport seen from a computer programmer's perspective

In this presentation, I give an elementary introduction to Optimal Transport, review some applications and explain how to compute it in 3D using a computer for the L_2 cost. Given two probability measures μ and ν, the problem consists in finding the application T that pushes μ on ν and that minimizes the transport cost $\int \|x - T(x)\|^2 d\mu$.

An efficient numerical solution mechanism for this problem is likely to facilitate the numerical simulation of some phenomena in physics, such as the evolution of the large scale density of matter in the universe [Frisch et.al, Nature 2002, Brenier et.al, MNRAS 2003]. The numerical methods used so far compute the optimal transport between two discrete measures (sums of Dirac masses), using a combinatorial algorithm, that does not scale-up well beyond a few tens of thousands Dirac masses.

In the case where μ has a density and ν is discrete ("semi-discrete" case), the pre-image of the Dirac masses through the optimal transport map T is a structure that is well known in computational geometry, called a power diagram. The parameters that define this power diagram (the weights) are determined as the unique maximum of a concave function [Aurenhammer et.al, Algorithmica 1998]. This characterization naturally leads to a numerical algorithm. Combined with a multi-scale approach, this algorithm makes it possible to efficiently compute the optimal transport in the plane [Merigot, Computer Graphics Forum 2011] and apply it to solving 2d Monge-Ampre type equations (Fokker planck, crowd dynamics, ...) [Benamou et.al, arXiv:1408.4536].

I show how to adapt this algorithm in 3d, in the case where μ has a piecewise-linear density, supported by a tetrahedral mesh, and where ν is discrete. The central part of the algorithm computes the intersection between a power diagram and a tetrahedral mesh by simultaneously propagating over both meshes. Degenerate configurations are handled by arbitrary precision computation methods [Shewchuk, DCG, 97] and symbolic perturbation [Edelsbrunner et.al, TOG, 90]. The parallel implementation of the algorithm computes optimal transport for problems with 1 million Dirac masses on a off-the-shelf PC.

The implementation of the algorithm is available at the following address:
`http://alice.loria.fr/software/geogram`

Daniel Matthes (*TU Munich*)

Discretizing Nonlinear Diffusion the Lagrangian Way

The interpretation of diffusion equations as gradient flows in suitable transportation metrics provides a "Lagrangian picture" of the evolution: the particle density moves along a (gradient) vector field that depends sensitively on the density itself.

We use that interpretation to define spatio-temporal "Lagrangian discretizations". That is, instead of calculating the change in density at given points, we trace the trajectories of "mass particles". The resulting schemes inherit various nice features of the original diffusion equations, like conservation of mass, preservation of positivity, energy dissipation and geodesic convexity.

Our main result concerns the rigorous analysis of the discrete-to-continuous-limit for a particular Lagrangian discretization of certain fourth order equations (like thin film and QDD) in one spatial dimension. The key estimates are obtained from the dissipation of an auxiliary discrete Lyapunov functional.

This is joint work with Horst Osberger (TU Muenchen).

Robert McCann (*Toronto*)

Unique optimal transport for smooth costs on manifolds with topology

The problem of characterizing extremal doubly stochastic measures dates back to Birkhoff (1946). It is closely connected with the question of which costs admit unique optimizers in the Monge-Kantorovich problem of optimal transport between arbitrary probability densities. For smooth costs on compact manifolds, the only known examples require at least one of the two underlying spaces to have the topology of the sphere. In this talk we explain how to construct a smooth cost on a pair of compact manifolds with arbitrary topology, so that the optimal transportation between any pair of probability densities is unique. This represents joint work with Ludovic Rifford (Nice).

Facundo Memoli (*Ohio*)

Persistence Diagrams of Metric Measure Spaces

We study the structure of collections of persistence diagrams that arise from taking the Vietoris-Rips filtration of all n-tuples of points from a given metric measure space. We consider what is the induced probability measure on that collection, and study stability of this measure in the Gromov-Wasserstein sense. These ideas provide a notion of statistics over persistence diagrams which is robust to perturbations in the input metric measure spaces.

Alexander Mielke (*WIAS Berlin*)

The chemical master equation as entropic gradient flow

The chemical master equation (CME) is the Kolmogorov forward equation for the time-continuous Markov process counting the number of molecules of the involved types of molecules, where the reaction rates are scaled by the total volume V. Under the assumption of detailed balance the nonlinear reaction-rate equation (RRE) of the macroscopic densities are given as a finite-dimensional entropic gradient flow, and as a consequence the CME is an infinite-dimensional entropic gradient system on the probability densities over a discrete lattice.

We discuss relation between the geodesic convexity properties of the the RRE and the CME. Using the energy-dissipation principle, we prove the evolutionary Gamma-convergence of the CME for volume $V \to \infty$ to the Liouville equation for the RRE.

This is joint work with Jan Maas, IST Austria.

Edouard Oudet (*Grenoble*)

Optimal networks and connectivity constraint

We first recall several problems and classical results related to optimal transport where the connectivity constraint appears. In a second part of this talk we focus on the simpler, but still delicate, Steiner tree problem. We describe the new research directions which have been explored last years to approximate optimal configurations. Finally, we introduced a new variational relaxation of Steiner's problem together with some partial numerical results.

Filippo Santambrogio (*Paris 11*)

Higher order optimal transport and gradient perturbations of the Monge problem

I will present several new variational problems in the class of transport maps T between two given measures, which involve not only the values of the map T, but also its derivatives, or its continuity and oscillations, together with their applications. Then, I will concentrate on a perturbation example, where we minimize the Monge cost $\int |T(x)-x|d\mu$ plus a gradient penalization $\epsilon \int |DT|^2 dx$, and study the limit as $\epsilon \to 0$

Giuseppe Savaré (*Pavia*)

Optimal transport and metric-Sobolev spaces

Optimal transport tools have been recently applied to obtain different characterizations and new properties of Sobolev spaces in the general framework of metric measure spaces (X,d,m). In particular the point of view of test plans (probability measures on the path space concentrated on absolutely continuous curves) provides a useful description of Sobolev spaces, that turns out to be equivalent to the usual ones based on the notion of p-Modulus (Fuglede, Koskela-Mac Manus and Shanmugalingham) and weak upper gradient (Cheeger). We will present an overview of some recent results in this direction, obtained in collaboration with L. Ambrosio, S. Di Marino and N. Gigli.

Carola Schönlieb (*University of Cambridge*)

A generalized model for optimal transport of images including dissipation and density modulation

In this talk I will present a new model in which the optimal transport and the metamorphosis perspectives are combined. For a pair of given input images geodesic paths in the space of images are defined as minimizers of a resulting path energy. To this end, the underlying Riemannian metric measures the rate of transport cost and the rate of viscous dissipation. Furthermore, the model is capable to deal with strongly varying image contrast and explicitly allows for sources and sinks in the transport equations which are incorporated in the metric related to the metamorphosis approach by Trouv'e and Younes. In the non-viscous case with source term existence of geodesic paths is proven in the space of measures. The proposed model is explored on the range from merely optimal transport to strongly dissipative dynamics. For this model a robust and effective variational time discretization of geodesic paths is proposed. This requires to minimize a discrete path energy consisting of a sum of consecutive image matching functionals. These functionals are defined on corresponding pairs of intensity functions and on associated pairwise matching deformations. Existence of time discrete geodesics is demonstrated. Furthermore, a finite element implementation is proposed and applied to instructive test cases and to real images. In the non-viscous case this is compared to the algorithm proposed by Benamou and Brenier including a discretization of the source term. Finally, the model is generalized to define discrete weighted barycentres with applications to textures and objects. This is joint work with Jan Maas, Martin Rumpf and Stefan Simon.

Christoph Schnörr (*Heidelberg*)

On image filtering by image patch assignment

Image denoising is mature field of research. The state of the art is consistently reflected both by the mathematical theory in terms of PDEs derived by the axiomatic approach, and by algorithms used in practice that exploit the geometry of image patches. In this talk, I will discuss a rudimentary step beyond iconic image processing motivated by the corresponding quest in computer vision. The focus is on the geometry of patch assignment and various links of the resulting image filtering approach to current work in computer vision and machine learning.

Takashi Shioya (*Tohoku University*)

Concentration, convergence and dissipation of spaces

Gromov introduced a new topology on the set of isomorphism classes of metric measure spaces, based on the idea of concentration of measure phenomenon due to Levy and Milman. This is a generalization of measured Gromov-Hausdorff topology. Different from the measured Gromov-Hausdorff topology, this new topology is suitable to study a non-GH-precompact family of spaces. In this talk, I show the study of convergence of spaces with unbounded dimension.

Anton Thalmaier (*Luxembourg*)

Local gradient-entropy estimates

Gradient estimates for positive solutions of heat equations on manifolds are a central theme at the crossroad of analysis, geometry, and probability theory. We report on recent progress and describe some of its applications.

Max von Renesse (*Leipzig*)

Modified Arratia Flow and Wasserstein Diffusion

We introduce a modified Arratia flow of sticky Brownian motions with masses under an additional preservation of total diffusivity of the full system. The induced measure valued flow is a weak solution to the SPDE of the Wasserstein diffusion but with a modified drift. Moreover, its large deviations in the short time asymptotics are given by the Wassserstein distance.